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The Medium Domination Number Of Cartesian Product Of Wheel Graph W_n And Fan Graph F_n

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Abstract

Graph theory, a major tool in mathematical research comprising of powerful methods, plays the most pivotal role in numerous sphere like social network and bio-logical science. The study of medium domination and medium dominating sets plays an imperative part in graph theory with applications in heterogeneous real world networks.
The paper presented here gives the overview of the historical roots of dominating sets and mainly delve in to the medium domination number of Cartesian product of wheel graph W_n and fan graph F_n.

Keywords: Domination number, Cartesian product, TDV, Medium domination number.

1. Introduction

In today's world, any situation can easily be framed diagrammatically with the set of points and lines. In the last three decades, a spectacular growth has been witnessed in graph theory due to its wide range of applications in optimization problems, combinatorial problems, networking problems etc. This is mainly due to the rise of a number of new-parameters that has been developed from the basic definition of domination.
Domination in graphs has many applications in numerous fields. Domination arises in facility location problems, where the number of facilities is fixed and one attempts to minimize the distance that a person wants to travel to get to the closest facility. Concepts from domination also appear in problems involving finding sets of representatives, in monitoring communication or electrical networks. The study of

dominations in graph improved the research led to different types of dominations in graphs such as total domination, weak domination, power domination etc. which in turn led to the concept of Medium domination numbers in graphs.

The research on dominating set in graph theory started around early sixties. However this branch of graph theory has its historical roots during 1862 when C. F. De Jaenisch [1] studied the problem of determine the minimum number of queens that are needed to dominate an nxn chessboard.

In 1892, W. W. Rouse Ball [2] has identified three basic types of chess problems namely covering problem, independent covering problem and independent problems related to dominating set. In 1958, Claude Berge [3] introduced the term domination number of a graph. Thereafter 1962, O. Ore[4] has used the terms dominating set and domination number for the same concept in graph theory.

In general a network is designed based on a graph where network design should be given more importance. Therefore graph model is an apt model to identify challenges and to find out remedial measures in graphical network design.

In social networks, we consider the hypothesis, that minimum order dominating sets contain agents with strong influence over the rest of the network. Thus domination theory is a special concept in graph theory. It is quite natural and appears in many situations.

A dominating set for a graph G=(V, E) is a subset D of V such that every vertex not in D is adjacent to at least one member of D. It is also used to protect the individual vertices in the graph but the medium dominating set is used to protect the pairs of vertices. In 2011, Duygu Vargor and Pinar D"undar[5] introduced the "medium domination number " of a graph.

2. Notations and Definitions:

Definition: 2. 1

A graph G consists of a pair (V(G), E(G)) where V(G) is a non-empty finite set whose elements are called points or vertices and E(G) is a set of unordered pairs of distinct elements of V(G). The elements of E(G) are called lines or edges of the graph G.

Definition: 2. 2

The degree of a vertex v in a graph G is the number of edges of G incident with v and it is denoted by deg(v_i). The maximum degree among the vertices of G is denoted by Δ(G). The minimum degree among the vertices of G is denoted by $\delta(G)$.

Definition: 2. 3

The graph G = (V, E) we mean a finite, undirected and connected graph with neither self loops nor multiple edges.

Definition: 2. 4

For G = (V, E) and $\forall u, v \in V$; if u and v are adjacent they dominate each other then at least $dom(u, v) = 1$.

Definition 2. 5
In the graph G = (V, E) and $\forall u, v \in V$; the total number of vertices that dominate every pair of vertices is defined as

$$TDV(G) = \sum_{\forall u,v \in V(G)} dom(u,v) \cdot$$

Definition 2. 6
In any connected, simple graph G of order n, the medium domination number of G is defined as $MD(G) = \frac{\mathrm{TDV}(\mathrm{G})}{nC_2}$

Definition 2. 7
For each $n \geq 4$, the wheel graph W_n with n vertices is defined to be the join K_1+C_{n-1} of an isolated vertex with a cycle of length n-1.

Definition 2. 8
The join K_1+P_{n-1} of K_1 and P_{n-1} is called a fan graph F_n. The vertex come from K_1 is called the core. The edges incident with the core are called spokes. The graph F_n has n vertices and 2n-3 edges.

3. Preliminary Notes on Medium Domination Number

Observation 3. 1:
dom(u, v) ≤ m(u, v).

Observation 3. 2:
dom(u, v) ≤min{deg(u), deg(v)}.

Observation 3. 3:
Let P_n be the path graph with n vertices; then we have $MD(P_n) = \frac{2\mathrm{n} - 3}{nC_2}$

Observation 3. 4:
Let C_n be the cycle graph with n vertices; then we have $MD(C_n) = \frac{2\mathrm{n}}{nC_2}$

Theorem 3. 1:
For G has n vertices, q edges and for deg(v_i)≥2; $TDV(G) = q + \sum_{v_i \in V} \left(\deg v_i C_2\right)$

Theorem 3. 2:
The medium domination number of the Cartesian product P_2 and P_n [6] is $MD(\mathrm{P}_2 \circ P_n) = \frac{9\mathrm{n} - 10}{2nC_2}, \; where\, n \geq 2$

Theorem 3. 3:
The medium domination number of Jahangir Graph $J_{2,\ m}$ [7] is
$$MD(J_{2,m}) = \frac{(m^2+13)/2}{(2m+1)C_2},\ where\ m \geq 3$$

Theorem 3. 4:
The medium domination number of Fan Graph F_n [8] is
$$MD(F_n) = \frac{(n^2+7n-18)/2}{nC_2},\ where\ n \geq 3$$

Theorem 3. 5:
The medium domination number of Jahangir Graph $J_{m,\ n}$ is [9] is
$$MD(J_{m,n}) = \frac{\{n[n+4(m+1)+1]\}/2}{(mn+1)C_2},\ where\ m \geq 3$$

4. Main Results on Medium Domination Number
Lemma: 4. 1
The medium domination number of the Cartesian product of W_4 and W_n is
$$MD(\mathrm{W}_4 \circ \mathrm{W}_n) = \frac{2n^2+80\mathrm{n}-64}{4nC_2},\ where\, n \geq 4$$

Proof:
We prove this lemma by method of induction.
Step i: The lemma is true for n=4

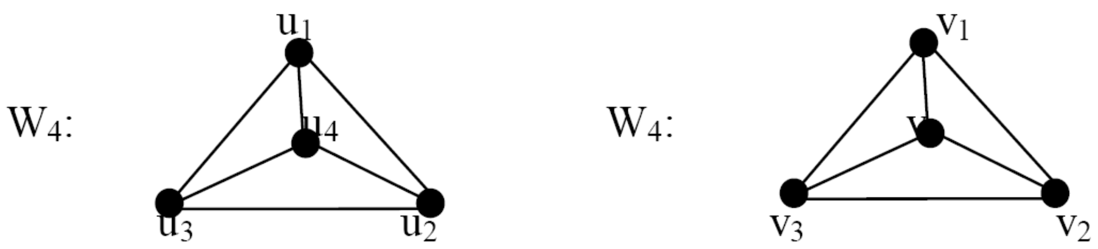

Figure:4. 1

The Cartesian product of W_4 and W_4 is

$W_4 \circ W_4$:

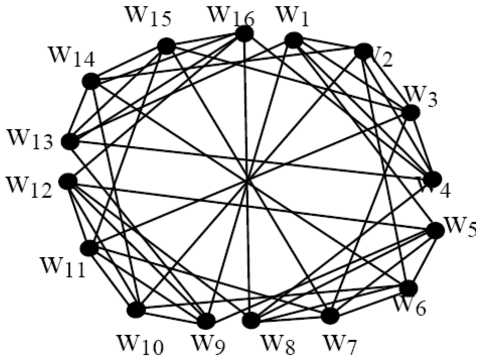

Figure:4. 2

The above graph $W_4°W_4$ has 16 vertices namely w_1, w_2, …w_{16}. All the vertices of degree 6 and the total number of edges is 48. The total number of vertices that dominate every pair of vertices is the sum of number of edges and the summation of $\deg v_i C_2$.

By theorem 3. 1, the total number of vertices that dominate every pair of vertices is

$$TDV(W_4 \circ W_4) = q + \sum_{v_i \in V} \left(\deg v_i C_2\right)$$

$$= 48 + 4[4(6C_2)]$$

$$= 288$$

For any connected, simple graph G of order n, the medium domination number of G is denoted as

$$MD(G) = \frac{\text{TDV (G)}}{nC_2}$$

$$\therefore MD(W_4 \circ W_4) = \frac{\text{TDV } (W_4 \circ W_4)}{nC_2} = \frac{288}{16C_2}$$

Step:ii

Assume that the result is true for $W_4°W_{n-1}$

The total number of vertices that dominates every pair of vertices is

$$TDV(W_4 \circ W_{n-1}) = 2(n-1)^2 + 80(n-1) - 64$$

Step iii:

Claim: The result is true for any n.

The graph $W_4 \circ W_{n-1}$, there are 4(n-1) vertices and 14n-22 edges. The total number of vertices that dominates every pair of vertices is split-up in to two different parts; identify the number of vertices of degree six and the vertices of degree n+1. Here there are 4(n-2) vertices of degree six and there are four vertices of degree n+1.

The graph $W_4 \circ W_n$, there are 4n vertices and 14n-8 edges. Here there are 4(n-1) vertices of degree six and there are four vertices of degree n+2. Any two consecutive stages, say ((n-1) and n) each graph is found with four additional vertices and 14 edges. This perhaps gives a conclusion that each additional vertices and edges formed at each stages are $14+4\{6C_2+(n+2)C_2-(n+1)C_2\}$.

The total number of vertices that dominates every pair of vertices is

$$TDV(W_4\circ W_n)=TDV(W_4\circ W_{n-1})+14+4\{6C_2+(n+2)C_2-(n+1)C_2\}$$

$$=2(n-1)^2+80(n-1)-64+14+4\{\frac{6(5)}{1(2)}+\frac{(n+1)(n+2)}{2}-\frac{(n+1)n}{2}\}$$

$$TDV(W_4\circ W_n)=2n^2+80n-64$$

$$MD(W_4\circ W_n)=\frac{TDV(W_4\circ W_n)}{4nC_2}$$

$$\therefore MD(W_4\circ W_n)=\frac{2n^2+80n-64}{4nC_2}$$

Lemma: 4. 2

The medium domination number of the Cartesian product of F_3 and F_n is

$$MD(F_3\circ F_n)=\frac{1.5n^2+(40.5)n-63}{3nC_2},\ \ where\, n\geq 3$$

Proof:

We prove this lemma by method of induction.

Step i: First we prove this result is true for n=3.

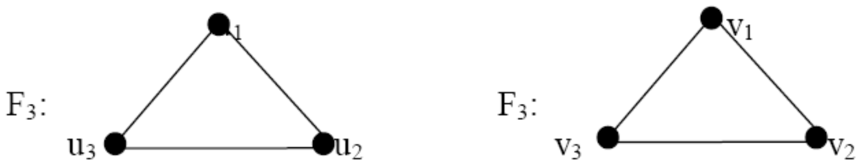

Figure:4. 3

The Cartesian product of F_3 and F_3 is

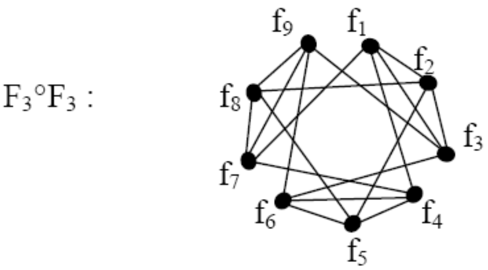

Figure:4. 4

In the above graph, there are nine vertices F_1, F_2, ….. F_9 with eighteen edges. Here all the vertices of degree four. The total numbers of vertices that dominate every pair of vertices is the sum of number of edges and the summation of $\deg v_i C_2$.

By theorem 3. 1, the total number of vertices that dominate every pair of vertices is

$$TDV(F_3 \circ F_3) = q_1 + \sum_{v_i \in V} (\deg v_i C_2)$$
$$= 18 + 3\{3(4C_2)\}$$
$$TDV(F_3 \circ F_3) = 72$$
$$MD(F_3 \circ F_3) = \frac{TDV(F_3 \circ F_3)}{9C_2}$$
$$\therefore MD(F_3 \circ F_3) = \frac{72}{9C_2}$$

Step:ii

Assume that the result is true for $F_3°F_{n-1}$

The total number of vertices that dominates every pair of vertices is

$$TDV(F_3 \circ F_{n-1}) = 1.5(n-1)^2 + 40.5(n-1) - 63$$

Step iii:Claim: The result is true for any n.

The graph $F_3 \circ F_{n-1,}$ there are 3(n-1) vertices and 9(n-1) edges. The total number of vertices that dominates every pair of vertices is split-up in to three different parts; identify the number of vertices of degree four and the number of vertices of degree five and the remaining vertices of degree n. Here there are six vertices of degree four and 3(n-4) vertices of degree five and three vertices of degree n.

The graph $F_3 \circ F_{n,}$ there are 3n vertices and 9n edges. In these 3n vertices, there are six vertices of degree four and 3(n-3) vertices of degree five and three vertices of degree n+1. For any two consecutive stages, say ((n-1) and n), each graph was found with three additional vertices and nine edges.

In each stage there are two types of vertices. There are n vertices are of odd degree and one vertex of degree (n-1). These additional vertices and edges made a impact on the total number of vertices that dominates every pair of vertices. So we come in to a conclusion that each additional vertices and edges formed at each stage will be $9 + 3\{5C_2 + (n+1)C_2 - nC_2\}$.

The total number of vertices that dominates every pair of vertices is

$$TDV(F_3 \circ F_n) = TDV(F_3 \circ F_{n-1}) + 9 + 3\{5C_2 + (n+1)C_2 - nC_2\}$$
$$= 1.5(n-1)^2 + 40.5(n-1) - 63 + 9 + 3\{5C_2 + (n+1)C_2 - nC_2\}$$
$$TDV(F_3 \circ F_n) = 1.5n^2 + 40.5n - 63$$
$$\therefore MD(F_3 \circ F_n) = \frac{TDV(F_3 \circ F_n)}{3nC_2} = \frac{1.5n^2 + (40.5)n - 63}{3nC_2}$$

Lemma: 4. 3

The medium domination number of the Cartesian product of W_4 and F_n is

$$MD(W_4 \circ \mathrm{F_n}) = \frac{2n^2 + 80\mathrm{n}\text{-}108}{4nC_2}, \ \ where\, n \geq 3$$

Proof:
We prove this lemma by method of induction.

Step i:
First we prove this result is true for n=3.

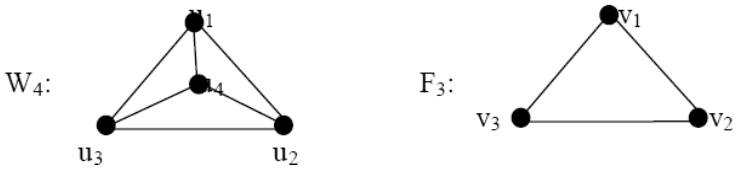

Figure:4. 5

The Cartesian product of W_4 and F_3 is

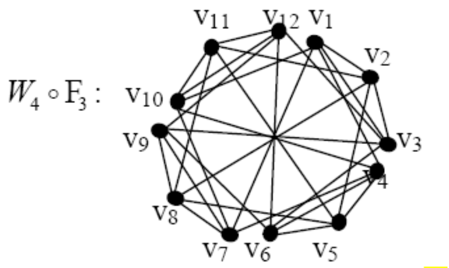

Figure:4. 6

In the above graph $W_4 \circ F_3$, there are twelve vertices V_1, V_2, ….. V_{12} with thirty edges. Here all the vertices of degree five. The total number of vertices that dominate every pair of vertices is the sum of number of edges and the summation of $\deg v_i C_2$.
By theorem 3. 1, the total number of vertices that dominate every pair of vertices is

$$TDV(W_4 \circ F_3) = q + \sum_{v_i \in V} \left(\deg v_i C_2 \right)$$

$$= 30 + 12(5C_2)$$

$$TDV(W_4 \circ F_3) = 150$$

$$MD(W_4 \circ F_3) = \frac{TDV(W_4 \circ F_3)}{12C_2}$$

$$\therefore MD(W_4 \circ F_3) = \frac{150}{12C_2}$$

Step:ii
Assume that the result is true for $W_4 \circ F_{n-1}$
The total number of vertices that dominates every pair of vertices is

$$TDV(W_4 \circ F_{n-1}) = 2(n-1)^2 + 80(n-1) - 108$$

Step iii:
Claim: The result is true for any n.
The graph $W_4 \circ F_{n-1,}$ there are 4(n-1) vertices and 14n-26 edges. The total number of vertices that dominates every pair of vertices is split-up in to two different parts; identify the number of vertices of degree five and number of vertices of degree six and the remaining vertices of degree n+2. Here there are eight vertices of degree five, 8 vertices of degree six and four vertices of degree n+2. The graph $W_4 \circ F_n$, there are 4n vertices and 14n-12 edges. For any two consecutive stages, say ((n-1) and n), each graph was found with four additional vertices and twelve edges.
These additional vertices and edges made a impact on the total number of vertices that dominates every pair of vertices. So we come in to a conclusion that each additional vertices and edges formed at each stage will be $14+4\{6C_2+(n+2)C_2-(n+1)C_2\}$.
The total number of vertices that dominates every pair of vertices is

$$TDV(W_4 \circ F_n) = TDV(F_3 \circ F_{n-1}) + 14 + 4\{6C_2 + (n+2)C_2 - (n+1)C_2\}$$

$$= 2(n-1)^2 + 80(n-1) - 108 + 14 + 4\{6C_2 + (n+2)C_2 - (n+1)C_2\}$$

$$= 2n^2 + 80n - 108$$

$$TDV(W_4 \circ F_n) = 2n^2 + 80n - 108$$

$$\therefore MD(W_4 \circ F_n) = \frac{TDV(W_4 \circ F_n)}{4nC_2} = \frac{2n^2 + 80n - 108}{4nC_2}$$

5. Conclusion

Graph theory has been applied to many areas like communication network, circuit analysis etc. There are different kinds of parameters in graphs; they are connectivity number, the edge-connectivity number, the independence number, the vertex domination number and the domination number. In a graph every vertices u, v Є V should be privileged and it is essential to scrutinize how many vertices are proficient of dominating both of u and v. We compute the total number of vertices that dominates all pairs of vertices and evaluate the average of this value and call it “the medium domination number” of graph. In this paper, we obtained the bound of the medium domination number of Cartesian product of wheel graph W_n and the fan graph F_n.

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