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On Rough Projective and Injective Modules

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Abstract

In this paper basic notions of Rough Set Theory (RST) will be given. Combining RST with algebra is a way to generalizing RST. Some papers proposed the concept of rough group, rough ring, rough module and rough ideals in approximate space and investigated their properties. In this paper, we shall discuss some properties of rough projective and rough injective modules.

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Keywords: Rough sets, Rough ring and Rough modules.

1. Introduction

The terminology of injective module and projective module was originated by H. Carten and S. Eilenberg [31] in 1956, to deal real life situations algebraically, and then the dual concept projective module and injective module have been covered in many text [32, 33, 35]. These terms are based on crisp set theory and can handle only exact situations. In Recent years it is seen that the most data sets are imprecise or the surrounding information is imprecise and our way of thinking or concluding depends on information. This means that to draw conclusions, we should able to process uncertain and/or incomplete information. To analyze any type of information, mathematical logics are most appropriate, so we should have to generalize the algebraic structures and the logic in sense of imprecise or vague. Rough set theory is a powerful mathematical tool to handle imprecise situations and rough algebraic structures can play a vital role to deal such situations.

In Pawlak rough set theory, the key concept is an equivalence relation and the building blocks for the construction of the lower and upper approximations are the equivalence

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classes. The lower approximation of the given set is the union of all the equivalence classes which are the subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set.

Z. Bonikowaski introduced the algebraic structures of rough sets [23]. R. Biswas and S. Nanda introduced the concept of rough group and rough subgroups [5]. N. Kuroki studied the rough ideals in semigroups [1]. B. Davvaz introduced the roughness in rings [3]. B. Davvaz, M. Mahdavipour introduced the roughness in module [11]. Rough modules and their some properties are also studied by Qun-Feng Zhang, Al-Min Fu and Shi-xin Zhao [10]. Standard sources for the algebraic theory of modules are [8, 9, 13]. One can find more on rough set and their algebraic structures in [2, 4, 6, 7, 15–18]. In recent years, there has been a fast growing interest in this new emerging theory, ranging frame work in pure theory, such as algebraic foundations and mathematical logic [6, 19, 20] to diverse areas of applications. Recently authors A.K. Sinha and Anand Prakash discussed on rough free module and rough projective modules in [14, 21, 22].

The aim of this paper is to investigate some properties of rough projective and Rough Injective Modules. The rest of the paper is organized as follows: In section 2, preliminaries are given. In section 3, we discuss the properties of rough projective and injective modules. Finally, our conclusions are presented. we are aspecting that reader is familiar with algebra and rough set theory, as we use the standard notations.

2. Preliminaries

For an equivalence relation θ on a set U(a universe), the set of the elements of U that are related to $x \in U$, is called the equivalence class of x, and is denoted by $[x]_\theta$. A pair (U, θ) where $U \neq \emptyset$ and θ is an equivalence relation on U, is called an approximation space.

Definition 2.1. [3] For an approximation space (U, θ), by a rough approximation in (U, θ) we mean a mapping $Apr : P(U) \to P(U) \times P(U)$ defined for every $X \in P(U)$ by

$$Apr(X) = (\underline{Apr}(X), \overline{Apr}(X)),$$

where $\underline{Apr}(X) = \{x \in U | [x]_\theta \subseteq X\}$, $\overline{Apr}(X) = \{x \in U | [x]_\theta \cap X \neq \phi\}$. $\underline{Apr}(X)$ is called the lower rough approximation of X in (U, θ), where as $\overline{Apr}(X)$ is called upper rough approximation of X in (U, θ).

Definition 2.2. [3] Given an approximation space (U, θ), a pair $(A, B) \in P(U) \times P(U)$ is called a rough subset in (U, θ) if and only if $(A, B) = Apr(X)$ for some $X \in P(U)$. A rough subset is also a rough set.

For the sake of illustration, Let (U, θ) is an approximation space, where $U = \{x_1, x_2, x_3, \ldots, x_8\}$ and an equivalence relation θ with the following equivalence classes:

$E_1 = \{x_1, x_4, x_8\}$

$E_2 = \{x_2, x_5, x_7\}$

$E_3 = \{x_3\}$

$E_5 = \{x_6\}$.

Let $X = \{x_3, x_5, x_7\}$, is a target set, then $\underline{Apr}(X) = \{x_3\}$ and $\overline{Apr}(X) = \{x_2, x_3, x_5, x_7\}$ and so $Apr(X) = (\{x_3\}, \{x_2, x_3, x_5, x_7\})$ is a rough set.

Definition 2.3. [3] Let $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ and $Apr(B) = (\underline{Apr}(B), \overline{Apr}(B))$ be any two rough sets in the approximation space (U, θ), then we define:

(1) $Apr(A) \cup Apr(B) = (\underline{Apr}(A) \cup \underline{Apr}(B),\ \overline{Apr}(A) \cup \overline{Apr}(B))$

(2) $Apr(A) \cap Apr(B) = (\underline{Apr}(A) \cap \underline{Apr}(B),\ \overline{Apr}(A) \cap \overline{Apr}(B))$

(3) $Apr(A) \subseteq Apr(B) \Leftrightarrow Apr(A) \cap Apr(B) = Apr(A)$.

As we know the mapping of $S \times S$ into S are called binary composition in the set S[8]. Let $S = (U, \theta)$ be a approximation space and $*$ be a binary composition, we define xy in stead of $x * y$, $\forall x, y \in U$.

Definition 2.4. [6] A subset $G(\neq \phi)$ of U is called a rough group if $Apr(G) = (\underline{G}, \overline{G})$ satisfies the following property:

(1) $xy \in \overline{G}$, $\forall x, y \in G$.

(2) $(xy)z = x(yz)$, $\forall x, y, z \in G$.

(3) $\exists, e \in \overline{G}$ such that $xe = ex = x$, $\forall x \in G$ the e is called the rough identity element.

(4) $\forall x \in G, \exists\ y \in G$ such that $xy = yx = e$; then y is called the rough inverse element of x in G.

[5] If $\phi \neq H \subseteq G \subseteq U$, and $(Apr(H), *)$ is a rough group, we call the rough subset $Apr(H)$ a rough subgroup of $Apr(G) = (\underline{G}, \overline{G})$, denoted as $Apr(H) \leq Apr(G)$.

Definition 2.5. [7] Let (U_1, θ) and (U_2, θ) be two approximation space, $*$ and $\overline{*}$ be two operations over U_1 and U_2, respectively. Let $G_1 \subseteq U_1$ and $G_2 \subseteq U_2$. $Apr(G_1)$ and $Apr(G_2)$ are called homomorphic rough set if there exists a mapping ϕ of G_1 into G_2 such that

$$\forall x, y \in \overline{G}, \qquad \phi(x * y) = \phi(x) \overline{*} \phi(y).$$

If ϕ is 1-1 mapping $Apr(G_1)$ and $Apr(G_2)$ are called isomorphic rough sets.

Definition 2.6. [6] An algebraic system $(Apr(R), +, *)$ is called rough ring if it satisfied:

(1) $(Apr(R), +)$ is a rough commutative addition group.

(2) $(Apr(R), *)$ is a rough multiplicative semi-group.

(3) $(x + y) * z = x * z + y * z$ and $x * (y + z) = x * y + x * z$
$\forall x, y, z \in Apr(R)$.

Definition 2.7. [10] Let $(Apr(R), +, *)$ be a rough ring with a unit, $(Apr(M), +)$ a rough commutative group. $Apr(M)$ is called a rough left module over the ring $Apr(R)$ if there is mapping $\overline{Apr}(R) \times \overline{Apr}(M) \to \overline{Apr}(M)$, $(a, x) \to ax$ such that

(1) $a(x + y) = ax + ay$, $a \in Apr(R)$, $x, y \in Apr(M)$

(2) $(a + b)x = ax + ay$, $a, b \in Apr(R)$, $x \in Apr(M)$

(3) $(ab)x = a(bx)$, $a, b \in Apr(R)$, $x \in Apr(M)$

(4) $1x = x$, 1 is a unit element of $Apr(R)$ and $x \in Apr(M)$.

A rough right module over the ring $Apr(R)$ can be defined similarly.

[10] A rough subset $Apr(N) \neq \phi$ of a rough module $Apr(M)$ is called rough submodule of $Apr(M)$, if $Apr(N)$ satisfied the following:

(1) $Apr(N)$ is a rough subgroup of $Apr(M)$

(2) $ay \in \overline{N}$, $\forall a \in Apr(R)$ and $y \in Apr(N)$.

Definition 2.8. [10] Let $Apr(M)$ and $Apr(M')$ be two rough R-module. If there exists a mapping η of M into M' such that

(1) η is a homomorphism of a rough group $Apr(M)$ into $Apr(M')$;

(2) $\eta(ax) = a\eta(x)$, $a \in Apr(R)$, $x \in Apr(M)$.

then η is called a homomorphism of rough module $Apr(M)$ into $Apr(M')$. If η is a 1-1 mapping, it is called an isomorphism of rough module $Apr(M)$ into $Apr(M')$.

3. On Rough Projective and Injective Modules

Theorem 3.1. Every rough module is homomorphic image of a rough module.

Proof. Let $Apr(M)$ be a rough module over rough ring $Apr(R)$. Then there exist a free module $Apr(F)$ and $Apr(R)$-module epimorphism $\phi : Apr(F) \to Apr(M)$. Now we consider the rough module $Apr(F)$, where $\mu = \chi_0$ then $Apr(F)$ is rough free module and $\phi^- : \mu_{Apr(F)} \to \mu_{Apr(R)}$ is the required rough homomorphism. ∎

Theorem 3.2. Let $Apr(I)$ be a rough module. Then the following conditions holds:

(i) if $\beta : Apr(I) \rightarrow Apr(B)$ is a rough homomorphism, then there exists a rough homomorphism $\alpha : Apr(B) \rightarrow Apr(I)$ such that $\alpha\beta = 1_{Apr(I)}$. Then (ii) holds.

(ii) $Apr(I)$ is a rough direct summand in every rough module which contains $Apr(I)$ as a submodule, then $Apr(I)$ is injective.

Proof. (i) Let $Apr(I)$ be a rough submodule of $Apr(A)$. Then the rough inclusion map $\alpha : Apr(I) \rightarrow Apr(A)$ is a rough homomorphism. Thus bye hypothesis there exist a rough homomorphism $\beta : Apr(A) \rightarrow Apr(I)$ such that $\alpha\beta = 1_{Apr(I)}$. So the sequence

$$0 \longrightarrow Apr(I) \xrightarrow{\beta} Apr(B) \xrightarrow{\pi} coker(\beta) \longrightarrow 0$$

is a rough short exact. There for we have $Apr(B) \cong Apr(I) \oplus \beta$. Thus (ii) holds.

(ii) By hypothesis we conclude that $Apr(I)$ is a direct summand in every module which contains $Apr(I)$ as a submodule. Therefor $Apr(I)$ is injective. now we show that $\mu = \chi_{Apr(I)}$. It is obvious that $\mu_{Apr(I)}$ is a rough submodule of $\chi_{Apr(I)}$.Hence bye hypothesis $\chi_{Apr(I)} = \mu \oplus \eta$, for some rough module $\eta_{Apr(K)}$. Let a be a nonzero element in $Apr(I)$. Then by definition of coproductwe have $1 = \chi(a) = (\mu \oplus \eta)(a, 0) = \mu(a)$, that is $\mu(a) = 1$, for all $a \in Apr(I)$. So $\mu = \chi_{Apr(I)}$. Therefore $Apr(I)$ is rough injective module. ■

Theorem 3.3. Let $Apr(P)$ be a rough $Apr(R)$-module. then the following conditions are equivalent:

(i) $Apr(P)$ is projective.

(ii) for each rough short exact sequence

$$0 \longrightarrow Apr(A) \xrightarrow{f} Apr(B) \xrightarrow{g} Apr(C) \longrightarrow 0$$

is exact.

(iii) if $\alpha : Apr(B) \rightarrow Apr(P)$ is a rough epimorphism, then there exist a rough homomorphism $\phi : Apr(P) \rightarrow Apr(B)$ such that $\alpha\phi = 1_{Apr(P)}$.

(iv) If $Apr(P)$ is the rough homomorphic image of a rough module μ_A then $Apr(P)$ is a direct summand of $\mu_{Apr(M)}$.

(v) $Apr(P)$ is a rough direct summand of a rough free $Apr(R)$-module.

Proof. $(i) \rightarrow (ii)$ Let the sequence

$$0 \longrightarrow Apr(A) \xrightarrow{f} Apr(B) \xrightarrow{g} Apr(C) \longrightarrow 0$$

be rough exact. Since $Apr(P)$ is projective in $Apr(R)$ fzmod, we conclude that $Apr(P)$ is projective $Apr(R)$-module and $Apr(Q) = \chi_0$, there for by theorem 2.9 we get that sequence (1) is exact.

$(ii) \to (iii)$ Let $\alpha : \beta \to Apr(P)$ be rough epimorphism, then bye ex 2.6, we have the following rough short exact sequence

$$0 \longrightarrow ker(\alpha) \xrightarrow{i} Apr(B) \xrightarrow{\alpha} \beta \longrightarrow 0$$

thus by hypothesis the sequence

$$0 \longrightarrow Apr(A) \xrightarrow{f} Apr(B) \xrightarrow{g} Apr(C) \longrightarrow 0$$

is exact. If we consider the rough homomorphism $1_{Apr(P)} : Apr(P) \to Apr(P)$ then sence α is onto there exist a rough homomorphism $\phi : Apr(P) \to Apr(B)$ such that $\alpha\phi = 1_{Apr(P)}$.

$(iii) \to (iv)$ Let $Apr(P)$ be a rough homomorphic image of a rough module $Apr(A)$, that is there exists a rough epimorphism $\phi : Apr(A) \to Apr(P)$, So bye hypothesis there exists a rough homomrophism $\phi : Apr(P) \to Apr(A)$ such that $\alpha\phi := 1_{Apr(P)}$and the sequence

$$0 \longrightarrow ker(\alpha) \xrightarrow{i} Apr(A) \xrightarrow{\alpha} Apr(P) \longrightarrow 0$$

is rough exact, where i is the inclusion map. Thus by the lemma 2.11, the above sequence is rough splitting i.e. $Apr(A) \cong Apr(P) \oplus ker(\alpha)$.

$(iv) \to (v)$ It is a easy consequence of theorem 3.1 and hypothesis.

$(v) \to (i)$ Let $Apr(P)$ is a rough direct summand of a rough $Apr(R)$-module then there exists a free module $Apr(F)$ such that $Apr(F) = Apr(P) \oplus Apr(Q)$. This implies $Apr(P)$ is projective, similarly the converse part. ■

Definition 3.4. An element x of the $Apr(R)$-module $Apr(M)$ is called a torsion element if its annihilator is non-zero, i.e., if there exists $a \neq 0$ in $Apr(R)$ such that $ax = 0$. An element of $Apr(M)$ which is not a torsion element is called a torsion free element; thus the element $x \in Apr(M)$ is torsion free if and only if the relation $ax = 0$ implies that $a = 0$, equivalently, if and only if the relation $ax = bx$ implies that $a = b$.

Example 3.5. Every non-zero element of the $Apr(Z)$-module $Apr(Z)$ is torsion free. More generally, if the ring $Apr(R)$ is without zero divisors, then every non-zero element of the $Apr(R)$-module $Apr(R_l)$ is torsion free. the element $1_{Apr(R)}$ of the $Apr(R)$-module $Apr(R)_l$ is always torsion free.

Corollary 3.6. If the $Apr(R)$-module $Apr(M)$ is free with basis $x_i, i \in Apr(I)$, then the $Apr(S^{-1})$ $Apr(R)$-module $Apr(S^{-1})Apr(M)$ is free with basis $x_i/1, i \in Apr(I)$.

Proof. Since $Apr(M)$ is a direct sum of submodules $Apr(R)x_i$, therefore $Apr(S^{-1})$ $Apr(M)$ is direct sum of its submodules

$$Apr(S^{-1})(Apr(R)x_i) = (Apr(S^{-1})Apr(R))(x_i/1)$$

thus it only remains to show that each $x_i/1$ is torsion free. Indeed, if $(a/Apr(S))(x_i/1) = 0$, there exists $t \in Apr(S)$ such that $tax_i = 0$; since x_i is a basis element, therefore $ta = 0$ and hence $(a/Apr(s)) = 0$. Therefore $x_i/1$ is torsion free and the proof is complete. ■

Theorem 3.7. Every rough module is isomorphic to a quotient of a free rough module.

Proof. Let $Apr(X)$ be a rough module and let $Apr(F)$ be the free rough module over the rough set μ. The identity homomorphism id_μ induces a unique homomorphism or rough module $h : Apr(F) \to Apr(X)$ such that $hoi = id_\mu$. The homomorphism h having a right inverse is into $Apr(X)$, implies μ is isomorphic to $\phi/ker(h)$. ■

Definition 3.8. An $Apr(R)$ module $Apr(M)$ is called a torsion module if every element of $Apr(M)$ is a torsion element, and a torsion-free module if every non-zero element of $Apr(M)$ is torsion free.

Lemma 3.9. If $Apr(R)$ is an integral domain, and $Apr(M)$ an $Apr(R)$-module, then the subset $Apr(T) = Apr(T(Apr(M)))$ of $Apr(M)$ consisting of the torsion elements of $Apr(M)$ is a submodule of $Apr(M)$ and the quotient module $Apr(M)/Apr(T)$ is torsion free.

Proof. Since $1_{Apr(R)}0 = 0$ and $1_{Apr(R)} \neq 0$, therefore $0 \in Apr(T)$; in particular, $Apr(T)$ is nonempty. If x, $Apr(Y)$ are two element of $Apr(T)$, there exists a non-zero element a, b of $Apr(R)$ such that $ax = 0, by = 0$; then ab is a non-zero element of $Apr(R)$ and $ab(x - y) = b(ax) - a(by) = 0$, so that $x - y \in Apr(T)$; if c is any element of $Apr(R)$, then $a(cx) = c(ax) = c0 = 0$, and hence $cx \in Apr(T)$. Thus $Apr(T)$ is a submodule of $Apr(M)$. suppose now that x^- is a non-zero element of $Apr(M^-) = Apr(M)/Apr(T)$, and let $ax^- = 0$. Then $ax \in Apr(T)$ and hence there exists $b \neq 0$ in $Apr(R)$ such that $bax = 0$. Since $x^- \neq 0$, therefore $x \neq Apr(T)$ and consequently $ba = 0$. Since $Apr(R)$ is an integral domain and $b \neq 0$, therefore $a = 0$; hence x^- is torsion free. Thus $Apr(M)/Apr(T)$ is torsion free. ■

Corollary 3.10. If $Apr(R)$ is a commutative and $Apr(P)$ is a projective $Apr(R)$-module, then for any multiplicative subset $Apr(S)$ of $Apr(R)$, the $Apr(S^{-1})$ $Apr(R)$-module $Apr(S^{-1})Apr(P)$ is projective.

Proof. Indeed, $Apr(P)$ is direct summand of a free $Apr(R)$-module $Apr(F)$, we write $Apr(F) = Apr(P) \oplus Apr(Q)$, we have $Apr(S^{-1})Apr(F) = Apr(S^{-1})Apr(P) \oplus Apr(S^{-1})Apr(Q)$, since the $Apr(S^{-1})$ $Apr(R)$-module $Apr(S^{-1})Apr(F)$ is free, therefore $Apr(S^{-1})Apr(P)$ is projective. ■

Lemma 3.11. If the $Apr(Z)$-module $Apr(Q)$ is injective, then the $Apr(R)$-module $Apr(H) = hom_{Apr(Z)}(Apr(A), Apr(Q))$ in injective.

Proof. Let $Apr(R)$-module $Apr(H)$ is a submodule of $Apr(R)$-module $Apr(M)$ then we show that $Apr(H)$ is a direct summand of $Apr(M)$. The mapping $u \to u(1)$ from $Apr(H)$ to $Apr(Q)$ is clearly additive. Since the $Apr(Z)$-module $Apr(Q)$ is injective, there exists a homomorphism $q : Apr(M) \to Apr(Q)$ of $Apr(Z)$-modules such that $q(u) = u(1)$ for all $u \in Apr(H)$ Define $p : Apr(M) \to Apr(H)$ by

$$(p(x))(a) = q(ax) \qquad (x \in Apr(M), a \in Apr(A))$$

the mapping $p(x) : Apr(R) \to Apr(Q)$ is z linear, therefore in $Apr(H)$. p is additive map. If $a \in Apr(A), x \in Apr(M)$, then for every $a' \in Apr(A)$,

$$(p(ax))(a') = q(a'ax) = (p(x))(a'a) = (ap(x))(a'),$$

hence $p(ax) = ap(x)$. Thus p is $Apr(R)$-linear. let $u \in Apr(H)$, then

$$(p(u))(a) = q(au) = (au)(1) = u(a)$$

for all $a \in Apr(R)$, and hence $p(u) = u$. Thus p is an $Apr(R)$-linear projection from $Apr(M)$ on $Apr(H)$. Hence $Apr(H)$ is a direct summand of $Apr(M)$. ■

Lemma 3.12. Every $Apr(Z)$-module can be embedded in an injective $Apr(Z)$-module.

Proof. Let $Apr(E)$ be a $Apr(Z)$-module. Write $Ar(E)$ as a $Apr(F)/Apr(N)$ with $Apr(F)$ a free $Apr(Z)$-module. Since $Apr(F)$ is a direct sum of copies of $Apr(Z)$, and since $Apr(Z)$ is a submodule of the divisible module $Apr(Q)$, therefore $Apr(F)$ is a submodule of a direct sum $Apr(G)$ of divisible $Apr(Z)$-modules. Then $Apr(E) = Apr(F)/Apr(N)$ is a submodule of $Apr(G)/Apr(N)$. since $Apr(G)$ is divisible, so is $Apr(G)/Apr(N)$, therefore $Apr(G)/Apr(N)$ in injective and this completes the proof. ■

4. Conclusion

Rough set theory is important in both pure and applied mathematics. The main objective of this paper is to point out some properties of rough projective and rough injective modules. Modules concepts are important and effective tools in linear algebra, vector spaces and physics. The lower and upper approximations are formulated in the context of module theory. We hope that this research may provide a powerfull tool in approximate reasoning. This work can also enrich the rough set theories and will be useful in the theory and applications of rough sets.

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