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Notes on degenerate tangent polynomials

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Abstract

In this paper, we consider the degenerate tangent numbers and polynomials. We also obtain some explicit formulas for degenerate tangent numbers and polynomials.

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1. Introduction

In [2], L. Carlitz introduced the degenerate Bernoulli polynomials. Recently, Feng Qi *et al.* [3] studied the partially degenerate Bernoull polynomials of the first kind in p-adic field. In this paper, we establish some interesting properties for degenerate tangent polynomials. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ denotes the set of complex numbers. The Tangent numbers are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \frac{2}{e^{2t}+1} \qquad \left(|t| < \frac{\pi}{2}\right). \tag{1.1}$$

Ryoo [6] defined Tangent polynomials by multiplying e^{xt} on the right side of the Eq. (1.1) as follows:

$$\sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!} = \frac{2}{e^{2t}+1} e^{xt} \qquad \left(|t| < \frac{\pi}{2}\right). \tag{1.2}$$

For more theoretical properties of the tangent numbers and polynomials , the readers may refer to [6, 7]. The notion of introducing of Tangent polynomials is similar to that of the generating functions of the Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ defined by

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt} \text{ and } \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt},$$

see, for details, [1-8] and cited references therein. We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations(see [8])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}. \tag{1.3}$$

We also have

$$\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.4}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.5}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$; we may also write

$$(x|\lambda)_n = \sum_{k=0}^{n} S_1(n,k)\lambda^{n-k}x^k. \tag{1.6}$$

Note that $(x|\lambda)$ is a homogeneous polynomials in λ and x of degree n, so if $\lambda \neq 0$ then $(x|\lambda)_n = \lambda^n(\lambda^{-1}x|1)_n$. Clearly $(x|0)_n = x^n$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \tag{1.7}$$

2. On the degenerate tangent polynomials

In this section, we define the degenerate tangent numbers and polynomials, and we obtain explicit formulas for them. For a variable t, we consider the degenerate tangent polynomials which are given by the generating function to be

$$\frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} T_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $T_{n,\lambda}(0) = T_{n,\lambda}$ are called the degenerate tangent numbers. Note that $(1+\lambda t)^{1/\lambda}$ tends to e^t as $\lambda \to 0$.

From (2.1) and (1.7), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} \lim_{\lambda\to 0} T_{n,\lambda}(x)\frac{t^n}{n!} &= \lim_{\lambda\to 0} \frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} \\ &= \sum_{n=0}^{\infty} T_n(x)\frac{t^n}{n!}.\end{aligned}$$

Thus, we get

$$\lim_{\lambda\to 0} T_{n,\lambda}(x) = T_n(x),\ (n \geq 0).$$

From (2.1) and (1.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} \\ &= \left(\sum_{m=0}^{\infty} T_{m,\lambda}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(x|\lambda)_l\frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T_{l,\lambda}(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.2}$$

Therefore, by (2.1) and (2.2), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$T_{n,\lambda}(x) = \sum_{l=0}^{n}\binom{n}{l}T_{l,\lambda}(x|\lambda)_{n-l}.$$

From (2.1), we can derive the following recurrence relation:

$$\begin{aligned}2 &= ((1+\lambda t)^{2/\lambda}+1)\sum_{n=0}^{\infty} T_{n,\lambda}\frac{t^n}{n!} \\ &= (1+\lambda t)^{2/\lambda}\sum_{n=0}^{\infty} T_{n,\lambda}\frac{t^n}{n!} + \sum_{n=0}^{\infty} T_{n,\lambda}\frac{t^n}{n!} \\ &= \left(\sum_{l=0}^{\infty}(2|\lambda)_l\frac{t^l}{l!}\sum_{m=0}^{\infty} T_{m,\lambda}\frac{t^m}{m!}\right) + \sum_{n=0}^{\infty} T_{n,\lambda}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(2|\lambda)_l T_{n-l,\lambda} + T_{n,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (2.3), we have the following theorem.

Theorem 2.2. For $n \in \mathbb{Z}_+$, we have

$$\sum_{l=0}^{n} \binom{n}{l} (2|\lambda)_l T_{n-l,\lambda} + T_{n,\lambda} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

By (2.1), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} T_{n,\lambda}(x+2)\frac{t^n}{n!} + \sum_{n=0}^{\infty} T_{n,\lambda}\frac{t^n}{n!} \\ &= \frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{(x+2)/\lambda} + \frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} \\ &= 2(1+\lambda t)^{x/\lambda} \\ &= 2\sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \end{aligned} \tag{2.4}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (2.4), we have the following theorem.

Theorem 2.3. For $n \in \mathbb{Z}_+$, we have

$$T_{n,\lambda}(x+2) + T_{n,\lambda}(x) = 2(x|\lambda)_n.$$

By (2.1), we get

$$\begin{aligned} \sum_{n=0}^{\infty} T_{n,-\lambda}(-x)\frac{t^n}{n!} &= \frac{2}{(1-\lambda t)^{-2/\lambda}+1}(1-\lambda t)^{x/\lambda} \\ &= \frac{2}{(1-\lambda t)^{2/\lambda}+1}(1-\lambda t)^{(x+2)/\lambda} \\ &= \sum_{n=0}^{\infty}(-1)^n T_{n,\lambda}(x+2)\frac{t^n}{n!}. \end{aligned} \tag{2.5}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (2.5), we have the following theorem.

Theorem 2.4. For $n \in \mathbb{Z}_+$, we have

$$T_{n,-\lambda}(-x) = (-1)^n T_{n,\lambda}(x+2),$$

In particular,

$$T_{n,-\lambda} = (-1)^n T_{n,\lambda}(2).$$

For $d \in \mathbb{N}$ with $d \equiv 1(\mathrm{mod} 2)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} \\ &= \frac{2}{(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{x/\lambda}\sum_{l=0}^{d-1}(-1)^l(1+\lambda t)^{2l/\lambda} \\ &= \sum_{n=0}^{\infty}\left(d^n\sum_{l=0}^{d-1}(-1)^l T_{n,\lambda/d}\left(\frac{2l+x}{d}\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

By comparing coefficients of $\frac{t^m}{m!}$ in the above equation, we have the following theorem:

Theorem 2.5. For $d \in \mathbb{N}$ with $d \equiv 1$ (mod 2) and $n \in \mathbb{Z}_+$, we have

$$T_{n,\lambda}(x) = d^n \sum_{l=0}^{d-1}(-1)^l T_{n,\lambda/d}\left(\frac{2l+x}{d}\right).$$

In particular,

$$T_{n,\lambda} = d^n \sum_{l=0}^{d-1}(-1)^l T_{n,\lambda/d}\left(\frac{2l}{d}\right).$$

From (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,\lambda}(x+y)\frac{t^n}{n!} &= \frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{(x+y)/\lambda} \\ &= \frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}(1+\lambda t)^{y/\lambda} \\ &= \left(\sum_{n=0}^{\infty} T_{n,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y|\lambda)_n\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T_{l,\lambda}(x)(y|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

Therefore, by (2.7), we have the following theorem.

Theorem 2.6. For $n \in \mathbb{Z}_+$, we have

$$T_{n,\lambda}(x+y) = \sum_{k=0}^{n}\binom{n}{k}T_{k,\lambda}(x)(y|\lambda)_{n-k}.$$

From Theorem 2.6, we note that $T_{n,\lambda}(x)$ is a Sheffer sequence.

By replacing t by $\frac{e^{\lambda t}-1}{\lambda}$ in (2.1), we obtain

$$\begin{aligned}\frac{2}{e^{2t}+1}e^{xt} &= \sum_{n=0}^{\infty} T_{n,\lambda}(x)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^n \frac{1}{n!}\\ &= \sum_{n=0}^{\infty} T_{n,\lambda}(x)\lambda^{-n}\sum_{m=n}^{\infty} S_2(m,n)\lambda^m \frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} T_{n,\lambda}(x)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.8}$$

Thus, by (2.8) and (1.2), we have the following theorem.

Theorem 2.7. For $n \in \mathbb{Z}_+$, we have

$$T_m(x) = \sum_{n=0}^{m} \lambda^{m-n} T_{n,\lambda}(x) S_2(m,n).$$

By replacing t by $\log(1+\lambda t)^{1/\lambda}$ in (1.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_n(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n \frac{1}{n!} &= \frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\ &= \sum_{m=0}^{\infty} T_{m,\lambda}(x)\frac{t^m}{m!},\end{aligned} \tag{2.9}$$

and

$$\sum_{n=0}^{\infty} T_n(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n \frac{1}{n!} = \quad = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} T_{n,\lambda}(x)\lambda^{m-n}S_1(m,n)\right)\frac{t^m}{m!}. \tag{2.10}$$

Thus, by (2.9) and (2.10), we have the following theorem.

Theorem 2.8. For $n \in \mathbb{Z}_+$, we have

$$T_{m,\lambda}(x) = \sum_{n=0}^{m} \lambda^{m-n} T_n(x) S_1(m,n).$$

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