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On certain estimates for Marcinkiewicz integrals with a rough kernel on product spaces

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Abstract

In this article, we establish L^p boundednesss of the Marcinkiewicz integral operators with rough kernels on $\mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_k}$ under some weak conditions on Ω and h. Our results are essential improvements and extensions of some known results on Marcinkiewicz integrals.

AMS subject classification:
Keywords: L^p boundedness, Marcinkiewicz integrals, rough kernels, product spaces.

1. Introduction

Throughout this article, let $n, m \geq 2$, and let $\mathbf{S}^{N-1}$ ($N = n$ or m) be the unit sphere in $\mathbf{R}^N$ which is equipped with the normalized Lebesgue surface measure $d\sigma = d\sigma(\cdot)$. Also, let $x' = x/|x|$ for $x \in \mathbf{R}^n \setminus \{0\}$, $y' = y/|y|$ for $y \in \mathbf{R}^m \setminus \{0\}$. Suppose that p' is denoted the exponent conjugate to p; that is $1/p + 1/p' = 1$.

Let $K_{\Omega,h}(u, v) = \dfrac{\Omega(u', v')}{|u|^{n-1}|v|^{m-1}} h(|u|, |v|)$, where h is a measurable function on $\mathbf{R}^+ \times \mathbf{R}^+$, Ω is a function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ with $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ satisfying the cancellation conditions:

$$\int_{\mathbf{S}^{n-1}} \Omega(x', .)d\sigma(x') = \int_{\mathbf{S}^{m-1}} \Omega(., y')d\sigma(y') = 0. \tag{1.1}$$

Let $d \neq 0$ and $\mathcal{H}_d$ be the class of all functions $\phi : (0, \infty) \to \mathbf{R}$ which are smooth and satisfy the following growth conditions:

$$|\phi(t)| \leq C_1 t^d, \qquad |\phi''(t)| \leq C_2 t^{d-2}, \qquad C_3 t^{d-1} \leq |\phi'(t)| \leq C_4 t^{d-1} \tag{1.2}$$

for $t \in (0, \infty)$, where C_1, C_2, C_3 and C_4 are positive constants independent of t.

For $\phi \in \mathcal{H}_{d_1}$, $\psi \in \mathcal{H}_{d_2}$ for some $d_1, d_2 \neq 0$, a measurable function h on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ and an Ω satisfying (1.1), we define the Marcinkiewicz integral operator $\mathcal{M}_{\Omega,h,\phi,\psi}$, initially for C_0^∞ on $\mathbf{R}^n \times \mathbf{R}^m$, by

$$\mathcal{M}_{\Omega,h,\phi,\psi} f(x) = \left(\int_0^\infty \int_0^\infty \left|F_{t,s}^{\phi,\psi}(x,y)\right|^2 \frac{dtds}{(ts)^3}\right)^{1/2}, \tag{1.3}$$

where

$$F_{t,s}^{\phi,\psi}(x,y) = \int_{|u|\le t} \int_{|v|\le s} f(x - \phi(|u|)u', y - \psi(|v|)v') K_{\Omega,h}(u,v)dudv. \tag{1.4}$$

When $\phi(t) = t$, $\psi(s) = s$, we denote $\mathcal{M}_{\Omega,h,\phi,\psi}$ by $\mathcal{M}_{\Omega,h}$.

The operators $\mathcal{M}_{\Omega,h,\phi,\psi}$ have their roots in the classical Marcinkiewicz integral operators $\mathcal{M}_{\Omega,1}$ which were introduced by Stein in [19]. The Marcinkiewicz integral operators have received much attention from many authors due to their powerful role in dealing with many significant problems arising in such parts of analysis as Poisson integrals, singular integrals and singular Radon transforms analysis (we refer the readers to [1, 3, 10, 12, 23, 24, 25] and the references therein), and they can also see [2, 4, 6, 9, 13, 14, 19, 21] for the corresponding results in the one parameter cases.

Before stating our result, we first recall the definition of the block space $B_q^{(0,\nu)}$ $(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. The special class of block space $B_q^{(0,\nu)}$ $(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$(for $\nu > -1$ and $q > 1$) was introduced by Jiang and Lu in the study of the singular integral operators (see [17]), and it is defined as follows: A q-block on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ is an L^q function $b(x,y)$ that satisfies (i) *supp* $(b) \subseteq I$, (ii) $\|b\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \le |I|^{-1/q'}$, where $|I| = \sigma(I)$ and $I = \left\{x' \in \mathbf{S}^{n-1} : \left|x' - x_0'\right| < \delta\right\} \times \left\{y' \in \mathbf{S}^{m-1} : \left|y' - y_0'\right| < \beta\right\}$ is a cap on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ for some $x_0' \in \mathbf{S}^{n-1}$, $y_0' \in \mathbf{S}^{m-1}$ and $\delta, \beta \in (0,1]$. The block space $B_q^{(0,\nu)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ is defined by

$$B_q^{(0,\nu)} = \{\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \colon \Omega = \sum_{\mu=1}^{\infty} C_\mu b_\mu \textit{ with } M_q^{(0,\nu)}\left(\{C_\mu\}\right) < \infty\},$$

where each C_μ is a complex number; each b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$, and

$$M_q^{(0,\nu)}\left(\{C_\mu\}\right) = \sum_{\mu=1}^{\infty} \left|C_\mu\right| \left(1 + \log^{(\nu+1)}\left(\left|I_\mu\right|^{-1}\right)\right).$$

Let $\|\Omega\|_{B_q^{(0,\nu)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} = \inf\{M_q^{(0,\nu)}\left(\{C_\mu\}\right) : \Omega = \sum_{\mu=1}^{\infty} C_\mu b_\mu$ and each b_μ is a q-block function supported on a cap I_μ on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\}$. Then $\|\cdot\|_{B_q^{(0,\nu)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}$ is a norm on the space $B_q^{(0,\nu)}\left(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}\right)$, and the space $\Big(B_q^{(0,\nu)}\left(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}\right),$ $\|\cdot\|_{B_q^{(0,\nu)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\Big)$ is a Banach space.

Employing the ideas of [18, 22] pointed out, for any $q>1$ and for any $\nu_2>\nu_1>-1$,

$$\cup_{r>1}L^r(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\subset B_q^{(0,\nu_2)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\subset B_q^{(0,\nu_1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}).$$

Our main concern in this work is in dealing with Marcinkiewicz operators $\mathcal{M}_{\Omega,h,\phi,\psi}$ under weak conditions on Ω as well as h. In fact, we will extend some known results (see [11, 25, 19]) to the case $\Omega\in B_q^{(0,0)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $q>1$ and $h\in\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)$ for some $\gamma>1$, where $\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)$ denote the set of all measurable functions h on $\mathbf{R}^+\times\mathbf{R}^+$ such that

$$\sup_{R_1R_2\in\mathbf{Z}}\frac{1}{R_1R_2}\left(\int_0^{R_2}\int_0^{R_1}|h(t,s)|^\gamma dtds\right)^{1/\gamma}<\infty.$$

Our main result is described as follows.

Theorem 1.1. Let $\Omega\in B_q^{(0,0)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $q>1$, $h\in\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)$ for some $\gamma>1$ and $\phi\in\mathcal{H}_{d_1}$, $\psi\in\mathcal{H}_{d_2}$ for some $d_1,d_2\neq 0$. Then there exists a constant C_p such that

$$\left\|\mathcal{M}_{\Omega,h,\phi,\psi}f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\leq C_p\,\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}$$

for any $f\in L^p(\mathbf{R}^n\times\mathbf{R}^m)$ and for any p satisfying $|1/p-1/2|<\min\{1/2,1/\gamma'\}$.

Throughout this paper, the letter C denotes a bounded positive constant that may vary at each occurrence but independent of the essential variables.

2. Definitions and Lemmas

In this section, we present some definitions and also establish some lemmas used in the sequel. Let us start this section by introducing the following:

Definition 2.1. Let $\mu\in\mathbf{N}\cup\{0\}$ and I_μ be an interval on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ with $\left|I_\mu\right|<e^{-1}$. Also, let $A_\mu=[\log\left|I_\mu\right|^{-1}]$ and $\omega_\mu=2^{A_\mu}$, where $[\cdot]$ is the greatest integer function. For suitable functions ϕ,ψ defined on $\mathbf{R}^+$ and $\tilde{b}_\mu\in L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, we define the family of measures $\{\sigma_{\tilde{b}_\mu,t,s}:t,s\in\mathbf{R}^+\}$ and the corresponding maximal operator $\sigma^*_{\tilde{b}_\mu,h,t,s}$ on $\mathbf{R}^n\times\mathbf{R}^m$ by

$$\begin{aligned}\int_{\mathbf{R}^n}\int_{\mathbf{R}^m}fd\sigma_{\tilde{b}_\mu,t,s}&=\frac{1}{ts}\int_{1/2t\leq|u|\leq t}\int_{1/2s\leq|v|\leq s}f(\phi(|u|)u',\psi(|v|)v')K_{\tilde{b}_\mu,h}(u,v)dudv,\\ \sigma^*_{\tilde{b}_\mu,h,t,s}f(x,y)&=\sup_{t,s\in\mathbf{R}^+}||\sigma_{\tilde{b}_\mu,t,s}|*f(x,y)|,\end{aligned}$$

where $|\sigma_{\tilde{b}_\mu,t,s}|$ is defined in the same way as $\sigma_{\tilde{b}_\mu,t,s}$, but with replacing $K_{\tilde{b}_\mu,h}$ by $|K_{\tilde{b}_\mu,h}|$. We write $t^{\pm\alpha}=\min\{t^{-\alpha},t^{+\alpha}\}$ and $\|\sigma\|$ for the total variation of σ.

In order to prove Theorem 1.1, it suffices to prove the following lemmas.

Lemma 2.2. Let $\mu \in \mathbf{N} \cup \{0\}$, $q > 1$, $h \in \triangle_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ for some $1 < \gamma \leq 2$, and $\phi \in \mathcal{H}_{d_1}$, $\psi \in \mathcal{H}_{d_2}$ for some $d_1, d_2 \neq 0$. Let $\tilde{b}_\mu$ be a function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ satisfying (i) $\left\|\tilde{b}_\mu\right\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \leq \left|I_\mu\right|^{-\frac{1}{q'}}$ for some interval I_μ on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ with $\left|I_\mu\right| < e^{-1}$; (ii) $\left\|\tilde{b}_\mu\right\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \leq 1$ and (iii) $\tilde{b}_\mu$ satisfies the vanishing conditions in (1.1) with Ω replaced by $\tilde{b}_\mu$. Then there are constants $\alpha, C > 0$ with $0 < \alpha < \dfrac{1}{2q'}$ such that

$$\left\|\sigma_{\tilde{b}_\mu,t,s}(\xi,\eta)\right\| \leq C; \tag{2.1}$$

$$\int_{\omega_\mu^i}^{\omega_\mu^{i+1}} \int_{\omega_\mu^j}^{\omega_\mu^{j+1}} \left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right|^2 \frac{dtds}{ts} \leq CA_\mu^2 \left|\xi\omega_\mu^{id_1}\right|^{\pm\frac{2\alpha}{\gamma' A_\mu}} \left|\eta\omega_\mu^{jd_2}\right|^{\pm\frac{2\alpha}{\gamma' A_\mu}} \tag{2.2}$$

hold for all $i, j \in \mathbf{Z}$. The constant C is independent of i, j, μ, ξ and η.

Proof. We prove our estimates only for $d_1, d_2 > 0$ because the proof for the other cases $d_1 < 0$ or $d_2 < 0$ is essentially the same and requires only minor modifications. Also we prove this lemma for the case $1 < q \leq 2$, since $L^q\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right) \subseteq L^2\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ for $q \geq 2$. By using condition (ii), it is easy to verify that (2.1) holds. By Hölder's inequality and a simple change of variables, we get that

$$\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| \leq \left(\int_{1/2t}^{t}\int_{1/2s}^{s} |h(r,k)|^\gamma \frac{drdk}{rk}\right)^{1/\gamma} \left(\int_{1/2}^{1}\int_{1/2}^{1} \left|L_{t,s}(r,k)\right|^{\gamma'} \frac{drdk}{rk}\right)^{1/\gamma'},$$

where

$$L_{t,s}(r,k) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} e^{-i(\phi(tr)x\cdot\xi + \psi(ks)y\cdot\eta)} \tilde{b}_\mu(x,y)\, d\sigma(x)\, d\sigma(y).$$

Since $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$, $1 < \gamma \leq 2$ and $|L_{t,s}(r,k)| \leq 1$, we obtain

$$\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| \leq C \left(\int_{1/2}^{1}\int_{1/2}^{1} \left|L_{t,s}(r,k)\right|^2 \frac{drdk}{rr}\right)^{1/\gamma'}.$$

By Schwarz inequality, we derive that

$$\begin{aligned} |L_{t,s}(r,k)|^2 &\le \int_{\mathbf{S}^{m-1}} \left| \int_{\mathbf{S}^{n-1}} e^{-i\phi(tr)x\cdot\xi}\, \tilde{b}_\mu(x,y)\, d\sigma(x) \right|^2 d\sigma(y) \\ &= \int_{\mathbf{S}^{m-1}} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} e^{-i\phi(tr)(x-u)\cdot\xi}\, \tilde{b}_\mu(x,y)\, \overline{\tilde{b}_\mu(u,y)} d\sigma(x)\, d\sigma(u) \right) d\sigma(y)\,. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| \\ &\quad\le C \left(\int_{\mathbf{S}^{m-1}} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} J(\xi,x,u)\tilde{b}_\mu(x,y)\, \overline{\tilde{b}_\mu(u,y)} d\sigma(x)\, d\sigma(u) \right) d\sigma(y) \right)^{1/\gamma'}, \end{aligned}$$

where $J(\xi,x,u) = \int_{1/2}^{1} e^{-i\phi(tr)\xi\cdot(x-u)} \frac{dr}{r}$. Write $J(\xi,x,u) = \int_{1/2}^{1} Y_t'(r) \frac{dr}{r}$, where

$$Y_t(r) = \int_{1/2}^{r} e^{-i\phi(tz)\xi\cdot(x-u)} dz,\ \ 1/2 \le z \le r \le 1.$$

By Van der Corput's lemma, the conditions on ϕ and integration by parts, we conclude

$$|J(\xi,x,u)| \le C \left| t^{d_1}\xi\cdot(x-u)\right|^{-1},$$

which when combined with the trivial estimate $|J(\xi,x,u)| \le C$ gives

$$|J(\xi,x,u)| \le C \left|t^{d_1}\xi\right|^{-\alpha} \left|\xi'\cdot(x-u)\right|^{-\alpha} \tag{2.3}$$

for any $0 < \alpha < 1$. Thus, by using Hölder's inequality, we have

$$\begin{aligned} \left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| &\le C \left|\xi t^{d_1}\right|^{\frac{-\alpha}{\gamma'}} \left\|\tilde{b}_\mu\right\|^{2/\gamma'}_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \\ &\quad\times \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \left|\xi'\cdot(x-u)\right|^{-\alpha q'} d\sigma(x) d\sigma(u) \right)^{1/q'\gamma'}. \end{aligned}$$

By choosing $0 < 2\alpha q' < 1$, we get that the last integral is finite. Hence, by the condition (i), we reach

$$\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| \leq C\left|\xi t^{d_1}\right|^{\frac{-\alpha}{\gamma'}}\left|I_\mu\right|^{-2/q'\gamma'}.$$

Combine this with the trivial estimate $\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| \leq C$ provides

$$\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| \leq C\left|\xi t^{d_1}\right|^{-\frac{\alpha}{\gamma' A_\mu}}. \tag{2.4}$$

Similarly, we derive

$$\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| \leq C\left|\eta s^{d_2}\right|^{-\frac{\alpha}{\gamma' A_\mu}}. \tag{2.5}$$

The other estimates in (2.2) can be obtained by using the cancellation property of $\tilde{b}_\mu$. By a change of variable plus the conditions on ϕ, we have that

$$\begin{aligned}&\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| \\ &\leq \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\int_{1/2}^{1}\int_{1/2}^{1}|e^{-i\phi(tr)\xi\cdot x}-1||\tilde{b}_\mu(x,y)||h(tr,ks)|\frac{drdk}{rk}d\sigma(x)\,d\sigma(y) \\ &\leq C\left|\xi t^{d_1}\right|,\end{aligned}$$

which when combined with the trivial estimate $\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| \leq C$ gives that

$$\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| \leq C\left|\xi t^{d_1}\right|^{\frac{\alpha}{\gamma' A_\mu}}. \tag{2.6}$$

Following the same manner, we attain

$$\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| \leq C\left|\eta s^{d_2}\right|^{\frac{\alpha}{\gamma' A_\mu}}. \tag{2.7}$$

Therefore, by combining (2.4)-(2.5) and (2.6)-(2.7), we acquire

$$\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right| \leq C\left|\xi t^{d_1}\right|^{\pm\frac{\alpha}{\gamma' A_\mu}}\left|\eta s^{d_2}\right|^{\pm\frac{\alpha}{\gamma' A_\mu}}, \tag{2.8}$$

and consequently,

$$\int_{\omega_\mu^i}^{\omega_\mu^{i+1}}\int_{\omega_\mu^j}^{\omega_\mu^{j+1}}\left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right|^2\frac{dtds}{ts} \leq CA_\mu^2\left|\xi\omega_\mu^{id_1}\right|^{\pm\frac{2\alpha}{\gamma' A_\mu}}\left|\eta\omega_\mu^{jd_2}\right|^{\pm\frac{2\alpha}{\gamma' A_\mu}}. \tag{2.9}$$

This completes the proof of the lemma. ■

We shall need the following lemma which can be found in [15].

Lemma 2.3. Let $\{\mu_i\colon i \in \mathbf{Z}\}$ be a sequence of nonnegative Borel measures on $\mathbf{R}^n$, and let $\{a_i\colon i \in \mathbf{Z}\}$ be lacunary sequence of positive numbers with $\inf_{i\in\mathbf{Z}}(a_{i+1}/a_i) \geq a$. Suppose that for all $i \in \mathbf{Z}$, $\xi \in \mathbf{R}^n$ and for some $C > 0$,

$$(i)\ \ \|\mu_i\| \leq C; \quad (ii)\left|\hat{\mu}_i\,(\xi)\right| \leq C\,|a_i\xi|^{-a}; \quad (iii)\left|\hat{\mu}_i\,(\xi) - 1\right| \leq C\,|a_{i+1}\xi|^{a}.$$

Then the inequality

$$\left\|\mu^*(f)\right\|_{L^p(\mathbf{R}^n)} \leq C\,\|f\|_{L^p(\mathbf{R}^n)}$$

holds for all $1 < p \leq \infty$ and $f \in L^p\left(\mathbf{R}^n\right)$.

The following result follows immediately from the Lemma 2.3.

Lemma 2.4. Let $\phi \in \mathcal{H}_d$ for some $d \neq 0$. Define the maximal function

$$M_{\phi,\xi}\,f(x) = \sup_{t\in\mathbf{R}^+}\frac{1}{t}\left|\int_{\frac{1}{2}t}^{t} f(x - \phi\,(s)\,\xi)ds\right|.$$

Then for $1 < p < \infty$, there exists a constant C_p such that

$$\left\|M_{\phi,\xi}(f)\right\|_{L^p(\mathbf{R}^n)} \leq C_p\,\|f\|_{L^p(\mathbf{R}^n)}$$

for any $f \in L^p(\mathbf{R}^n)$.

Proof. It is clear that $M_{\phi,\xi}\,f(x) \leq C\sup_{i\in\mathbf{Z}}\left|\int_{2^i}^{2^{i+1}} f(x - \phi\,(s)\,\xi)\frac{ds}{s}\right|$. Define a sequence of measures ν_i on $\mathbf{R}$ by

$$\hat{\nu}_i(\xi) = \int_{2^i}^{2^{i+1}} e^{-i\phi(s)\xi}\,\frac{ds}{s}.$$

Following the same arguments used in proof of Lemma 2.2, we achieve that

$$\begin{cases} \left|\hat{\nu}_i(\xi)\right| \leq C; \\ \left|\hat{\nu}_i(\xi) - \hat{\nu}_i(0)\right| \leq C\left|2^{id}\xi\right|; \\ \left|\hat{\nu}_i(\xi)\right| \leq C\left|2^{id}\xi\right|^{-1}. \end{cases} \tag{2.10}$$

By this and Lemma 2.3, we finish the proof. ■

Lemma 2.5. Let $\mu \in \mathbf{N} \cup \{0\}$, $1 < q \leq 2$, $h \in \triangle_{\gamma}(\mathbf{R}^{+} \times \mathbf{R}^{+})$ for some $1 < \gamma \leq 2$ and $\phi \in \mathcal{H}_{d_1}$, $\psi \in \mathcal{H}_{d_2}$ for some $d_1, d_2 \neq 0$. Let $\tilde{b}_{\mu}$ be a given function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ as in Lemma 2.2. Let $M_{\tilde{b}_{\mu},\phi,\psi}$ be the maximal function defined on $\mathbf{R}^{n} \times \mathbf{R}^{m}$ by

$$M_{\tilde{b}_{\mu},\phi,\psi} f(x, y) = \sup_{t,s \in \mathbf{R}^{+}} \frac{1}{ts} \left| \int_{\frac{1}{2}t}^{t} \int_{\frac{1}{2}s}^{s} f(x - \phi(|u|)\, u', y - \psi(|v|)\, v') \frac{\tilde{b}_{\mu}(u, v)}{|u|^{n-1}|v|^{m-1}} du dv \right|.$$

Then there exists a constant C_p such that

$$\left\| M_{\tilde{b}_{\mu},\phi,\psi}(f) \right\|_{L^{p}(\mathbf{R}^{n} \times \mathbf{R}^{m})} \leq C_p \, \| f \|_{L^{p}(\mathbf{R}^{n} \times \mathbf{R}^{m})}$$

for any $1 < p < \infty$ and $f \in L^{p}(\mathbf{R}^{n} \times \mathbf{R}^{m})$.

Proof. By using Hölder's inequality, we have

$$\begin{aligned} \left\| M_{\tilde{b}_{\mu},\phi,\psi} f \right\|_{L^{p}(\mathbf{R}^{n} \times \mathbf{R}^{m})} &\leq \left\| \tilde{b}_{\mu} \right\|_{L^{1}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \\ &\times \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} \left\| M^{1}_{\phi,u'} \odot M^{2}_{\psi,v'} f \right\|_{L^{p}(\mathbf{R}^{n} \times \mathbf{R}^{m})} d\sigma(u') d\sigma(v'), \end{aligned}$$

where $M^{1}_{\phi,u'} f(x, y) = M_{\phi,u'} f(\cdot, y)(x)$, $M^{2}_{\psi,v'} f(x, y) = M_{\psi,v'} f(x, \cdot)(y)$ and $\odot$ denotes the composition of operators. Therefore, we get the result by using Lemma 2.4. ∎

Lemma 2.6. Let $\mu \in \mathbf{N} \cup \{0\}$, $1 < q \leq 2$, $h \in \triangle_{\gamma}(\mathbf{R}^{+} \times \mathbf{R}^{+})$ for some $1 < \gamma \leq 2$ and $\phi \in \mathcal{H}_{d_1}$, $\psi \in \mathcal{H}_{d_2}$ for some $d_1, d_2 \neq 0$. Assume that $\tilde{b}_{\mu}$ is a given function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ as in Lemma 2.2, and $\sigma^{*}_{\tilde{b}_{\mu},h,t,s}$ is given as in Definition 2.1. Then for any $f \in L^{p}(\mathbf{R}^{n} \times \mathbf{R}^{m})$ with $p > \gamma'$, there exists a constant C_p such that

$$\|\sigma^{*}_{\tilde{b}_{\mu},h,t,s} f(x, y)\|_{L^{p}(\mathbf{R}^{n} \times \mathbf{R}^{m})} \leq C_p \|f\|_{L^{p}(\mathbf{R}^{n} \times \mathbf{R}^{m})}. \tag{2.11}$$

Proof. By Hölder's inequality, we obtain

$$\begin{aligned} ||\sigma_{\tilde{b}_{\mu},t,s}| * f(x, y)| &\leq C \left\| \tilde{b}_{\mu} \right\|^{1/\gamma}_{L^{1}(\mathbf{S}^{n-1} \times \mathbf{S}^{n-1})} \\ &\times sup_{t,s \in \mathbf{R}^{+}} \left(\int_{\frac{t}{2}}^{t} \int_{\frac{s}{2}}^{s} \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} |\tilde{b}_{\mu}(u, v)||f(x - \phi(r)u, y - \phi(k)v)|^{\gamma'} d\sigma(u) d\sigma(v) \frac{drdk}{rk} \right)^{1/\gamma'}. \end{aligned}$$

Use Minkowski's inequality for integrals and assumptions of $\tilde{b}_\mu$, we get

$$\begin{aligned}
&\|\sigma^*_{\tilde{b}_\mu,t,s,h} f(x,y)\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \\
\leq\ & C\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\tilde{b}_\mu(u,v)| \left(\|M_{\tilde{b}_\mu,\phi,\psi}\left(|f|^{\gamma'}\right)\|_{L^{p/\gamma'}(\mathbf{R}^n\times\mathbf{R}^m)}\right) d\sigma(u)d\sigma(v)\right)^{1/\gamma'}.
\end{aligned}$$

By this and Lemma 2.5, we finish the proof. ■

By tracking the constants, we have the following.

Lemma 2.7. Let $\mu \in \mathbf{N}\cup\{0\}$, $q > 1$, $h \in \triangle_\gamma(\mathbf{R}^+\times\mathbf{R}^+)$ for some $1 < \gamma \leq 2$ and $\phi \in \mathcal{H}_{d_1}$, $\psi \in \mathcal{H}_{d_2}$ for some $d_1, d_2 \neq 0$. If $\tilde{b}_\mu$ is a given function on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ as in Lemma 2.2, then for any p satisfying $|1/p - 1/2| < 1/\gamma'$, there exists a positive constant C_p such that

$$\left\|\left(\sum_{i,j\in\mathbf{Z}} \int_{\omega_\mu^i}^{\omega_\mu^{i+1}} \int_{\omega_\mu^j}^{\omega_\mu^{j+1}} \left|\sigma_{\tilde{b}_\mu,t,s} * g_{i,j}\right|^2 \frac{dtds}{ts}\right)^{1/2}\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \leq C_p A_\mu \left\|\left(\sum_{i,j\in\mathbf{Z}} \left|g_{i,j}\right|^2\right)^{1/2}\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}$$

holds for arbitrary measurable functions $\left\{g_{i,j}(\cdot,\cdot)\right\}_{i,j\in\mathbf{Z}}$ on $\mathbf{R}^n\times\mathbf{R}^m$. The constant C_p is independent of μ.

Proof. We employ some ideas from [2, 16]. By Schwarz's inequality, we obtain

$$\begin{aligned}
\left|\sigma_{\tilde{b}_\mu,t,s} * g_{i,j}\right|^2 \leq\ & C\int_{\frac{1}{2}t}^{t}\int_{\frac{1}{2}s}^{s}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \left|g_{i,j}(x-\phi(r)u, y-\psi(k)v)\right|^2 \\
&\times \left|\tilde{b}_\mu(u,v)\right| |h(r,k)|^{2-\gamma} d\sigma(u)d\sigma(v)\frac{drdk}{rk}.
\end{aligned}$$

Let us first prove this lemma for the case $2 \leq p < \dfrac{2\gamma}{2-\gamma}$. By duality, there is a non-negative function $\Lambda \in L^{(p/2)'}(\mathbf{R}^n\times\mathbf{R}^m)$ with $\|\Lambda\|_{L^{(p/2)'}(\mathbf{R}^n\times\mathbf{R}^m)} \leq 1$ such that

$$\begin{aligned}
&\left\|\left(\sum_{i,j\in\mathbf{Z}} \int_{\omega_\mu^i}^{\omega_\mu^{i+1}} \int_{\omega_\mu^j}^{\omega_\mu^{j+1}} \left|\sigma_{\tilde{b}_\mu,t,s} * g_{i,j}\right|^2 \frac{dtds}{ts}\right)^{1/2}\right\|^2_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \\
=\ & \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{i,j\in\mathbf{Z}} \int_{\omega_\mu^i}^{\omega_\mu^{i+1}} \int_{\omega_\mu^j}^{\omega_\mu^{j+1}} \left|\sigma_{\tilde{b}_\mu,t,s} * g_{i,j}(x,y)\right|^2 \frac{dtds}{ts}\Lambda(x,y)dxdy.
\end{aligned}$$

Thus, by a change of variable we derive

$$\left\| \left(\sum_{i,j\in\mathbf{Z}} \int_{\omega_\mu^i}^{\omega_\mu^{i+1}} \int_{\omega_\mu^j}^{\omega_\mu^{j+1}} \left| \sigma_{\tilde{b}_\mu,t,s} * g_{i,j} \right|^2 \frac{dtds}{ts} \right)^{1/2} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}^2$$

$$\leq \; CA_\mu^2 \int_{\mathbf{R}^n\times\mathbf{R}^m} \left(\sum_{i,j\in\mathbf{Z}} \left| g_{i,j}(x,y) \right|^2 \right) \sigma^*_{\tilde{b}_\mu,|h|^{2-\gamma},t,s} \Lambda(-x,-y) dxdy.$$

Since $h(\cdot,\cdot) \in L^\gamma(\mathbf{R}^+ \times \mathbf{R}^+, \frac{dtds}{ts})$, then $|h(\cdot,\cdot)|^{2-\gamma} \in L^{\gamma/(2-\gamma)}(\mathbf{R}^+ \times \mathbf{R}^+, \frac{dtds}{ts})$, and since $\left(\frac{p}{2}\right)' > \left(\frac{\gamma}{2-\gamma}\right)'$, then by Lemma 2.6 and Hölder's inequality, we achieve that

$$\left\| \left(\sum_{i,j\in\mathbf{Z}} \int_{\omega_\mu^i}^{\omega_\mu^{i+1}} \int_{\omega_\mu^j}^{\omega_\mu^{j+1}} \left| \sigma_{\tilde{b}_\mu,t,s} * g_{i,j} \right|^2 \frac{dtds}{ts} \right)^{1/2} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}^2$$

$$\leq \; CA_\mu^2 \left\| \left(\sum_{i,j\in\mathbf{Z}} \left| g_{i,j} \right|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}^2 \left\| \sigma^*_{\tilde{b}_\mu,|h|^{2-\gamma},t,s} \Lambda(-x,-y) \right\|_{L^{(p/2)'}(\mathbf{R}^n\times\mathbf{R}^m)}$$

$$\leq \; C_p A_\mu^2 \left\| \left(\sum_{i,j\in\mathbf{Z}} \left| g_{i,j} \right|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}^2 .$$

For the case $\frac{2\gamma}{3\gamma-2} < p < 2$, by the duality, there are functions $\zeta = \zeta_{i,j}(x,y,t,s)$ defined on $\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^+ \times \mathbf{R}^+$ with $\left\| \left\| \|\zeta_{i,j}\|_{L^2([\omega_\mu^i,\omega_\mu^{i+1}]\times[\omega_\mu^j,\omega_\mu^{j+1}],\frac{dtds}{ts})} \right\|_{l^2} \right\|_{L^{p'}(\mathbf{R}^n\times\mathbf{R}^m)} \leq 1$ such that

$$\left\| \left(\sum_{i,j\in\mathbf{Z}} \int_{\omega_\mu^i}^{\omega_\mu^{i+1}} \int_{\omega_\mu^j}^{\omega_\mu^{j+1}} \left| \sigma_{\tilde{b}_\mu,t,s} * g_{i,j} \right|^2 \frac{dtds}{ts} \right)^{1/2} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}$$

$$\leq \; C_p \left\| (\Upsilon(\zeta))^{1/2} \right\|_{L^{p'}(\mathbf{R}^n\times\mathbf{R}^m)} \left\| \left(\sum_{i,j\in\mathbf{Z}} \left| g_{i,j} \right|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} ,$$

where

$$\Upsilon(\zeta)=\sum_{i,j\in\mathbf{Z}}\int_{\omega_\mu^i}^{\omega_\mu^{i+1}}\int_{\omega_\mu^j}^{\omega_\mu^{j+1}}\left|\sigma_{\tilde{b}_\mu,t,s}*\zeta_{i,j}(x,y,t,s)\right|^2\frac{dtds}{ts}.$$

Applying the above procedure, we reach that

$$\begin{aligned}\|\Upsilon(\zeta)\|_{L^{(p'/2)}(\mathbf{R}^n\times\mathbf{R}^m)} &\leq C_p\left\|\left(\sum_{i,j\in\mathbf{Z}}\int_{\omega_\mu^i}^{\omega_\mu^{i+1}}\int_{\omega_\mu^j}^{\omega_\mu^{j+1}}\left|\zeta_{i,j}(\cdot,\cdot,t,s)\right|^2\frac{dtds}{ts}\right)\right\|_{L^{(p'/2)}(\mathbf{R}^n\times\mathbf{R}^m)}\\ &\times\left\|\sigma^*{}_{\tilde{b}_\mu,s,t,|h|^{2-\gamma}}(\vartheta)\right\|_{L^{(p'/2)'}(\mathbf{R}^n\times\mathbf{R}^m)}\leq CA_\mu,\end{aligned}$$

where ϑ is a function in $L^{(p'/2)'}(\mathbf{R}^n\times\mathbf{R}^m)$ with $\|\vartheta\|_{L^{(p'/2)'}(\mathbf{R}^n\times\mathbf{R}^m)}\leq 1$. By this, we get our desired for $\frac{2\gamma}{3\gamma-2}\leq p<2$. This completes the proof of Lemma 2.7. ■

3. Proof of the main Result

Assume that $\Omega\in B_q^{(0,0)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $q>1$ and satisfies (1.1). Thus Ω can be written as $\Omega=\sum_{\mu=1}^{\infty}C_\mu b_\mu$, where $C_\mu\in\mathbf{C}$, b_μ is a q-block supported on an interval I_μ on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$, and $M_q^{(0,0)}\left(\{C_\mu\}\right)<\infty$. For each block function $b_\mu(x,y)$, let $\tilde{b}_\mu(x,y)$ be a function defined by

$$\begin{aligned}\tilde{b}_\mu(x,y) &= b_\mu(x,y)-\int_{\mathbf{S}^{n-1}}b_\mu(u,y)\,d\sigma(u)-\int_{\mathbf{S}^{m-1}}b_\mu(x,v)\,d\sigma(v)\\ &+\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}b_\mu(u,v)\,d\sigma(u)\,d\sigma(v).\end{aligned}\tag{3.1}$$

Let $\mathbf{D}=\{\mu\in\mathbf{N}:|I_\mu|<e^{-1}\}$, and let $\tilde{b}_0=\Omega-\sum_{\mu\in\mathbf{D}}C_\mu\tilde{b}_\mu$. Then it is easy to show that, for each $\mu\in\mathbf{D}\cup\{0\}$, $\tilde{b}_\mu(x,y)$ has the following properties: sss

$$\int_{\mathbf{S}^{n-1}}\tilde{b}_\mu(u,\cdot)\,d\sigma(u)=\int_{\mathbf{S}^{m-1}}\tilde{b}_\mu(\cdot,v)\,d\sigma(v)=0,$$

$$\left\|\tilde{b}_\mu\right\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\leq C\left|I_\mu\right|^{-\frac{1}{q'}},$$

and

$$\left\|\tilde{b}_\mu\right\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \leq C,$$

where I_0 is an interval on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ with $|I_0| = e^{-2}$ and C is a positive constant independent of μ. Using the assumption that Ω satisfies the vanishing conditions (1.1) and the definition of $\tilde{b}_\mu$; we deduce that Ω can be written as $\Omega = \sum\limits_{\mu\in\mathbf{D}\cup\{0\}} C_\mu \tilde{b}_\mu$, which in turn implies

$$\left\|\mathcal{M}_{\Omega,h,\phi,\psi}\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \leq \sum_{\mu\in\mathbf{D}\cup\{0\}} |C_\mu| \left\|\mathcal{M}_{\tilde{b}_\mu,h,\phi,\psi}\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}. \tag{3.2}$$

Therefore, to prove our theorem, it is enough to show

$$\left\|\mathcal{M}_{\tilde{b}_\mu,\phi,\psi}\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \leq C_p A_\mu \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}. \tag{3.3}$$

We establish the inequality (3.3) by applying the same approaches that Al-Qassem [2] as well as Fan and Pan [16] used. Without loss of generality we may assume that $h \in L^\gamma(\mathbf{R}^+\times\mathbf{R}^+, \frac{dtds}{ts})$ for some $1 < \gamma \leq 2$ and $\phi \in \mathcal{H}_{d_1}$, $\psi \in \mathcal{H}_{d_2}$ for some $d_1, d_2 > 0$. For $i \in \mathbf{Z}$ and $\mu \in \mathbf{N}$, let $\left\{\Lambda_{i,\mu}\right\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the interval $\mathcal{I}_{i,\mu} = [\omega_\mu^{-id_1-|d_1|}, \omega_\mu^{-id_1+|d_1|}]$. More precisely, we require the following:

$$\begin{aligned} \Lambda_{i,\mu} &\in C^\infty,\ 0 \leq \Lambda_{i,\mu} \leq 1,\ \sum_i \Lambda_{i,\mu}(t) = 1, \\ \operatorname{supp}\Lambda_{i,\mu} &\subseteq \mathcal{I}_{i,\mu},\ \textit{and}\ \left|\frac{d^s\Lambda_{i,\mu}(t)}{dt^s}\right| \leq \frac{C_s}{t^s}, \end{aligned}$$

where C_s is independent of ω_μ. Define the multiplier operators $M_{i,j,\mu}$ on $\mathbf{R}^n\times\mathbf{R}^m$ by $(\widehat{M_{i,j,\mu}f})(\xi,\eta) = \Lambda_{i,\mu}(|\xi|)\Lambda_{j,\mu}(|\eta|)\hat{f}(\xi,\eta)$. By Minkowski's inequality, we get that

$$\begin{aligned} \mathcal{M}_{\tilde{b}_\mu,h,\phi,\psi}f(x,y) &= \left(\int_{\mathbf{R}^+\times\mathbf{R}^+}\left|\sum_{i,j=0}^{\infty} 2^{-(i+j)}\int_{2^{-i-1}t<|u|\leq 2^{-i}t}\int_{2^{-j-1}s<|v|\leq 2^{-j}s}\right.\right. \\ &\times \left.\left. f(x-\phi(|u|)u', y-\psi(|v|)v')K_{\tilde{b}_\mu,h}(u,v)dudv\right|^2 \frac{dtds}{(ts)^3}\right)^{1/2} \\ &\leq \sum_{i,j=0}^{\infty} 2^{-(i+j)}\left(\int_{\mathbf{R}^+\times\mathbf{R}^+}\left|\int_{1/2t<|u|\leq t}\int_{1/2s<|v|\leq s}\right.\right. \\ &\times \left.\left. f(x-\phi(|u|)u', y-\psi(|v|)v')K_{\tilde{b}_\mu,h}(u,v)dudv\right|^2 \frac{dtds}{(ts)^3}\right)^{1/2} \\ &\leq 4\left(\int_{\mathbf{R}^+\times\mathbf{R}^+}\left|\sigma_{\tilde{b}_\mu,t,s} * f(x,y)\right|^2 \frac{dtds}{ts}\right)^{1/2}. \end{aligned} \tag{3.4}$$

Decompose $\sigma_{\tilde{b}_\mu,t,s} * f(x,y) = \sum_{a,b\in\mathbf{Z}} Y_{\mu,a,b}(x,y,t,s)$, where

$$Y_{\mu,a,b}(x,y,t,s) = \sum_{i,j\in\mathbf{Z}} \sigma_{\tilde{b}_\mu,t,s} * M_{i+a,j+b,\mu} f(x,y)\chi_{[\omega_\mu^i,\omega_\mu^{i+1})\times[\omega_\mu^j,\omega_\mu^{j+1})}(t,s).$$

For any $f \in \mathcal{S}(\mathbf{R}^n\times\mathbf{R}^m)$, define $S_{\mu,a,b}f(x,y) = \left(\int_0^\infty\int_0^\infty \left|Y_{\mu,a,b}(x,y,t,s)\right|^2 \frac{dtds}{ts}\right)^{1/2}$.

Hence,

$$\mathcal{M}_{\tilde{b}_\mu,t,s} f(x,y) \le C \sum_{a,b\in\mathbf{Z}} S_{\mu,a,b}(f). \tag{3.5}$$

Let us first compute the L^2-norm of $S_{\mu,a,b}(f)$. By using Plancherel's theorem and Lemma 2.2, we obtain that

$$\begin{aligned}
&\left\|S_{\mu,a,b}(f)\right\|^2_{L^2(\mathbf{R}^n\times\mathbf{R}^m)}\\
\le\ & \sum_{i,j\in\mathbf{Z}} \int_{\Gamma_{i+a,j+b,\mu}} \left(\int_{\omega_\mu^i}^{\omega_\mu^{i+1}}\int_{\omega_\mu^j}^{\omega_\mu^{j+1}} \left|\hat{\sigma}_{\tilde{b}_\mu,t,s}(\xi,\eta)\right|^2 \frac{dtds}{ts}\right) \left|\hat{f}(\xi,\eta)\right|^2 d\xi d\eta\\
\le\ & CA_\mu^2 \sum_{i,j\in\mathbf{Z}} \int_{\Gamma_{i+a,j+b,\mu}} \left(\left|\omega_\mu^{id_1}\xi\right|^{\pm\frac{2\alpha}{A_\mu}} \left|\omega_\mu^{jd_2}\eta\right|^{\pm\frac{2\alpha}{A_\mu}}\right) \left|\hat{f}(\xi,\eta)\right|^2 d\xi d\eta\\
\le\ & CA_\mu^2 2^{-\alpha(|a|+|b|)} \sum_{i,j\in\mathbf{Z}} \int_{\Gamma_{i+a,j+b,\mu}} \left|\hat{f}(\xi,\eta)\right|^2 d\xi d\eta\\
\le\ & CA_\mu^2 2^{-\alpha(|a|+|b|)} \left\|f\right\|^2_{L^2(\mathbf{R}^n\times\mathbf{R}^m)},
\end{aligned}$$

where $\Gamma_{i,j,\mu} = \left\{(\xi,\eta)\in\mathbf{R}^n\times\mathbf{R}^m : (|\xi|,|\eta|)\in\mathcal{I}_{i,\mu}\times\mathcal{I}_{j,\mu}\right\}$. Thus,

$$\left\|S_{\mu,a,b}(f)\right\|_{L^2(\mathbf{R}^n\times\mathbf{R}^m)} \le CA_\mu 2^{\frac{-\alpha(|a|+|b|)}{2}} \left\|f\right\|_{L^2(\mathbf{R}^n\times\mathbf{R}^m)}. \tag{3.6}$$

Applying the Littlewood-Paley theory and Theorem 3 along with the remark that follows its statement in [[20], p. 96], plus using Lemma 2.7, we obtain

$$\left\|S_{\mu,a,b}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \le C_p A_\mu \left\|f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \tag{3.7}$$

for any p satisfying $|1/p - 1/2| < 1/\gamma'$. By interpolation between (3.6) and (3.7) we reach

$$\left\|S_{\mu,a,b}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \le CA_\mu 2^{\frac{-\alpha(|a|+|b|)}{2}} \left\|f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}. \tag{3.8}$$

Consequently, by (3.3)-(3.5) and (3.8), we get our result. This completes the proof of Theorem 1.1.

4. Concluding Remarks

Let $k \in \mathbf{N}$ and $n_1 \dots n_k \geq 2$. Assume that for $j = 1, \dots, k$, $\phi_j \in \mathcal{H}_{d_j}$ for some $d_j \neq 0$, a measurable function h on $\mathbf{R}^+ \times \dots \times \mathbf{R}^+$ (k-times), and $\Omega(x_1', \dots, x_k')$ be an integrable function on $\mathbf{S}^{n_1-1} \times \dots \times \mathbf{S}^{n_k-1}$ satisfying the following condition:

$$\int_{\mathbf{S}^{n_j-1}} \Omega(x_1', \dots, x_k') d\sigma(x_j') = 0 \quad for \quad j = 1, \dots, k.$$

Define the corresponding Marcinkiewicz integral operator on $\mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_k}$ by

$$\mathcal{M}_{\Omega,h,\phi_1,\dots,\phi_k} f(x_1, \dots, x_k) = \left(\int_0^\infty \dots \int_0^\infty \left| F_{t_1,\dots,t_k}^{\phi_1,\dots\phi_k}(x_1, \dots, x_k) \right|^2 \frac{dt_1 \dots dt_k}{(dt_1 \dots dt_k)^3} \right)^{1/2}, \tag{4.1}$$

where

$$F_{t_1,\dots,t_k}^{\phi_1,\dots\phi_k}(x_1, \dots, x_k) = \int_{|u_1| \leq t_1} \dots \int_{|u_k| \leq t_k} f(x_1 - \phi_1(|u_1|)u_1', \dots, x_k - \phi_k(|u_k|)u_k')$$
$$\times \quad \frac{\Omega(u_1', \dots, u_k') h(|u_1|, \dots, |u_k|)}{|u_1|^{n_1-1}, \dots, |u_k|^{n_k-1}} du_1 \dots du_k.$$

By following the above procedure, we derive the following corollary.

Corollary 4.1. Let $k \in \mathbf{N}$, $\Omega \in B_q^{(0,k/2-1)}(\mathbf{S}^{n_1-1} \times \dots \times \mathbf{S}^{n_k-1})$ for some $q > 1$. Assume that $h \in \triangle_\gamma(\mathbf{R}^+ \times \dots \times \mathbf{R}^+)$ for some $\gamma > 1$, and for $j = 1, \dots, k$, $\phi_j \in \mathcal{H}_{d_j}$ for some $d_j \neq 0$. Then there exists a constant C_p such that

$$\left\| \mathcal{M}_{\Omega,h,\phi_1,\dots,\phi_k} f \right\|_{L^p(\mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_k})} \leq C_p \left\| f \right\|_{L^p(\mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_k})}$$

for any $f \in L^p(\mathbf{R}^{n_1} \times \dots \times \mathbf{R}^{n_k})$ and for any p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

We point out that for $k = 1$ (the underlying space is not a product space), the L^p boundedness of $\mathcal{M}_{\Omega,h,\phi_1}$ was obtained in [7], and for $k = 2$ the boundedness of $\mathcal{M}_{\Omega,h,\phi_1}$ was established in [8] by using the extrapolation argument found in [5].

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