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On The Evaluation of Integrals for the Solution of the Modified Helmholtz and Heat Equations

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Abstract

We present new fourth order accurate methods for evaluating certain volume and surface integrals which satisfy the modified Helmholtz equation on general two dimensional regions with smooth boundaries. The integrals are computed by fast finite difference methods on a rectangular region in which integration region is embedded at a cost of $O(n^2 \log n)$ operations where n is the number of mesh points in each direction in the embedding region. We apply these techniques to the numerical solution of the modified Helmholtz equation and the heat equation on general regions using integral equation methods, and indicate how these methods could be combined with mesh refinement. Computational results are provided.

Keywords. Helmholtz equation, heat equation, integral equation method

AMS (MOS) subject classification: 65E05, 65R20.

1 Introduction

In this paper we present rapid new high order accurate methods for evaluating volume integrals whose kernels are the fundamental solution of the modified Helmholtz equation and surface integrals whose kernels are the normal derivative of the fundamental solution. The regions of integration can be any smooth curve or region in two dimensions, and the cost of evaluating either integral is essentially equal to the cost of inverting a discrete approximation to the modified Helmholtz operator on a rectangular two dimensional region. Thus, by using Fourier methods [11] both types of integrals can be evaluated in $O(n^2 \log n)$ operations. We note that other rapid methods have been developed for solving the modified Helmholtz equation on rectangular regions [6] and can equally well be combined with our method. Our method is an extension of one we developed previously for solving Poisson's and the biharmonic equation on general regions. See, for example, [9],[16],[17].

The essential idea is the following. Let L denote the modified Helmholtz operator, and suppose the integral W is such that $LW = f$ in D and $LW = 0$ outside. (f can be 0 if W is a surface integral) We evaluate an approximation to W by first embedding D in a larger rectangular region R with a uniform mesh, and computing a fourth order accurate approximation to LhW, the discrete modified Helmholtz operator Lh applied to W, at all the mesh points of R. Once we have done this, we apply an operator that inverts Lh on R to obtain an approximation to W. It is easy to compute such an approximation at most mesh points of R. At mesh points inside D that have all their neighboring mesh points inside D we approximate LhW by f since $LW = f$ on D. Similarly, at mesh points outside D whose neighboring mesh points are also outside D, we set $LhW = 0$ since $LW = 0$ outside D. The difficulty arises at the other, "irregular" mesh points. Because of the discontinuities in the derivatives of W across the boundary of D the discrete modified Helmholtz operator is not well approximated by the continuous Helmholtz operator. It turns out, however, that one can compute an approximation using the discontinuities in W, which depend on f and its derivatives, and information about the boundary of D.

In order to solve the modified Helmholtz equation we use an integral equation formulation also used by Quaife and Kropinski [14] in which one assumes the solution is the sum of a volume integral and a surface integral. The volume integral satisfies the inhomogeneous differential equation, and the surface integral is chosen so that the boundary condition is satisfied. The density of the surface integral is the solution of a boundary integral equation of the second kind with bounded kernel similar to one used for solving Laplace's equation. (The kernel is the normal derivative of the fundamental solution of the Helmholtz equation in free space.) Quaife and Kropinski have solved this equation using a fast multipole method, and for the discretization they used hybrid gauss-trapezoidal quadrature rules developed by Alpert [2] for evaluating integrals with logarithmic singularities. If there are N discretization points on the boundary of the irregular region then they can solve the integral equation using only $O(N)$ operations. Since we were only interested in the accuracy of evaluating the volume and surface integrals, not the speed of solving the integral equation, in our numerical experiments we discretized the integral equation using a Nystrom method with the trapezoid rule as the quadrature method, and solved the resulting linear system of equations using a preconditioned conjugate gradient method [18]. Our cost of solving the integral equation was therefore $O(N^2)$. Our method of evaluating integrals can and should, of course, be combined with FMM type methods for solving integral equations [19], [20]. For simplicity we also used the same discretization in time of the heat equation as Quaife and Kropinski, and combined it with our method of evaluating volume and surface integrals.

We note that integral equation formulations are particularly convenient for solving problems on exterior regions. Once a particular solution has been evaluated, the computational problem is reduced to the solution of a homogeneous differential equation, which only requires the solution of an integral equation on the surface of the region. We also note that integral equation methods for solving differential equations

have become increasingly popular in part due to rapid methods for solving certain integral equations Most prominent among these are those based on the fast multipole method, which have made the solution of problems on very complex geometries possible. Other methods based on wavelets, singular value decompositions and other sparse representations are commonly used [1], [4]. Despite these advances, fewer effective methods have been developed for solving realistic inhomogeneous equations which require the evaluation of volume integrals. The most common methods involve direct application of some quadrature formula. See, for example [13]. In particular, a total of $O(n^4)$ operations are needed to evaluate the integral at every point of an n by n grid, since evaluating each integral requires $O(n^2)$ operations. In contrast, our method only requires $O(n^2 \log n)$ operations.
Another difficulty encountered when using quadrature formulas in a straight- forward way is that fundamental solutions of the modified Helmholtz equation are discontinuous and have discontinuities in their derivatives as the point at which one is evaluating the integral nears a point of the region of integration. That is, the function $K0(r)$ and its derivatives are unbounded at the origin. Therefore, it is difficult compute these integrals very accurately at points in, or near, the region of integration. The method we use does not have these problems. We note other rapid and sophisticated methods have been developed for evaluating volume integrals, but these require smooth extension of the inhomogeneous term from the irregular region to the rest of the rectangular embedding region [8], [14]. Our methods have no such requirement.
The organization of this paper is as follows. In Section 2 we show how to compute the volume and surface integrals, in Section 3 we show how to use these methods in the solution of the modified Helmholtz and heat equations on general regions and discuss mesh refinement. In Section 4 we provide results of numerical experiments.

2 Evaluation of volume and surface integrals

In this section we present our method for evaluating an integral whose kernel $1/2\pi a^2)K0(r/a) + g$ is a fundamental solution of the modified Helmholtz equation

$\Delta u - a^2 u = \delta$

Here $K0(r)$ is the zeroth order modified Bessel function of the second kind and δ is the Dirac function. We later show how to compute surface integrals. Initially we ignore boundary conditions since we often only seek a particular solution of the equation.

2.1 Evaluation of volume integrals

We start by embedding the irregular region D over which we are evaluating the integral in a larger rectangular region R with a uniform grid, say with mesh width h, which ignores the boundary of D.

$$W(x,y) = \frac{1}{2\pi a^2} \int K_0\left(\frac{r}{a}\right) f(x',y')\,dV \quad (2.1)$$

where $r = \sqrt{(x-x')^2+(y-y')^2}$

As noted, we evaluate this integral by computing an approximation to L_hW where L_h is a discrete approximation to the modified Helmholtz operator at all the mesh points of R. Once we have done that, we apply an operator, L_h^{-1} which inverts L_h on R to obtain an approximation to W, de- spite the fact that W is not smooth. We note that inverting L_h on a grid with n points only requires $O(n \log n)$ operations [11].

Since $\Delta W - a^2 W = f$ in D and $\Delta W - a^2 W = 0$ outside D (2.2)

at mesh points inside D, which have all their neighboring mesh points inside D we set $L_hW = f$, and at points outside D, with all their neighbors outside we set $L_hW = 0$.

The problem then reduces to computing an approximation to L_hW at the other mesh points, the points which are in one region, but have neighboring mesh points in the other region.

It turns out that in order to be able to compute an approximation to L_hW at these points it is sufficient to know what the discontinuities in the derivatives of W in the coordinate directions are at the boundary of the region D. We now show how to find these discontinuities.

Suppose the boundary of D is given by $(x(t), y(t))$.

For a function g defined on R let $[g(p)]$ denote the discontinuity in g at a point p on ∂D. An integral of the form (2.1) and its normal derivative are continuous across ∂D. Therefore, for p in ∂D

$$[W(p)] = 0, \qquad (2.3)$$

and

$$[W_n(p)] = \dot{y}(t)[W_x(p)] - \dot{x}(t)[W_y(p)] = 0 \qquad (2.4)$$

By differentiating (2.3) in the tangential direction t, we see

$$[W_t(p)] = \dot{x}(t)[W_x(p)] + \dot{y}(t)[W_y(p)] = 0, \qquad (2.5)$$

so $[W_x] = [W_y] = 0$.

By (2.2) the second derivatives of W are discontinuous and

$$[W_{xx}] + [W_{yy}] = f \qquad (2.6)$$

By differentiating (2.4) and (2.5) in the tangential direction we see

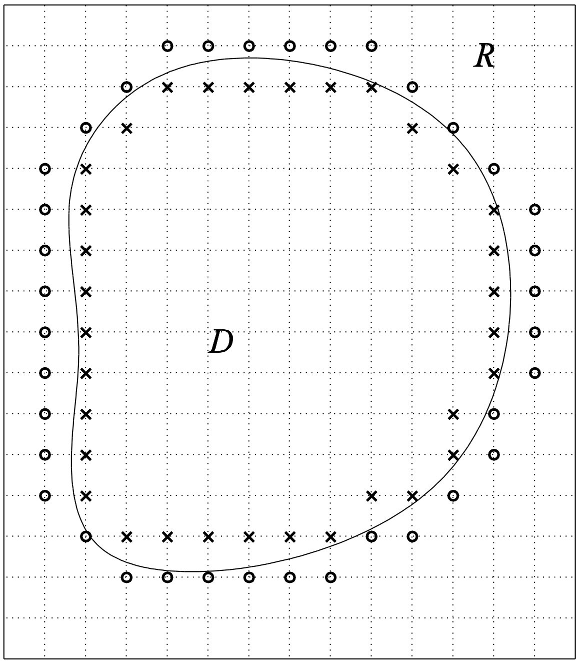
$$\dot{x}(t)^2[W_{xx}] + \dot{y}(t)^2[W_{yy}] + 2\dot{x}(t)\dot{y}(t)[W_{xy}] = 0. \qquad (2.7)$$

$$\dot{x}(t)\dot{y}(t)[W_{xx}] - \dot{x}(t)\dot{y}(t)[W_{yy}] + (\dot{y}^2(t) - \dot{x}^2(t))[W_{xy}] = 0. \qquad (2.8)$$

Equations (2.6), (2.7) and (2.8) determine $[W_{xx}]$, $[W_{yy}]$ and $[W_{xy}]$. We use similar methods in order to determine the discontinuities in third and higher order derivatives. See [6].

Now we show how to use these discontinuities to approximate L_hW.

Let $\overline{w}(p)$ denote the values of $W(p)$ at points p outside D, $w(p)$ denote the values of $W(p)$ at points p inside D, and let B be the set of irregular mesh points, that is the set of points which have one of their neighboring mesh points on the opposite side of D.



B consists of × *and*∘

Figure 1

We need to show how to approximate the discrete modified Helmholtz operator at points of *B*. Let

where $L_h^9 W = \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} W - a^2 W$

Suppose a point *p* is in *D*, and the point to the right, *pE,* is not. Let p^* be the point on the line between *p* and *pE* which intersects *D*, let $h1$ be the distance between *pE* and p^*, and let $h2 = h - h1$.

By manipulating the Taylor series centered at *p* and *pE* and evaluated at p^* we can derive the following expression for
$\overline{w}(p_E) - w(p)$. (For details see [6]).

$$\overline{w}(p^*) - w(p^*) - h_1[\overline{w}_x(p^*) - w_x(p^*)] + \frac{h_1^2}{2}[\overline{w}_{xx}(p^*) - w_{xx}(p^*)]$$
$$\frac{h_1^3}{6}[\overline{w}_{xxx}(p^*) - w_{xxx}(p^*)] + \frac{h_1^4}{24}[\overline{w}_{xxxx}(p^*) - w_{xxxx}(p^*)]$$
$$-hw_x(p) + \frac{h^2}{2}w_{xx}(p) - \frac{h^3}{6}w_{xxx}(p) + \frac{h^4}{24}w_{xxxx}(p) + O(h^5) \qquad (2.9)$$

Note that the first five terms depend on the discontinuities between *w* and *w*˜ and in their *x* derivatives across *D*. The other terms are the usual Taylor series terms.

Therefore, the right hand side of (2.9) is a sum of terms we can evaluate in terms of the discontinuities between w and $\tilde{w}$ and their x derivatives, and terms we would have if the boundary of D did not pass between p and pE.

Now let pW, pN, pS be the mesh points to the left of, above, and below

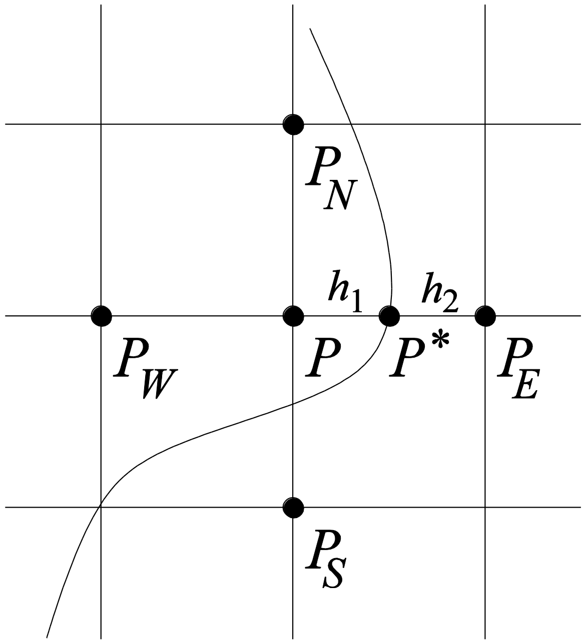


FIGURE 2

We get the same type of expressions for the differences between the value of W at p and at its other neighbors, that is $W(pW) - W(p)$, $W(pN) - W(p)$, $W(pS) - W(p)$, except that there will not be any boundary terms unless ∂D passes between p and that neighbor. Therefore, we can compute an approximation to $L^5\, W$, which is just the sum of the above four differences divided by h^2 minus $a^2 W(p)$

$$L_h^5 W = \frac{W(p_E) + W(p_S) + W(p_N) + W(p_W) - 4W(p)}{h^2} - a^2 W(p)$$

at all the irregular points when we know the derivatives of the boundary curve $(x(t), y(t))$ and the derivatives of f accurately enough.

More precisely, for mesh points $p \in B$ we define the mesh function $m(p)$ to be the value of the extra terms in $L^5_h\, W(p)$ due to the discontinuities in W and its derivatives. We define Wh to be the solution of the following equations:

$$L_h^5 W_h(p) = f(p) \; for\; p \in D - B$$
$$L_h^5 W_h(p) = f(p) + m(p) \; for\; p \in B \cap D$$
$$L_h^5 W_h(p) = m(p) \; for\; p \in B \cap D^c$$
$$L_h W_h(p) = 0 \; for\; p \in R - D \cap B$$

$Wh(p) = W(p)\; p \in \partial R$

How accurately we compute the terms $m(p)$ determines the accuracy of the solution we obtain after applying a fast solver. In particular, if the values of $m(p)$ are first order accurate then W^h will be a second order accurate approximation to W. For a proof see [3].

We can also compute an approximation to a fourth order accurate 9 point approximation to L. Then, by applying an operator which inverts the discrete operator L^9 we can obtain a fourth order accurate solution.

Specifically, in order to evaluate the surface and volume integrals we use the fourth order accurate approximation to the modified Helmholtz operator

$$\left(\Delta_h^9 - \left(a^2 + \frac{a^4 h^2}{12}\right)\right) W = f + \frac{h^2}{12}(f + \Delta f)$$

where

$$\Delta_h^9 W = \frac{1}{6h^2}\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} W$$

To see that this operator is fourth order accurate we note that for a smooth function ω

$$\Delta_h^9\, \omega = \Delta\, \omega + \frac{h^2}{12}\, \Delta\, (\Delta \omega) + O(h^4)$$

So, if $\Delta\omega = a^2\omega + f$, then

$$\Delta_h^9\, \omega = a^2 \omega + f + \frac{h^2}{12}\, \Delta(a^2\, \omega + f) + O(h^4)$$

$$= a^2\, \omega + f + \frac{h^2}{12}\, (a^2(a^2\omega + h) + \Delta g) + O(h^4)$$

$$= \left(a^2 + \frac{a^4 h^2}{12}\right)\omega + f + \frac{h^2}{12}(f + \Delta f) + O(h^4)$$

To compute an approximation to $\Delta_h^9 W$ we use the fact that the stencil is a linear combination of two second order accurate stencils, i.e.

$$\Delta_h^9 = \frac{2}{3}\Delta_h^5 + \frac{1}{3}\,\Delta_h^{\times}$$

where $\Delta_h^{\times} = \frac{1}{h^2}\begin{bmatrix} 1 & & 1 \\ & -4 & \\ 1 & & 1 \end{bmatrix}$

In order to approximate $\Delta^x{}_h W$ we need to know the discontinuities of W in u and v directions where $u = \frac{x+y}{\sqrt{2}}$ and $v = \frac{x-y}{\sqrt{2}}$ The discontinuities can, of course, be computed in terms of the discontinuities in the x and y directions by using the chain rule. For example, if p^* is on ∂D, then

$$[W_u(p^*)] = \frac{[W_x(p^*)]}{\sqrt{2}} + \frac{[W_y(p^*)]}{\sqrt{2}}.$$

Once we know these discontinuities we may, in the same way as before, use them to compute a higher order accurate approximation to $\Delta^x W$.
The accuracy of the resulting method is same as the accuracy to which the discrete Helmholtz operator is computed. That is, the errors in computed values of the potential Wh are bounded by a constant times the maximum truncation error, and if the potential function which is being computed is smooth, the errors are $O(h^2)$ or $O(h^4)$.

2.2 Evaluation of surface integral

We use essentially the same method we use to compute volume integrals to compute surface integrals of the form

$$U(x,y) = \frac{1}{2\pi a^2}\int \mu(s)\frac{\partial K_0\left(\frac{r}{a}\right)}{\partial n_s}\,ds$$

As when we evaluate volume integrals, the problem of evaluating an integral of this type reduces to evaluating LU in the regions inside and outside D, and evaluating the discontinuities in U and its derivatives across the boundary of D. We first note that LU vanishes in the two regions, i.e.

$(\Delta - a^2)U(x, y) = 0$ inside D, and $(\Delta - a^2)U(x, y) = 0$ outside D. (2.10)

It is also known that such a surface integral is continuous in the normal direction, and has a discontinuity equal to the density μ in the tangential direction. That is

$[U] = \mu$ (2.11)

and

It follows that

$[Un] = 0.$ (2.12)

To find the discontinuities in the second derivatives of U we note that by (2.10)

$[\Delta U] = a^2\,[U] = a^2\mu$

We also note that by (2.11) and (2.12)

$[U_{tt}] = \ddot{\mu}$
$[U_{nt}] = 0$
The above three equations determine $[U_{xx}]$, $[U_{xy}]$ and $[U_{yy}]$.
To find the discontinuities in the four third derivatives we use the follow- ing four equations:
$[(\Delta U)_n] = a^2 [U_n] = 0,$
$[(\Delta U)_t] = a^2 [U_t] = a^2 \dot{\mu}$
...
$[U_{ttt}] = \mu$

$[U_{ntt}] = 0.$
As before, once they are determined we use these discontinuities in the Derivatives of U to compute approximations to $L_h^5 U$ or $L_h^9 U$ at the irregular mesh points of R, and then apply a fast solver to obtain an approximation to U.

Boundary Conditions
When computing volume and surface integrals it is, of course, necessary to provide boundary conditions at the edge of the computational region R before inverting L_h. If we only require a particular solution of the modi- fied Helmholtz equation then it makes no difference which are prescribed, since the discontinuities in the derivatives of the integrals are the same, and therefore $L_h V$, is the same, independent of which fundamental solution of the modified Helmholtz equation is the kernel. The integral we obtain an approximation to with this method is the one associated with the same boundary conditions as the fast Helmholtz solver we use. For example, if we use a doubly periodic Helmholtz solver, then we obtain an approximation to the integral whose kernel is the doubly periodic Green's function for the modified Helmholtz equation on R. If we need an integral with a specific kernel, then we use the corresponding solver.
It is also possible to obtain approximations to integrals which satisfy free space boundary conditions. In that case we can use a method originally developed by Hockney [11] and later improved by James [12] where one cal- culates the boundary potential by finding a set of correction charges on the boundary of the embedding region, and then convolves them with a suitable
Green's function. This method, however, is more expensive than the others since it requires two applications of the operator L^{-1}.

3 Solution of modified Helmholtz and Heat equations
We use the volume and surface integrals discussed in the previous section to solve the modified Helmholtz and heat equations on smooth two dimensional regions.

3.1 Modified Helmholtz equation

As in [14], in order to solve the inhomogeneous equation $(\Delta - a^2)U(x, y) = f(x, y)$ when Dirichlet boundary conditions $U(x, y) = g(x, y)$ on ∂D are prescribed we use the representation of the solution as the sum of a volume integral and an integral of a double layer density function:

$$U(x,y) = W(x,y) + U_S(x,y)$$

$$= \frac{1}{2\pi a^2}\int f(x',y')K_0\left(\frac{r}{a}\right)dV + \frac{1}{2\pi a^2}\int \mu(s)\frac{\partial K_0\left(\frac{r}{a}\right)}{\partial n_s}ds \quad (3.1)$$

where K_0 is the zeroth order modified Bessel function of the second kind. We first evaluate W_h, the approximation to the volume integral W at mesh points of R, and then we interpolate its values onto the discretization points of ∂D using the known discontinuities in its derivatives. More precisely, we use the values of the mesh function W_h at points on both sides of ∂D and the discontinuities in the second and third derivatives of W to compute the extension of the inside function w to nearby points of R outside D. We then interpolate values of the extended function w onto ∂D using fourth order Lagrange interpolation.

Also, as in [14], in order to determine the density $\mu(s)$ we solve the integral equation

$$-\frac{\mu(t)}{2a^2} + \frac{1}{2\pi a^2}\int_{\partial D}\frac{\partial}{\partial n_s}K_0\left(\frac{r}{a}\right)\mu(s)ds = \omega(t) \quad (3.2)$$

where $\omega(t) = g(t) - w(t)$

As noted, in our experiments we discretized the above equation using a Nystrom method with the trapezoid rule as the quadrature formula, and we chose the discretization points to be equally spaced. By the Euler Maclaurin formula [7] the trapezoid rule is highly accurate for smooth functions on periodic regions, and we needed few mesh points to solve the integral equation very accurately on the simple test regions we used. For calculations on more general regions one should use multipole methods [14].

Once we have solved the integral equation we evaluate the surface integral U_S using the method described in the previous section.

3.2 Heat Equation

We also used the surface and volume integrals to solve the heat equation

$u_t(x, y, t) - \Delta u(x, y, t) = F(x, y, t)$, $(x, y) \in D$, $0 \le t \le t_f$

with Dirichlet boundary conditions

$u(x, y, t) = f(x, y, t)$ $(x, y) \in \partial D$, $t \in [0, t_f]$ and initial conditions

$u(x, y, 0) = u_0(x, y)$

prescribed where D is a two dimensional region with smooth boundary.

Instead of using an integral equation approach based on a fundamental solution of the heat equation we first discretized with respect to time. Specifically, as in [14] we used the implicit second order accurate extrapolated Gear method:

$$\Delta u^{N+1} - \frac{1}{\alpha^2} u^{N+1} = \frac{1}{3\alpha^2}(u^{N-1} - 4u^N + 4\delta t F^N - 2\delta t F^{N-1}) = B^N \tag{3.3}$$

where δt is the time step and $\alpha^2 = \frac{2}{3}\delta t$.

At the N th time step the solution u^N is represented as the sum of a volume integral W^N and a surface integral U^N :

$$u^N = W^N + U_S^N$$

where

$$\Delta W^N - \frac{1}{\alpha^2} W^N = B^N, \tag{3.4}$$

and

$$\Delta U_S^N - \frac{1}{\alpha^2} U_S^N = 0$$

$$U^N(x, y, t^N) = f(x, y, t^N) - W^N(x, y, t^N),$$

Thus, at each time step we must solve one integral equation and evaluate two integrals.

We can accurately approximate the right hand side of (3.4) and it's Laplacian. However, since we cannot approximate the normal derivative of the right hand side accurately enough, we cannot accurately approximate the discontinuities in the third and higher order derivatives of W^N. Therefore the method should only be second order accurate in space. In practice, how- ever, we have found the method to be somewhat more accurate.

Mesh refinement

When approximating integrals it is often not necessary to use as refined a mesh at all points of the computational region. That is, it may be desirable to evaluate volume and surface integrals on regions composed of subregions with different mesh widths. There are several recently developed accurate methods for solving elliptic differential equations on regions with locally refined grids that can be combined with our method. See, for example, [6].

However, it is also possible to perform calculations on different subregions independently. For example, suppose we want to approximate an integral U_S on a rectangular region R^S disjoint from D, which has a different (presumably coarser) mesh than the one on the region covering D.

Since $(\Delta - a^2)\, U_S$ is known in R^S, in order to approximate U_S we only need to evaluate it at mesh points on the boundary of R^S and then invert L_h on R^S. Although one could evaluate the integral at all boundary points by quadrature, we have instead done the following. We only evaluated it by quadrature at the four corner mesh points of R^S and at their neighboring points. At mesh points on the line between two consecutive corner mesh points we required the fourth derivative of W to be 0. Thus, finding the values at all points on the boundary of R^S only requires solving four 5 diagonal linear system of equations, one for each side of the rectangle R^S, and therefore the cost was $O(n)$.

Conclusion

We have presented rapid, fourth order accurate numerical methods for evaluating volume integrals whose kernels are a fundamental solution of the modified Helmholtz equation and surface integrals whose kernels are the nor- mal derivatives of such functions. We have also shown how these methods can be used as part of efficient numerical methods for solving both the modified Helmholz and heat equations on general two dimensional regions in space. In addition, we have indicated how these methods can be combined with locally refined meshes.

4 Numerical Experiments

In this section we report on results of numerical experiments in which we tested the accuracy of our methods for evaluating surface and volume integrals, and for solving the inhomogeneous modified Helmholtz equation and the heat equation.

In all experiments we embedded the irregular region D in a square of side 1.6 in the x direction and y directions. In the tables nx denotes the number of mesh points in the x direction, and ny denotes the number of mesh points in the y direction.

In our first set of numerical experiments we tested our method of evaluating volume integrals $W(x, y)$. The test region D was the unit disc, and we chose $W(x, y)$ so that

$$\Delta W - a^2 W = 4b$$

where $b = -\frac{K_1(ad)}{2da}$

When D is a disc of radius d the analytic value of the integral is known:

$W = b(r^2 - d^2) + K0\,(ad)$ for $r \leq d$

And

$K0\,(ar)$ for $r > d$

The results in Table 1 are for a =.45, d = 1, the errors are the maximum relative errors, and the numbers in the last column are the ratios of consecutive errors.

Table 1

nx	*ny*	rel. error	rate
17	17	0.279E-03	.
33	33	0.205E-04	13.61
65	65	0.145E-05	14.14
129	129	0.101E-06	14.36
257	257	0.711E-08	14.20

In Table 2 the errors are maximum absolute errors for $a = 5/6$, $d = 1$, and in Table 3 the errors are maximum relative errors for $a = 5$, $d = .5$.

Table 2			
nx	*ny*	abs. error	rate
17	17	0.181E-03	.
33	33	0.123E-04	14.72
65	65	0.822E-06	14.96
129	129	0.557E-07	14.76
257	257	0.371E-08	15.02

Table 3			
nx	*ny*	abs. error	rate
17	17	0.568E-02	.
33	33	0.465E-03	12.23
65	65	0.910E-04	5.10
129	129	0.665E-05	13.68
257	257	0.328E-06	20.27

These numbers confirm that our method of evaluating volume integrals method is essentially fourth order accurate.
In our next set of experiments we tested the accuracy of our method for solving the integral equation and evaluating the surface integral. We chose the boundary values
$g(x, y) = I1\,(ar)x/r$,
and let the region D be the disc of radius d. In this case both the density function and the values of the surface integral inside and outside D are known:
$\mu = (I1(ad) - cK1(ad))\, x/r$
where

$$c = \frac{I_0(ad) - \frac{1}{ad} I_1(ad)}{-K_0(ad) + \frac{1}{ad} K_1(ad)}$$

and
$US\,(r) = I1(ar)x/r$ for $r \leq d$
$US\,(r) = cK1(ar)x/r$ for $r > d$.
In the tables 4, 5 and 6 *ns* is the number of discretization points on ∂D, and the errors are the maximum relative errors. Results in Table 4 are for $a = .1$, $d = 1$, those in Table 5 are for $a = 10.0$, $d = 1$, and those in Table 6 are for $a = 1$, $d = .5$.

Table 4			
nx	*ns*	error	rate
17	50	0.328E-05	.
33	100	0.185E-06	17.73
65	200	0.502E-08	38.85
129	400	0.635E-10	80.10

Table 5			
nx	*ns*	rel. error	rate
17	50	0.110E+00	.
33	100	0.156E-01	7.05
65	200	0.120E-02	13.40
129	400	0.809E-04	14.83
257	800	0.532E-05	15.20

Table 6			
nx	*ny*	rel. error	rate
17	50	0.821E-04	.
33	100	0.568E-05	14.45
65	200	0.232E-06	24.48
129	400	0.169E-07	13.73

We next tested the accuracy of our method of solving the heat equation. We give the results of solving the homogeneous equation whose exact solution is $e^{-t}\cos(x)$ on the unit disc for $0 \leq t \leq 1$ using 32 time steps in Table 7. The errors are r.m.s. errors at $t = 1$.

Table 7			
nx	*ns*	error	rate
17	60	0.936E-03	.
33	120	0.266E-03	3.52
65	240	0.410E-04	6.51
129	480	0.563E-05	7.28

In Table 8 we give our results of solving the inhomogeneous heat equation whose solution is $\sin(t)\sin(x)\sin(y)$ on the unit disc for $0 \leq t \leq 1$. The numbers *nt* are the number of time steps and the errors are the rms. errors at $t = 1$.

Table 8				
nx	*ns*	nt	error	rate
17	50	12	0.966E-03	.
33	100	12	0.160E-03	6.04
65	200	12	0.168E-04	9.52
129	400	12	0.221E-05	7.60
17	50	24	0.198E-03	.
33	100	24	0.332E-04	5.95
65	200	24	0.430E-05	7.72
129	400	24	0.581E-06	7.41

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