Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 5 (2015), pp. 3529-3542

http://www.ripublication.com

On The Limit Cycles Of The Floquet Differential Systems

[1]Amar Makhlouf and [2]Zeyneb Bouderbala

[1 2] DEPARTMENT OF MATHEMATICS, LABORATORY OF APPLIED MATHEMATICS (LMA), UNIVERSITY OF ANNABA, ELHADJAR, 23 ANNABA, ALGERIA
E-mail address: makhloufamar@yahoo.fr
E-mail address: zeynebbouderbala@yahoo.fr

Abstract

In this paper we study the existence of periodic solutions of the Floquet differential systems

$\dot{x} = Ax(t) + b(t) + \varepsilon B(t)x(t),$ (1)

where $x(t)$ and $b(t)$ are column vectors of length n, A is a constant $(n \times n)$ matrix and B(t) is $(n \times n)$ matrix for $n = 2$ and 3. The components of b(t) and B(t) are T-periodic.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Floquet theory is concerned with the study of linear differential equations with periodic coefficients see [3, 4, 5, 10, 12, 19]. It is very important for the study of dynamical systems. These differential systems have been studied intensively and have many applications see for instance the papers [16, 6, 17, 18] and references quoted therein.

The linear first order differential system

$\dot{x} = Ax(t) + b(t).$ (2)

Where x(t) and b(t) are column vectors of lenghth n, A and B(t) are $(n \times n)$ matrix, B(t) and b(t) are periodic with period T, is called a Floquet differential system.

A limit cycle of the differential system (2) is a periodic orbit isolated in the set of all the periodic orbits of the same differential system. To obtain analytically limit cycles of a differential system is in general a very difficult problem, many times impossible. If the averaging theory can be applied to the differential system (1), then it reduces this difficult problem to find the zeros of a nonlinear function. It is known that in general the averaging theory for finding limit cycles does not provide all the limit cycles of the diffrential system.

The averaging theory (see for instance [15]) gives a quantitative relation between the solutions of some nonautonomous differential system and the solutions of its autonoumous averaged differential system. In particular, it allows to study the periodic orbits of a non-autonomous differential system in function of the periodic orbits of the averaged one, see for more details [1, 2, 8, 9, 15,19]. For more information about the averaging theory see section 2.

In the paper [7], the authors studied the limit cycles of the homogenous perturbed linear system

$\dot{x} = Ax(t) + \varepsilon(B(t)x(t) + b(t))$

Here, we consider the nonhomogenous perturbed linear system (1) for $n = 2$ and $n = 3$.

Our main result on the periodic solutions of the second-order non-autonomous differential system (1), where

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

$b(t) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$,

$B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix}$,

Is the following one.

Theorem1. We define

$$\mathcal{F}_1(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} (\cos(t)[b_{11}(t)C(t) + b_{12}(t)D(t)] + \sin(t)[b_{21}(t)C(t) + b_{22}(t)D(t)])dt,$$

$$\mathcal{F}_2(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} (-\sin(t)[b_{11}(t)C(t) + b_{12}(t)D(t)] + \cos(t)[b_{21}(t)C(t) + b_{22}(t)D(t)])dt,$$

where

$$C(t) = x_0\cos(t) - y_0\sin(t) + \int_0^t (-b_2(\tau)\sin(t-\tau) + b_1(\tau)\cos(t-\tau))d\tau,$$

$$D(t) = x_0\sin(t) + y_0\cos(t) + \int_0^t (b_1(\tau)\sin(t-\tau) + b_2(\tau)\cos(t-\tau))d\tau,$$

If

$$\int_0^{2\pi} (\cos(\tau)b_1(\tau) + \sin(\tau)b_2(\tau))d\tau = 0,$$

$$\int_0^{2\pi} (-sin(\tau)b_1(\tau) + \cos(t)b_2(\tau))d\tau = 0,$$

Then for every (x_0^*, y_0^*) solution of the system

$\mathcal{F}_k(x_0, y_0) = 0, \qquad k = 1,2,$

satisfying

$$det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(x_0, y_0)}\bigg|_{(x_0,y_0)=(x_0^*,y_0^*)}\right) \neq 0,$$

The differential system (1) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ tending to the solution

$$\begin{pmatrix} x(t,0) \\ y(t,0) \end{pmatrix} = \begin{pmatrix} x_0^* \cos(t) - y_0^* \sin(t) + \int_0^t (-b_2(\tau)\sin(t-\tau) + b_1(\tau)\cos(t-\tau))d\tau \\ x_0^* \sin(t) + y_0^* \cos(t) + \int_0^t (b_1(\tau)\sin(t-\tau) + b_2(\tau)\cos(t-\tau))d\tau \end{pmatrix} \quad (3)$$

Of the system (2) when $\varepsilon \to 0$.

Consider the case $n = 3$. Our main results on the periodic solutions of the third-order differential system (1) where

$$A(t) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \qquad B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) & b_{13}(t) \\ b_{21}(t) & b_{22}(t) & b_{23}(t) \\ b_{31}(t) & b_{32}(t) & b_{33}(t) \end{pmatrix}$$

and

$$b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \end{pmatrix}$$

are the following.

Theorem2. Consider the case $\lambda = 0$. We define

$$\mathcal{F}_1(x_0, y_0, z_0) = \frac{1}{2\pi}\int_0^{2\pi} (\cos(t)[b_{11}(t)C(t) + b_{12}(t)D(t) + b_{13}(t)E(t)] + \sin(t)[b_{21}(t)C(t) + b_{22}(t)D(t) + b_{23}(t)E(t)])dt,$$

$$\mathcal{F}_2(x_0, y_0, z_0) = \frac{1}{2\pi}\int_0^{2\pi} (-\sin(t)[b_{11}(t)C(t) + b_{12}(t)D(t) + b_{13}(t)E(t)] + \cos(t)[b_{21}(t)C(t) + b_{22}(t)D(t) + b_{23}(t)E(t)])dt,$$

$$\mathcal{F}_3(x_0, y_0, z_0) = \frac{1}{2\pi}\int_0^{2\pi} (b_{31}(t)C(t) + b_{32}(t)D(t) + b_{33}(t)E(t))dt,$$

where

$$C(t) = x_0\cos(t) - y_0\sin(t) + \int_0^t (-b_2(\tau)\sin(t-\tau) + b_1(\tau)\cos(t-\tau))d\tau,$$

$$D(t) = x_0\sin(t) + y_0\cos(t) + \int_0^t (b_1(\tau)\sin(t-\tau) + b_2(\tau)\cos(t-\tau))d\tau,$$

$$E(t) = z_0 + \int_0^t b_3(\tau)d\tau,$$

If we have

$$\int_0^{2\pi} (\cos(\tau)b_1(\tau) + \sin(\tau)\, b_2(\tau))d\tau = 0,$$

$$\int_0^{2\pi} (-\sin(\tau)b_1(\tau) + \cos(t)b_2(\tau))d\tau = 0,$$

$$\int_0^{2\pi} b_3(\tau)d\tau = 0,$$

then for every (x_0^*, y_0^*, z_0^*) solution of the system $\mathcal{F}_k(x_0, y_0, z_0) = 0,\ k = 1,2,3,$ satisfying

$$det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)}{\partial(x_0,y_0,z_0)}\right|_{(x_0,y_0,z_0)=(x_0^*,y_0^*,z_0^*)}\right)\neq 0,$$

The differential system (1) has a periodic solution $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ tending to the solution

$$\begin{pmatrix} x(t,0)\\ y(t,0)\\ z(t,0)\end{pmatrix}=\begin{pmatrix} x_0^*\cos(t)-y_0^*\sin(t)+\int_0^t(-b_2(\tau)\sin(t-\tau)+b_1(\tau)\cos(t-\tau))d\tau\\ x_0^*\sin(t)+y_0^*\cos(t)+\int_0^t(b_1(\tau)\sin(t-\tau)+b_2(\tau)\cos(t-\tau))d\tau\\ z_0^*+\int_0^t b_3(\tau)\,d\tau\end{pmatrix}\quad(4)$$

of the system (2) when $\varepsilon\to 0$.

Theorem3. Consider the case $\lambda\neq 0$. We define

$$\mathcal{F}_1(x_0,y_0)=\frac{1}{2\pi}\int_0^{2\pi}(\cos(t)[b_{11}(t)C(t)+b_{12}(t)D(t)+b_{13}(t)E(t)]+\sin(t)[b_{21}(t)C(t)+b_{22}(t)D(t)+b_{23}(t)E(t)])dt,$$

$$\mathcal{F}_2(x_0,y_0)=\frac{1}{2\pi}\int_0^{2\pi}(-\sin(t)[b_{11}(t)C(t)+b_{12}(t)D(t)+b_{13}(t)E(t)]+\cos(t)[b_{21}(t)C(t)+b_{22}(t)D(t)+b_{23}(t)E(t)])dt,$$

where

$$C(t)=x_0\cos(t)-y_0\sin(t)+\int_0^t(-b_2(\tau)\sin(t-\tau)+b_1(\tau)\cos(t-\tau))d\tau,$$

$$D(t)=x_0\sin(t)+y_0\cos(t)+\int_0^t(b_1(\tau)\sin(t-\tau)+b_2(\tau)\cos(t-\tau))d\tau,$$

$$E(t)=z_0+\int_0^t b_3(\tau)d\tau,$$

If we have

$$\int_0^{2\pi}(\cos(\tau)b_1(\tau)+\sin(\tau)\,b_2(\tau))d\tau=0,$$

$$\int_0^{2\pi}(-\sin(\tau)b_1(\tau)+\cos(t)b_2(\tau))d\tau=0,$$

$$z_0^*=\frac{e^{2\pi\lambda}}{1-e^{2\pi\lambda}}\int_0^{2\pi}e^{-\lambda\tau}b_3(\tau)d\tau,$$

then for every (x_0^*,y_0^*) solution of the system $\mathcal{F}_k(x_0,y_0)=0,\ k=1,2,$ satisfying

$$det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2)}{\partial(x_0,y_0)}\right|_{(x_0,y_0)=(x_0^*,y_0^*)}\right)\neq 0,$$

The differential system (1) has a periodic solution $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ tending to the solution

$$\begin{pmatrix} x(t,0) \\ y(t,0) \\ z(t,0) \end{pmatrix} = \begin{pmatrix} x_0^* \cos(t) - y_0^* \sin(t) + \int_0^t (-b_2(\tau)\sin(t-\tau) + b_1(\tau)\cos(t-\tau))d\tau \\ x_0^* \sin(t) + y_0^* \cos(t) + \int_0^t (b_1(\tau)\sin(t-\tau) + b_2(\tau)\cos(t-\tau))d\tau \\ \frac{e^{\lambda t}\int_0^{2\pi} e^{\lambda(2\pi-\tau)} b_3(\tau)d\tau}{1-e^{2\pi\lambda}} + \int_0^t e^{\lambda(t-\tau)} b_3(\tau)\, d\tau \end{pmatrix} \quad (5)$$

of the differential system (2) when $\varepsilon \to 0$.

Theorem 1, 2 and 3 are proved in section 3. Their proofs are based on the averaging theory for computing periodic solutions, see section 2. For others applications of the averaging theory to the study of periodic solutions, see [11] and [13].

Applications of Theorem 1, 2 and 3 are the following ones.

Corollary 1. Consider the Floquet differential system (1) in $\mathbb{R}^2$ with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$b(t) = \begin{pmatrix} \sin(2t) \\ \cos(t) \end{pmatrix}$$

$$B(t) = \begin{pmatrix} \cos(t) & \sin(2t) \\ \sin(t) & \cos(3t) \end{pmatrix}.$$

Then for $\varepsilon \neq 0$ sufficiently small the differential system (1) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ tending to the periodic solution $\left(-\frac{17}{6}\cos(t) - \frac{1}{6}\cos(2t) + \frac{1}{2}, -\frac{11}{6}\sin(t) + \frac{1}{6}\sin(2t)\right)$ of the differential system

$$\begin{cases} \dot{x} = -y + \sin(t) \\ \dot{y} = x + \cos(t) \end{cases}$$

When $\varepsilon \to 0$.

Corollary 1 is proved in section 4.

Corolloray 2. Consider the Floquet differential system (1) in $\mathbb{R}^3$ with

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$b(t) = \begin{pmatrix} sin(t) \\ cos(t) \\ sin(2t) \end{pmatrix}$$

$$B(t) = \begin{pmatrix} \cos(t) & 1 & \sin(t) \\ 2 & \sin(2t) & 1 \\ \sin(t) & 1 & \cos(3t) \end{pmatrix}$$

then for $\varepsilon \neq 0$ sufficiently small the differential system (1) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ tending to the periodic solution $\left(-\cos(t), 0, \frac{1}{4} - \frac{1}{2}\cos(2)\right)$ of the differential system

$$\begin{cases} \dot{x} = -y + \sin(t) \\ \dot{y} = x + \cos(t) \\ \dot{z} = -\sin(2t) \end{cases}$$

when $\varepsilon \to 0$.
Corollary 2 is proved in section 4.

Corolloray 3. Consider the Floquet differential system (1) in $\mathbb{R}^3$ with

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$b(t) = \begin{pmatrix} sin(t) \\ 0 \\ sin(t) \end{pmatrix}$$

$$B(t) = \begin{pmatrix} 0 & 0 & \sin(t) \\ \sin(2t) & \sin(t) & \cos(t) \\ \cos(t) & 0 & \cos(t) \end{pmatrix}$$

then for $\varepsilon \neq 0$ sufficiently small the differential system (1) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ tending to the periodic solution

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} -\dfrac{2\lambda(e^{2\pi\lambda} - 1)(\sin(t)\,\lambda - 2\cos(t))}{\pi(\lambda^4 + 5\lambda^2 + 4)} \\ \dfrac{(2e^{2\pi\lambda}\lambda^2 - 2\lambda^2)\cos(t)}{\pi(\lambda^4 + 5\lambda^2 + 4)} + \dfrac{(\pi\lambda^4 + 5\pi\lambda^2 + 4e^{2\pi\lambda}\lambda + 4\pi - 4\lambda)\sin(t)}{\pi(\lambda^4 + 5\lambda^2 + 4)} \\ \dfrac{-\lambda\cos(t) + \sin(t)}{\lambda^2 + 1} \end{pmatrix}$$

of the differential system

$$\begin{cases} \dot{x} = -y + \sin(t) \\ \dot{y} = x \\ \dot{z} = \lambda z + \sin(t) \end{cases}$$

when $\varepsilon \to 0$.
Corollary 3 is proved in section 4.

2. BASIC RESULTS ON AVERAGING THEORY

In this section we present the basic results from the averaging theory that we shall need for proving the main results of this paper.
We consider the problem of the bifurcation of T-periodic solutions from differential systems of the form

$$\mathbf{x}' = F_0(\mathrm{t}, \mathbf{x}) + \varepsilon F_1(\mathrm{t}, \mathbf{x}) + \varepsilon^2 F_2(\mathrm{t}, \mathbf{x}, \varepsilon) \qquad (6)$$

With $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are C^2 functions, T-periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is thet the unperturbed system

$$\mathrm{x}' = F_0(\mathrm{t}, \mathrm{x}), \qquad (7)$$

Has a submanifold of dimension n of periodic solutions. A solution of this problem is

given using the averaging theory.
Let x(t,z,0) be the solution of the system (7) such that x(0,z,0)=z. We write the linearization of the unperturbed system along the periodic solution x(t,z,0) as

$$y' = D_x F_0\big(t, x(t,z,0)\big)y. \qquad (8)$$

In what follows we denote by $M_z(t)$ some fundamental matrix of the linear differential system (8), and by $\xi\colon \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates ; i.e. $\xi(x_1, \dots, x_n) = (x_1, \dots, x_k)$.
We assume that there exists an open set V with $Cl(V) \subset \Omega$ such that for each $\boldsymbol{z} \in Cl(V)$, x(t, **z**, 0) is T-periodic. The set $Cl(V)$ is isochronous for the system (6) ; i.e it is a set formed only by periodic orbits, all of them having the same period.
Then, an answer to the problem of the bifurcation of T-periodic solutions from the periodic solutions x(t,z,0) contained in $Cl(V)$ is given in the following result.

Theorem 4. Let V be an open and bounded subset of $\mathbb{R}^k$, and let $\beta : Cl(V) \to \mathbb{R}^{n-k}$ be a C^2 function. We assume that

i. $\boldsymbol{Z} = \{\boldsymbol{z}_\alpha = (\alpha, \beta(\alpha)),\ \alpha \in Cl(V)\} \subset \Omega$ and that for each $\boldsymbol{z}_\alpha \in \boldsymbol{Z}$ the solution $x(t, \boldsymbol{z}_\alpha)$ of (5) is T-periodic ;
ii. For each $\boldsymbol{z}_\alpha \in \boldsymbol{Z}$ there is a fundamental matrix $M_{\boldsymbol{z}_\alpha}(t)$ of (6) such that the matrix $M_{\boldsymbol{z}_\alpha}^{-1}(0) - M_{\boldsymbol{z}_\alpha}^{-1}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.

We consider the function $\mathcal{F} : Cl(V) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi\left(\frac{1}{T}\int_0^T M_{\boldsymbol{z}_\alpha}^{-1}(t) F_1\big(t, x(t, \boldsymbol{z}_\alpha)\big)dt\right). \qquad (9)$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\big((d\mathcal{F}/d\alpha)\,(a)\big) \neq 0$, then there is a T-periodic solution $\varphi(t,\varepsilon)$ of system (4) such that $\varphi(0,\varepsilon) \to \boldsymbol{z}_\alpha$ as $\varepsilon \to 0$.

Theorem 5. (Perturbations of an isochronous set). We assume that there exists an open and bounded set V with $Cl(V) \subset \Omega$ such that for each $\boldsymbol{z} \in Cl(V)$, the solution $x(t, \boldsymbol{z}, 0)$ is T-periodic, then we consider the function $\mathcal{F} : Cl(V) \to \mathbb{R}^n$

$$\mathcal{F}(\alpha) = \int_0^T M_{\boldsymbol{z}}^{-1}(t) F_1\big(t, x(t, \boldsymbol{z}, 0)\big)dt. \qquad (10)$$

If there exists $\alpha \in V$ with $\mathcal{F}(\alpha) = 0$ and $\det\left(\left(\frac{d\mathcal{F}}{d\boldsymbol{z}}\right)(\alpha)\right) \neq 0$, then there exists a T-periodic solution $\varphi(t,\varepsilon)$ of system (6) such that $\varphi(0,\varepsilon) \to \alpha$ as $\varepsilon \to 0$.

3. PROOF OF THEOREM 1, 2 AND 3

Proof of theorem 1. We shall study the periodic solutions of system (7), i.e. the periodic solutions of the system (1) with $\varepsilon = 0$. The solution of the system (2) such that $\big(x(0), y(0)\big) = (x_0, y_0)$ is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \int_0^t e^{A(t-\tau)}\begin{pmatrix} b_1(\tau) \\ b_2(\tau) \end{pmatrix} d\tau \,,$$

where

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

so

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \cos(t) - y_0 \sin(t) + \int_0^t (b_1(\tau)\cos(t-\tau) - b_2(\tau)\sin(t-\tau))d\tau \\ x_0 \sin(t) + y_0 \cos(t) + \int_0^t (b_1(\tau)\sin(t-\tau) + b_2(\tau)\cos(t-\tau))d\tau \end{pmatrix} \quad (11)$$

These solutions are 2π-periodic if and only if

$(x(2\pi), y(2\pi)) = (x(0), y(0))$.

We obtain the following periodicity conditions

$$\int_0^{2\pi} (\cos(\tau) b_1(\tau) + \sin(\tau) b_2(\tau)) d\tau = 0,$$

$$\int_0^{2\pi} (-\sin(\tau) b_1(\tau) + \cos(t) b_2(\tau)) d\tau = 0,$$

We shall apply Theorem 4 to the differential system (1). It can be written as system (6) taking

x = $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, F_0(t , x) = $\begin{pmatrix} -y(t) + b_1(t) \\ x(t) + b_2(t) \end{pmatrix}$. F_1(t, x) = $\begin{pmatrix} b_{11}x(t) + b_{12}y(t) \\ b_{21}x(t) + b_{22}y(t) \end{pmatrix}$

The set of the periodic solutions (11) has dimension two, To look for the periodic solutions of our system (1) we must calculate the zeros $\mathbf{z} = (x_0, y_0)$ of the system $\mathcal{F}(\mathbf{z}) = 0$, where $\mathcal{F}(\mathbf{z})$ is given by (10). The fundamental matrix M(t) of the differential system (8) is

M (t) = M_z(t) = $\begin{pmatrix} \cos(t) & -sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$

Consequently all the assumptions of Theorem 4 are satisfied. Therefore we must study the zeros of the system $\mathcal{F}(\mathbf{z}) = 0$ of two equations with two unknowns, where $\mathcal{F}$ is given in the statement of theorem (4). More precisely, we have $\mathcal{F}(\mathbf{z}) = (\mathcal{F}_1(x_0, y_0), \mathcal{F}_2(x_0, y_0))$ where $\mathcal{F}_1(x_0, y_0)$, $\mathcal{F}_2(x_0, y_0)$ are defined as in the statement of Theorem 3. The zeros (x_0^*, y_0^*) of system

$$\begin{pmatrix} \mathcal{F}_1(x_0, y_0) \\ \mathcal{F}_2(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (12)$$

with respect to the variables x_0 and y_0 provide periodic orbits of system (1) with $\varepsilon \neq 0$ sufficiently small if they are simple, i.e. if

$$det \left(\left. \frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(x_0, y_0)} \right|_{(x_0, y_0) = (x_0^*, y_0^*)} \right) \neq 0.$$

For every simple zeros (x_0^*, y_0^*) of system (12), we obtain a 2π-periodic solution $(x(t, \varepsilon), y(t, \varepsilon))$ of the differential system (1) for $\varepsilon \neq 0$ sufficiently small which tends to the periodic solution (3) of the differential system

$$\begin{cases} \dot{x} = -y + b_1(t) \\ \dot{y} = x + b_2(t) \end{cases}$$

When $\varepsilon \to 0$. This completes the proof of Theorem 1.

Proof of theorem 2 and 3. The solution of the system (1) with $\varepsilon = 0$ such that $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{At} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \int_0^t e^{A(t-\tau)} \begin{pmatrix} b_1(\tau) \\ b_2(\tau) \\ b_3(\tau) \end{pmatrix} \mathrm{d}\tau.$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

we obtain

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 \cos(t) - y_0 \sin(t) + \int_0^t (-b_2(\tau)\sin(t-\tau) + b_1(\tau)\cos(t-\tau))d\tau \\ x_0 \sin(t) + y_0 \cos(t) + \int_0^t (b_1(\tau)\sin(t-\tau) + b_2(\tau)\cos(t-\tau))d\tau \\ e^{\lambda t} z_0 + \int_0^t e^{\lambda(t-\tau)} b_3(\tau)\, d\tau \end{pmatrix}$$

For studying the periodicity of this solution, we distinguish two cases: $\lambda = 0$ and $\lambda \neq 0$. these two cases will be studied respectively in Theorem 2 and Theorem 3.

Proof of theorem 2. We will apply the averaging theory described in section 2 for studying the limit cycles of system (2). More precisely we shall analyze which periodic orbits of system (2) can be continued to limit cycles of system (1) with $\varepsilon \neq 0$ sufficiently small. Now we define the elements of section 2 and of Theorem 4 corresponding to our differential system (1). We have that $\Omega = \mathbb{R}^3$ and $T = 2\pi$ we write (1) in the form (6) and we get that F_0, F_1 and F_2 are given by

$F_0(t,\mathrm{x}) = \mathrm{A}\, x + b(t)$,

$F_1(t,\mathrm{x}) = \mathrm{B}(\mathrm{t})\, x$,

$F_2(t,\mathrm{x}) = 0$.

We shall study the periodic solutions of system (7) in our case, i.e. the periodic solutions of the system (1) with $\varepsilon = 0$.

These solutions such that $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ are

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 \cos(t) - y_0 \sin(t) + \int_0^t (-b_2(\tau)\sin(t-\tau) + b_1(\tau)\cos(t-\tau))d\tau \\ x_0 \sin(t) + y_0 \cos(t) + \int_0^t (b_1(\tau)\sin(t-\tau) + b_2(\tau)\cos(t-\tau))d\tau \\ z_0 + \int_0^t b_3(\tau)\, d\tau \end{pmatrix} \quad (13)$$

These solutions are 2π-periodic if and only if

$(x(2\pi), y(2\pi), z(2\pi)) = (x(0), y(0), z(0))$

We obtain the following periodicity conditions

$$\int_0^{2\pi} (\cos(\tau) b_1(\tau) + \sin(\tau)\, b_2(\tau))d\tau = 0,$$
$$\int_0^{2\pi} (-\sin(\tau) b_1(\tau) + \cos(t) b_2(\tau))d\tau = 0,$$
$$\int_0^{2\pi} b_3(\tau)d\tau = 0,$$

The set of periodic solutions (13) has dimension 3. To look for the periodic solutions of our system (1) we must calculate the zeros $\boldsymbol{z} = (x_0, y_0, z_0)$ of the system $\mathcal{F}(\boldsymbol{z}) = 0$, where $\mathcal{F}(\boldsymbol{z})$ is given by (10). The fundamental matrix M(t) of the differential system (8) is

$$M(t) = M_z(t) = e^{At} = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore we must study the zeros of the system $\mathcal{F}(\boldsymbol{z}) = 0$ where $\mathcal{F}(\mathbf{z}) = (\mathcal{F}_1(x_0, y_0, z_0), \mathcal{F}_2(x_0, y_0, z_0), \mathcal{F}_3(x_0, y_0, z_0))$ are given in the statement of Theorem 2. More precisely, the zeros (x_0^*, y_0^*, z_0^*) of the system

$$\begin{pmatrix} \mathcal{F}_1(x_0, y_0, z_0) \\ \mathcal{F}_2(x_0, y_0, z_0) \\ \mathcal{F}_3(x_0, y_0, z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (14)$$

With respect to the variables x_0, y_0 and z_0 provide periodic orbits of system (1) with $\varepsilon \neq 0$ sufficiently small if they are simple, i.e. if

$$det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)}{\partial(x_0, y_0, z_0)}\Big|_{(x_0, y_0, z_0) = (x_0^*, y_0^*, z_0^*)} \right) \neq 0.$$

For every simple zero (x_0^*, y_0^*, z_0^*) of system (14),we obtain a 2π-periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ of the differential system (1) for $\varepsilon \neq 0$ sufficiently small such that $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ tends to the periodic solution (4) of (1) when $\varepsilon \to 0$. Note that this solution is periodic of period 2π. This completes the proof of Theorem 2.

Proof of theorem 3. We shall study the periodic solutions of system (7), i.e. the periodic solution of the system (2) with $\varepsilon = 0$. The solution of the system (2) with $\lambda \neq 0$ such that $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 \cos(t) - y_0 \sin(t) + \int_0^t (-b_2(\tau)\sin(t-\tau) + b_1(\tau)\cos(t-\tau))d\tau \\ x_0 \sin(t) + y_0 \cos(t) + \int_0^t (b_1(\tau)\sin(t-\tau) + b_2(\tau)\cos(t-\tau))d\tau \\ e^{\lambda t} z_0 + \int_0^t e^{\lambda(t-\tau)} b_3(\tau)\, d\tau \end{pmatrix}$$

these solutions are 2π-periodic if and only if

$(x(2\pi), y(2\pi), z(2\pi)) = (x(0), y(0), z(0))$.

We obtain the following periodicity conditions

$$\int_0^{2\pi} (\cos(\tau) b_1(\tau) + \sin(\tau)\, b_2(\tau))d\tau = 0,$$
$$\int_0^{2\pi} (-\sin(\tau) b_1(\tau) + \cos(t) b_2(\tau))d\tau = 0,$$

$$z_0^* = \frac{e^{2\pi\lambda}}{1-e^{2\pi\lambda}}\int_0^{2\pi} e^{-\lambda\tau} b_3(\tau)d\tau.$$

We shall apply Theorem 4 to the differential system (1) with $\lambda \neq 0$. It can be written as system (6) taking

$$\mathrm{x}=\begin{pmatrix} x(t)\\ y(t)\\ z(t)\end{pmatrix},\ \mathrm{F_0(t\,,x)}=\begin{pmatrix} -y(t)+b_1(t)\\ x(t)+\ b_2(t)\\ \lambda z(t)+b_3(t)\end{pmatrix},$$

$$\mathrm{F_1(t{,}x)}=\begin{pmatrix} b_{11}(t)x(t)+\ b_{12}(t)y(t)+b_{13}(t)z(t)\\ b_{21}(t)x(t)+\ b_{22}(t)y(t)+b_{23}(t)z(t)\\ b_{31}(t)x(t)+\ b_{32}(t)y(t)+b_{33}(t)z(t)\end{pmatrix}$$

The set of the periodic solutions becomes

$$\begin{pmatrix} x(t)\\ y(t)\\ z(t)\end{pmatrix}=\begin{pmatrix} x_0\cos(t)-y_0\sin(t)+\int_0^t(-b_2(\tau)\sin(t-\tau)+b_1(\tau)\cos(t-\tau))d\tau\\ x_0\sin(t)+y_0\cos(t)+\int_0^t(b_1(\tau)\sin(t-\tau)+b_2(\tau)\cos(t-\tau))d\tau\\ \frac{e^{\lambda t}\int_0^{2\pi}e^{\lambda(2\pi-\tau)}b_3(\tau)d\tau}{1-e^{2\pi\lambda}}+\int_0^t e^{\lambda(t-\tau)}b_3(\tau)\,d\tau\end{pmatrix}.$$

This set of the periodic solutions has dimension two. To look for the periodic solutions of our system (1) we must calculate the zeros $\boldsymbol{z}=(x_0,y_0)$ of the system $\mathcal{F}(\boldsymbol{z})=0$, where $\mathcal{F}(\boldsymbol{z})$ is given by (10). The fundamental matrix M(t) of the differential system (8) is therefore

$$M(t)=M_z(t)=\begin{pmatrix}\cos(t) & -\sin(t) & 0\\ \sin(t) & \cos(t) & 0\\ 0 & 0 & e^{\lambda t}\end{pmatrix}.$$

It verifies

$$M^{-1}(0)-M^{-1}(2\pi)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1-e^{-2\pi\lambda}\end{pmatrix}.$$

Consequently all the assumptions of Theorem 4 are satisfied. Therefore we must study the zeros of the system $\mathcal{F}(\boldsymbol{z})=0$ of two equations with two unknowns. More precisely, we have $\mathcal{F}(\mathbf{z})=\left(\mathcal{F}_1(x_0,y_0),\mathcal{F}_2(x_0,y_0)\right)$ where $\mathcal{F}_1(x_0,y_0)$, $\mathcal{F}_2(x_0,y_0)$ are defined as in the statement of Theorem 3. The zeros $(x_0^*\,,y_0^*)$ of system

$$\begin{pmatrix}\mathcal{F}_1(x_0,y_0)\\ \mathcal{F}_2(x_0,y_0)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\qquad(15)$$

with respect to the variable x_0 and y_0 provide periodic orbits of system (1) with $\varepsilon\neq 0$ sufficiently small if they are simple, i.e. if

$$det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2)}{\partial(x_0,y_0)}\right|_{(x_0,y_0)=(x_0^*,y_0^*)}\right)\neq 0.$$

For every simple zeros $(x_0^*,\ y_0^*)$ of system (15), we obtain a 2π-periodic solution $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ of differential system (1) for $\varepsilon\neq 0$ sufficiently small wich tends to the periodic solution (5) of the differential system

$$\dot{x}=-y(t)+b_1(t)$$
$$\dot{y}=x+\mathrm{b}_2\,(t)$$
$$\dot{z}=\lambda z(t)+b_3(t)$$

when $\varepsilon \to 0$. This completes the proof of Theorem 2.

4. PROOF OF COROLLARIES 1, 2 AND 3

Proof of corollary 1. We consider the differential system (1) with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} \cos(t) & \sin(2t) \\ \sin(t) & \cos(3t) \end{pmatrix}$$

and

$$b(t) = \begin{pmatrix} \sin(2t) \\ \cos(t) \end{pmatrix}.$$

Computing the functions $\mathcal{F}_1, \mathcal{F}_2$ of theorem 1, we obtain

$$\mathcal{F}_1(x_0, y_0) = \frac{1}{4}x_0 + \frac{5}{8}$$

$$\mathcal{F}_2(x_0, y_0) = \frac{1}{4}y_0$$

The system $\mathcal{F}_1 = \mathcal{F}_2 = 0$ has the solution $(x_0^*, \ y_0^*) = \left(-\frac{5}{2}, 0\right)$. Since the Jocabian

$$det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(x_0, y_0)}\bigg|_{(x_0, y_0) = \left(-\frac{5}{2}, 0\right)}\right) = \frac{1}{16}$$

Then this differential system has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ tending to the solution given in the statement of the corollary (1) when $\varepsilon \to 0$.

Proof of corollary 2. We must apply Theorem 2 with *b(t)* and *B*(t) are defined in the statement of corollary (2). We can verify easily the periodicity conditions

$$\int_0^{2\pi} [\cos(\tau) sin(\tau) + \sin(\tau) cos(\tau)] d\tau = 0,$$

$$\int_0^{2\pi} [-\sin^2(\tau) + \cos^2(\tau)] d\tau = 0,$$

$$\int_0^{2\pi} sin(\tau) d\tau = 0.$$

After computations of the functions $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ of Theorem 2, we obtain

$$\mathcal{F}_1 = \frac{1.178097245}{\pi} y_0$$

$$\mathcal{F}_2 = \frac{1}{2\pi}(-0{,}7853981634 + 3.141592654\, y_0 - 0.7853981634 x_0)$$

$$\mathcal{F}_3 = \frac{1}{2\pi}(3.926990817 + 3.141592654 z_0 + 3.141592654\, x_0$$

The system $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = 0$ has one real solution given by

$$(x_0^*, y_0^*, z_0^*) = \left(-1, 0, -\frac{1}{4}\right)$$

Since the Jacobian

$$det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)}{\partial(x_0, y_0, z_0)}\bigg|_{(x_0, y_0, z_0) = \left(-1, 0, -\frac{1}{4}\right)}\right) = \frac{0.7267096096}{\pi^3}.$$

Using Theorem 2 we obtain the periodic solution given in the statement of the corollary (2).

***Proof of corollary* 3.** We must apply Theorem 3 with *b(t)* and *B(t)* are defined in the statement of corollary**.** We can verify easily the periodicity conditions

$$\int_0^{2\pi} [2\cos(\tau) sin(\tau)] d\tau = 0,$$

$$\int_0^{2\pi} [-\sin^2(\tau) + \cos^2(\tau)] d\tau = 0.$$

After computations of the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ of Theorem 3, we obtain

$$\mathcal{F}_1 = \frac{(\lambda^4\pi + 5\lambda^2\pi + 4\pi)x_0 - 4e^{2\pi\lambda} + 4\lambda}{4\pi(\lambda^4 + 5\lambda^2 + 4)}$$

$$\mathcal{F}_2 = \frac{(\lambda^4\pi + 5\lambda^2\pi + 4\pi)y_0 - 2e^{2\pi\lambda}\lambda^2 + 2\lambda^2}{4\pi(\lambda^4 + 5\lambda^2 + 4)}$$

The system $\mathcal{F}_1 = \mathcal{F}_2 = 0$ has only one real solution given by

$$(x_0^*, y_0^*) = \left(\frac{4\lambda(e^{2\pi\lambda} - 1)}{\pi(\lambda^4 + 5\lambda^2 + 4)}, \frac{2\lambda^2(e^{2\pi\lambda} - 1)}{\pi(\lambda^4 + 5\lambda^2 + 4)}\right).$$

Since the Jacobian

$$det\left(\left.\frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(x_0, y_0)}\right|_{(x_0, y_0) = \left(\frac{4\lambda(e^{2\pi\lambda} - 1)}{\pi(\lambda^4 + 5\lambda^2 + 4)}, \frac{2\lambda^2(e^{2\pi\lambda} - 1)}{\pi(\lambda^4 + 5\lambda^2 + 4)}\right)}\right) = \frac{-1}{16}$$

using Theorem 3 we obtain the periodic solution given in the statement of the corollary (3).

REFERENCES

[1] V. I. Arnold, V. V, Kozlonov and A. I. Neishtadt, Mathematical Aspects of classical and Celestial Mechanics, second printing, Springer-Verlag, Berlin, 1997.

[2] A. Buicà, J. P. Françoise and J. LIibre, Periodic solutions of nonlinear periodic differential systems with a small parameter, Communication on Pure and Applied Analysis, 6 (2007), 103-111.

[3] C. Chicone, Ordinary Differential Equations with Applications, Springer-Verlag, New York, 1999.

[4] M. Farkas, Periodic Motions. Vol. 104, Applied Mathematical sciences. New York (NY): Springer-Verlag; 1994.

[5] G. Floquet, Sur les équations différentielles linéaires à coefficients périodiques [On linear differential equations with periodic coefficients]. Ann. Ecole Norm. Sup. 1883;12:47-49.

[6] C. A. Klausmeier, Floquet theory: a useful tool for understanding nonequilibrium dynamics, Theo Ecol (2008) 1:153-161. Doi 10. 1007/s12080-008-0016-2.

[7] J. Llibre and A. Rodrigues. On the limit cycles of the Floquet differential equation, discrete and continuous dynamical systems series B volume 19, number 4, June 2014.

[8] J. Llibre, M. A. Teixeira and J. Torregrosa, Limit Cycles Bifurcating from a K-dimensional isochronous set center contained in $\mathbb{R}^n$ with k≤n, Math. Phys. Anal. Geom, 10 (2007), 237-249.

[9] P. Lochak and C. Meunier, multiphase Averaging for Classical Systems, Appl. Math. Sciences 72, Springer-Verlag, New York, 1988.

[10] W. Magnus, S. Winkler. Hill's Equation, Dover-Phoenix Editions, ISBN 0-486-49565-5.

[11] I. G. Malkin, Some Problems of the theory of nonlinear oscillations, Gosudarstv. Izdat. Tehn-Teor. Lit. Moscow, 1956 (in Russian).

[12] L. Perko, Differential equations and dynamical systems. 3rd ed. Vol. 7, Texts in applied New York (NY): Springer-Verlag; 2001.

[13] M. Roseau, Vibrations non linéaires et théorie de la stabilité, Springer Tracts in Natural Philosophy, Vol. 8, Springer, New York, 1985.

[14] J. A. Sanders and F. Verhulst, Averaging Methods in Nonlinear Dynamical Systems, Applied Mathematical Sciences 59, Springer, 1985.

[15] J. A. Sanders and F. Verhulst, and J. Murdock, Averaging Methods in Nonlinear Dynamical Systems, Second edition, Applied Mathematical Sci., 59, Springer-Verlag, New York, 2007.

[16] J. P. Tian and J. Wang, Some results in Floquet theory, with application to periodic epidemic models. Applicable Analysis, 2014 http://dx.doi.org/10.1080/00036811.2014.918606.

[17] W. F. Trench, On nonautonomous linear systems of differential and difference equations with R-symetric coefficient matrices, Linear Algebra Appl., 431 (2009), 2109-2117.

[18] W. F. Trench, Asymptotic preconditioning of linear homogenous systems of differential equations, Linear Algebra Appl., 431 (2011), 1631-1637.

[19] F. Verhulst, Nonlinear Differential Equations and Dynamical Systems, Universitext, Springer, New York, 1996.