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On Generalized Order Statistics From Marshall-Olkin Log-Logistic Distribution

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Abstract

In this small note we have established some new explicit expressions and recurrence relations for single and product moments of generalized order statistics for the Marshall-Olkin log-logistic distribution. These explicit expressions and recurrence relations can be used to develop the relationship for moments of ordinary order statistics, record statistics and other ordered random variable techniques. Further, a characterization result of this distribution has been considered on using the conditional moment of the generalized order statistics.

Keywords Generalized order statistics, order statistics, record values, Marshall-Olkin log-logistic distribution, single and product moment, recurrence relations and characterization.

Mathematics Subject Classification 2010 62G30, 62E10, 62E15

1. Introduction

The concept of generalized order statistics (gos) was introduced by Kamps [1]. Several models of ordered random variables such as order statistics, record values, sequential order statistics, progressive type II censored order statistics and Pfeifer's record values can be discussed as special cases of the gos. Suppose $X(1,n,m,k),\dots,X(n,n,m,k)$, $(k\geq 1,\ m$ is a real number), are $n\ gos$ from an absolutely continuous distribution function (df) $F(x)$ with probability density function (pdf) $f(x)$, if their joint pdf is of the form

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$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^m f(x_i)\right)(1-F(x_n))^{k-1} f(x_n) \tag{1.1}$$

on the cone $F^{-1}(0) < x_1 \le \ldots \le x_n < F^{-1}(1)$, where $\gamma_j = k+(n-j)(m+1) > 0$ for all j, $1 \le j \le n$, k is a positive integer and $m \ge -1$.

If $m = 0$ and $k = 1$, then this model reduces to the ordinary $r-$th order statistic and (1.1) will be the joint *pdf* of n order statistics $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ from *df* $F(x)$. If $k = 1$ and $m = -1$, then (1.1) will be the joint *pdf* of the first n record values of the identically and independently distributed (iid) random variables with *df* $F(x)$ and corresponding *pdf* $f(x)$.

In view of (1.1), the marginal *pdf* of the $r-$th *gos*, $X(r,n,m,k)$, $1 \le r \le n$, is

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x)), \tag{1.2}$$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is

$$f_{X(r,n,m,k)\,X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x)\, g_m^{r-1}(F(x))$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(y),\ x < y, \tag{1.3}$$

where

$$\bar{F}(x) = 1 - F(x),\ C_{r-1} = \prod_{i=1}^{r}\gamma_i$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \ne -1 \\ -\ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0),\ x \in [0,1).$$

The general theory for distributions of sequential and *gos* have been developed by Cramer and Kamps (2003) without imposing and condition on the parameters. Some characterization of probability distributions via regression of *gos* has been obtained by Bieniek and Szynal (2003) and Cramer *et al.* (2004). Some recurrence for moments of *gos* for different distributions have been considered by Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000), Habibullah and Ahsanullah (2000), Pawlas and Szynal (2001), Raqab (2001), Kamps and Cramer (2001), Ahmad and Fawzy (2003), Saran and Pandey (2003), Al-Hussaini, *et al.* (2005), Ahmad (2007, 2008) and Kumar (2013) among others.

In this paper, we are concerned with some new expressions for the exact moments and recurrence relations of *gos* from Marshall-Olkin log-logistic distribution. Section 2, gives a exact moments and recurrence relations for single moments *gos* which can be applied to obtained relations for order statistics and record values. In Section 3, some exact moments and recurrence relations for product moments *gos* are derived and we show that result for order statistics and record values are deduced as special cases. Finally in last section, of the paper we present a characterization of this distribution is obtained by using conditional moment of *gos* we also use the explicit expression to calculate the mean and variance of order statistics and record values.

A random variable X is said to have Marshall-Olkin log-logistic distribution if its *pdf* is of the form

$$f(x)=\frac{\beta\lambda\alpha^{\beta}x^{\beta-1}}{(x^{\beta}+\lambda\alpha^{\beta})^{2}},\ x>0,\ \alpha,\ \beta,\lambda>0 \tag{1.4}$$

and the corresponding survival function is

$$\bar{F}(x)=\frac{\lambda\alpha^{\beta}}{x^{\beta}+\lambda\alpha^{\beta}},\ x>0,\ \alpha,\ \beta,\lambda>0. \tag{1.5}$$

Note that for Marshall-Olkin log-logistic distribution defined in (1.5)

$$\beta\bar{F}(x)=x\left(1+\frac{\lambda\alpha^{\beta}}{x^{\beta}}\right)f(x). \tag{1.6}$$

Log-logistic distribution is consider as a special case of Marshall-Olkin log-logistic distribution when $\lambda=1$.

2 Relations for single moments

For the Marshall-Olkin log-logistic distribution as given in (1.5), the $j-$th moments of $X(r,n,m,k)$ is given as

$$E[X^{j}(r,n,m,k)]=\int_0^{\infty}x^{j}f_{X(r,n,m,k)}(x)\,dx$$

$$=\frac{C_{r-1}}{(r-1)!}\int_0^{\infty}x^{j}[\bar{F}(x)]^{\gamma_r-1}f(x)\,g_m^{r-1}(F(x))\,dx \tag{2.1}$$

Further, on using the binomial expansion, we can rewrite (2.1) as

$$E[X^{j}(r,n,m,k)]=\frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{u=0}^{r-1}(-1)^{u}\binom{r-1}{u}$$

$$\times\int_0^{\infty}x^{j}[\bar{F}(x)]^{\gamma_{r-u}-1}f(x)\,dx. \tag{2.2}$$

Now letting $t=[\bar{F}(x)]$ in (2.2), we get

$$E[X^{j}(r,n,m,k)]=\frac{(\alpha\lambda^{1/\beta})^{j}C_{r-1}}{(r-1)!(m+1)^{r}}\sum_{u=0}^{r-1}\sum_{p=0}^{\infty}(-1)^{u+p}\binom{r-1}{u}\frac{\Gamma\left(\frac{j}{\beta}+1\right)}{p!\Gamma\left(\frac{j}{\beta}+1-p\right)}$$

$$\times B\left(\frac{k}{m+1}+(n-r)+u+\frac{p-(j/\beta)}{m+1},\ 1\right). \tag{2.3}$$

Since

$$\sum_{a=0}^{b}(-1)^a\binom{b}{a}B(a+k,\ c)=B(k,\ c+b) \tag{2.4}$$

where $B(a,\, b)$ is the complete beta function. Therefore,

$$E[X^j(r,n,m,k)]=\frac{(\alpha\lambda^{1/\beta})^j C_{r-1}}{(m+1)^r}\sum_{p=0}^{\infty}(-1)^p\frac{\Gamma\left(\frac{j}{\beta}+1\right)}{p!\Gamma\left(\frac{j}{\beta}+1-p\right)}$$

$$\times\frac{\Gamma\left(\frac{k+(n-r)(m+1)+p-(j/\beta)}{m+1}\right)}{\Gamma\left(\frac{k+n(m+1)+p-(j/\beta)}{m+1}\right)} \tag{2.5}$$

$$=(\alpha\lambda^{1/\beta})^j\sum_{p=0}^{\infty}(-1)^p\frac{\Gamma\left(\frac{j}{\beta}+1\right)}{p!\Gamma\left(\frac{j}{\beta}+1-p\right)}\frac{1}{\prod_{i=1}^{r}\left(1+\frac{p-(j/\beta)}{\gamma_i}\right)}. \tag{2.6}$$

Special cases

i) Putting $m=0$, $k=1$ in (2.5), we get moments of order statistics from Marshall-Olkin log-logistic distribution as;

$$E[X_{r:n}^j]=\frac{(\alpha\lambda^{1/\beta})^j n!}{(n-r)!}\sum_{p=0}^{\infty}(-1)^p\frac{\Gamma\left(\frac{j}{\beta}+1\right)}{p!\Gamma\left(\frac{j}{\beta}+1-p\right)}\frac{\Gamma[n-r+1+p-(j/\beta)]}{\Gamma[n+1+p-(j/\beta)]},$$

for $r=1$

$$E[X_{1:n}^j]=n(\alpha\lambda^{1/\beta})^j\sum_{p=0}^{\infty}(-1)^p\frac{\Gamma\left(\frac{j}{\beta}+1\right)}{p!\Gamma\left(\frac{j}{\beta}+1-p\right)[n+p-(j/\beta)]}$$

ii) Putting $m=-1$ in (2.6), to get moments of $k-$th record value from Marshall-Olkin log-logistic distribution as;

$$E[(Z_r^{(k)})^j]=(\alpha\lambda^{1/\beta})^j\sum_{p=0}^{\infty}(-1)^p\frac{\Gamma\left(\frac{j}{\beta}+1\right)}{p!\Gamma\left(\frac{j}{\beta}+1-p\right)}\frac{1}{\left(1+\frac{p-(j/\beta)}{k}\right)^r}.$$

A recurrence relation for single moment of *gos* from *df* (1.5) is obtained in the following theorem.

Theorem 2.1 For the distribution as given in (1.5) and for $2 \le r \le n$, $n \ge 2$ and $k = 1,2,\ldots$

$$\left(1-\frac{j}{\beta\gamma_r}\right)E[X^j(r,n,m,k)]$$

$$= E[X^j(r-1,n,m,k)] + \frac{j\lambda\alpha^\beta}{\beta\gamma_r}E[X^{j-\beta}(r,n,m,k)]. \quad (2.7)$$

Proof: From (1.2), we have

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r-1} f(x)\, g_m^{r-1}(F(x))\,dx.$$

Integrating by parts treating $[\bar{F}(x)]^{\gamma_r-1} f(x)$ for integration and rest of the integrand for differentiation, we get

$$E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)]$$

$$+\frac{j\,C_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}[\bar{F}(x)]^{\gamma_r}\, g_m^{r-1}(F(x))\,dx. \quad (2.8)$$

Upon substituting for $\bar{F}(x)$ from (1.6) in (2.8) and simplifying the resulting expression, we derive the relation in Theorem 2.1.

Remark 2.1 Putting $m=0$, $k=1$, in (2.7), we obtain a recurrence relation for single moment of order statistics of the Marshall-Olkin log-logistic distribution in the form

$$\left(1-\frac{j}{\beta(n-r+1)}\right)E[X_{r:n}^j] = E[X_{r-1:n}^j] + \frac{j\lambda\alpha^\beta}{\beta(n-r+1)}E[X_{r:n}^{j-\beta}].$$

Remark 2.2 Setting $m=-1$ and $k \ge 1$ in Theorem 2.1, we get a recurrence relation for single moment of $k-$th record values from Marshall-Olkin log-logistic distribution in the form

$$\left(1-\frac{j}{\beta k}\right)E[(Z_r^{(k)})^j] = E[(Z_{r-1}^{(k)})^j] + \frac{j\lambda\alpha^\beta}{\beta k}E[(Z_r^{(k)})^{j-\beta}].$$

Table 2.1: Mean of order statistics

n	r	$\beta=3$			
		$\lambda=1$		$\lambda=2$	
		$\alpha=1$	$\alpha=2$	$\alpha=1$	$\alpha=2$
1	1	1.209402	2.418805	1.523752	3.047503
2	1	0.806537	1.613075	1.016173	2.032347
	2	1.612267	3.224535	2.031330	4.062660
3	1	0.672382	1.344764	0.847148	1.694296
	2	1.074848	2.149696	1.354224	2.708448

	3	1.880977	3.761954	2.369883	4.739765
4	1	0.597939	1.195878	0.753356	1.506711
	2	0.895712	1.791423	1.128526	2.257052
	3	1.253985	2.507970	1.579922	3.159844
	4	2.089975	4.179949	2.633203	5.266406
5	1	0.548383	1.096766	0.690919	1.381838
	2	0.796195	1.592389	1.003142	2.006285
	3	1.044987	2.089975	1.316602	2.633203
	4	1.393316	2.786633	1.755469	3.510937
	5	2.264139	4.528278	2.852637	5.705273
n	r	$\beta=4$			
		$\lambda=1$		$\lambda=2$	
		$\alpha=1$	$\alpha=2$	$\alpha=1$	$\alpha=2$
1	1	1.110995	2.221990	1.321203	2.642407
2	1	0.833588	1.667175	0.991308	1.982616
	2	1.388403	2.776805	1.651098	3.302197
3	1	0.729728	1.459456	0.867798	1.735596
	2	1.041306	2.082612	1.238329	2.476657
	3	1.561951	3.123902	1.857483	3.714967
4	1	0.669255	1.338509	0.795882	1.591765
	2	0.911149	1.822298	1.083545	2.167089
	3	1.171463	2.342927	1.393113	2.786225
	4	1.692114	3.384227	2.012274	4.024547
5	1	0.627772	1.255543	0.746550	1.493101
	2	0.835228	1.670456	0.993259	1.986518
	3	1.025031	2.050061	1.218974	2.437947
	4	1.269085	2.538170	1.509205	3.018410
	5	1.797871	3.595741	2.138041	4.276081
n	r	$\beta=5$			
		$\lambda=1$		$\lambda=2$	
		$\alpha=1$	$\alpha=2$	$\alpha=1$	$\alpha=2$
1	1	1.069270	2.138540	1.228269	2.456537
2	1	0.855787	1.711573	0.983041	1.966082
	2	1.282753	2.565506	1.473497	2.946993
3	1	0.770577	1.541153	0.885160	1.770320
	2	1.026207	2.052414	1.178802	2.357605
	3	1.411026	2.822053	1.620844	3.241687
4	1	0.719571	1.439142	0.826570	1.653141
	2	0.923593	1.847186	1.060930	2.121859
	3	1.128821	2.257642	1.296675	2.593350
	4	1.505095	3.010189	1.728900	3.457800
5	1	0.683968	1.367936	0.785673	1.571346
	2	0.862029	1.724058	0.990211	1.980422

	3	1.015939	2.031878	1.167008	2.334015
	4	1.204076	2.408152	1.383120	2.766240
	5	1.580349	3.160699	1.815345	3.630690

Table 2.2: Variance of order statistics

n	r	$\beta = 3$			
		$\lambda = 1$		$\lambda = 2$	
		$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
1	1	0.955776	3.823098	1.517197	6.068793
2	1	0.155691	0.622760	0.247143	0.988571
	2	1.431260	5.725036	2.271981	9.087924
3	1	0.085413	0.341655	0.135586	0.542345
	2	0.188258	0.753030	0.298839	1.195356
	3	1.836146	7.344583	2.914698	11.65880
4	1	0.060582	0.242330	0.096169	0.384677
	2	0.093405	0.373623	0.148272	0.593086
	3	0.218929	0.875713	0.347527	1.390110
	4	2.200496	8.801992	3.493072	13.97229
5	1	0.047754	0.191017	0.075806	0.303222
	2	0.062734	0.250939	0.099585	0.398338
	3	0.102273	0.409089	0.162346	0.649392
	4	0.248168	0.992666	0.393939	1.575761
	5	2.536915	10.14766	4.027102	16.1084
n	r	$\beta = 4$			
		$\lambda = 1$		$\lambda = 2$	
		$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
1	1	0.336575	1.346298	0.475989	1.903950
2	1	0.090705	0.362824	0.128278	0.513112
	2	0.428532	1.714135	0.606038	2.424147
3	1	0.056809	0.227234	0.080339	0.321357
	2	0.093781	0.375124	0.132625	0.530505
	3	0.505552	2.022208	0.714960	2.859831
4	1	0.043321	0.173287	0.061267	0.245062
	2	0.053384	0.213537	0.075496	0.301990
	3	0.100296	0.401179	0.141837	0.567356
	4	0.572867	2.291476	0.810156	3.240634
5	1	0.035856	0.143427	0.050709	0.202835
	2	0.038711	0.154845	0.054747	0.218984
	3	0.053777	0.215115	0.076053	0.304219
	4	0.107481	0.429927	0.152002	0.608007
	5	0.633292	2.533173	0.895610	3.582445
n	r	$\beta = 5$			

		$\lambda = 1$		$\lambda = 2$	
		$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
1	1	0.178118	0.712473	0.235027	0.940115
2	1	0.060712	0.242849	0.080109	0.320438
	2	0.204375	0.817499	0.269672	1.078693
3	1	0.040885	0.163543	0.053949	0.215797
	2	0.056799	0.227199	0.074948	0.299789
	3	0.228801	0.915196	0.301902	1.207612
4	1	0.032475	0.129899	0.042851	0.171401
	2	0.034900	0.139600	0.046051	0.184205
	3	0.057640	0.230561	0.076056	0.304224
	4	0.250456	1.001832	0.330480	1.321919
5	1	0.027631	0.110523	0.036459	0.145836
	2	0.026445	0.105781	0.034895	0.139581
	3	0.033370	0.133478	0.044030	0.176125
	4	0.059661	0.238646	0.078724	0.314896
	5	0.269841	1.079358	0.356055	1.424220

Table 2.3: Mean of record statistics

n	$\beta = 3$			
	$\lambda = 1$		$\lambda = 2$	
	$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
1	1.209402	2.418805	1.523752	3.047503
2	2.105240	4.210480	2.652436	5.304872
3	3.295122	6.590244	4.151593	8.303187
4	5.016621	10.033242	6.320546	12.641092
5	7.566886	15.133772	9.533679	19.067358
6	11.37473	22.74947	14.33127	28.66253
7	17.07648	34.15297	21.51502	43.03004
8	25.62326	51.24652	32.28329	64.56657
9	38.43998	76.87997	48.43134	96.86269
10	57.66302	115.32604	72.65085	145.30170
n	$\beta = 4$			
	$\lambda = 1$		$\lambda = 2$	
	$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
1	1.110995	2.221990	1.321203	2.642407
2	1.675684	3.351369	1.992736	3.985472
3	2.317494	4.634989	2.755981	5.511962
4	3.131800	6.263601	3.724359	7.448718
5	4.198067	8.396134	4.992371	9.984743
6	5.609711	11.219422	6.671108	13.342216
7	7.486482	14.972965	8.902978	17.805956

8	9.985849	19.971698	11.875243	23.750485
9	13.31666	26.63332	15.83627	31.67253
10	17.75680	35.51359	21.11651	42.23301
n	$\beta=5$			
	$\lambda=1$		$\lambda=2$	
	$\alpha=1$	$\alpha=2$	$\alpha=1$	$\alpha=2$
1	1.069270	2.138540	1.228269	2.456537
2	1.483495	2.966990	1.704088	3.408177
3	1.913709	3.827419	2.198275	4.396550
4	2.420716	4.841433	2.780673	5.561346
5	3.040628	6.081257	3.492765	6.985529
6	3.808631	7.617263	4.374969	8.749937
7	4.765041	9.530081	5.473594	10.947189
8	5.958627	11.917254	6.844665	13.689330
9	7.449564	14.899129	8.557302	17.114605
10	9.312663	18.625326	10.697440	21.394881

Table 2.2: Variance of record statistics

n	$\beta=3$			
	$\lambda=1$		$\lambda=2$	
	$\alpha=1$	$\alpha=2$	$\alpha=1$	$\alpha=2$
1	0.955776	3.823098	1.517197	6.068793
2	4.164934	16.65974	6.611421	26.44569
3	15.85024	63.40095	25.1607	100.6427
4	55.61831	222.4733	88.2886	353.1543
5	185.5822	742.3291	294.5935	1178.374
6	599.4961	2397.984	951.6406	3806.563
7	1895.305	7581.218	3008.608	12034.43
8	5904.382	23617.53	9372.621	37490.49
9	18205.32	72821.27	28899.15	115596.6
10	55723.94	222895.8	88456.24	353825.0
n	$\beta=4$			
	$\lambda=1$		$\lambda=2$	
	$\alpha=1$	$\alpha=2$	$\alpha=1$	$\alpha=2$
1	0.336575	1.346298	0.475989	1.903950
2	0.940466	3.761857	1.330017	5.320068
3	2.470848	9.883383	3.494304	13.97721
4	6.089299	24.35718	8.61157	34.44627
5	14.30896	57.23584	20.23593	80.94370
6	32.48669	129.9468	45.94313	183.7725
7	71.92309	287.6924	101.7146	406.8585
8	156.2632	625.0529	220.9896	883.9583

9	334.6536	1338.614	473.2715	1893.086
10	708.6871	2834.75	1002.235	4008.943
n	$\beta=5$			
	$\lambda=1$		$\lambda=2$	
	$\alpha=1$	$\alpha=2$	$\alpha=1$	$\alpha=2$
1	0.178118	0.712473	0.235027	0.940115
2	0.393501	1.574003	0.519228	2.076907
3	0.860701	3.442797	1.135699	4.542796
4	1.791993	7.167965	2.364547	9.458187
5	3.575371	14.30147	4.717723	18.87092
6	6.90354	27.61412	9.109256	36.43706
7	13.00177	52.00712	17.15596	68.62379
8	24.0228	96.09124	31.69829	126.7931
9	43.72719	174.9087	57.69836	230.7934
10	78.65241	314.6095	103.7825	415.1298

3 Relations for product moments

For Marshall-Olkin log-logistic distribution, the product moments of $X(r,n,m,k)$ and $X(s,n,m,k)$ is given as $E[X^i(r,n,m,k)\ X^j(s,n,m,k)]=\int_0^\infty\int_x^\infty x^i y^j f_{X(r,n,m,k)\ X(s,n,m,k)}(x,y)\,dxdy$.

On using (1.3) and binomial expansion, we have

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)]$$

$$=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$

$$\times\int_0^\infty x^i[\bar{F}(x)]^{(s-r+u-v)(m+1)-1}f(x)I(x)\,dx,\ \ x>y, \tag{3.1}$$

where

$$I(x)=\int_x^\infty y^j[\bar{F}(y)]^{\gamma_{s-v}-1}f(y)\,dy. \tag{3.2}$$

By setting $t=[\bar{F}(y)]$ in (3.2), we obtain

$$I(x)=(\alpha\lambda^{1/\beta})^j\sum_{p=0}^{\infty}(-1)^p\frac{\Gamma\left(\frac{j}{\beta}+1\right)}{p!\Gamma\left(\frac{j}{\beta}+1-p\right)}\frac{[\bar{F}(x)]^{\gamma_{s-v}+p-(j/\beta)}}{[\gamma_{s-v}+p-(j/\beta)]}.$$

On substituting the above expression of $I(x)$ in (3.1), we find that

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)]$$

$$=\frac{(\alpha\lambda^{1/\beta})^j C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{p=0}^{\infty}(-1)^{u+v+p}\binom{r-1}{u}\binom{s-r-1}{v}$$

$$\times \frac{\Gamma\left(\frac{j}{\beta}+1\right)}{p!\,\Gamma\left(\frac{j}{\beta}+1-p\right)[\gamma_{s-v}+p-(j/\beta)]} \int_0^\infty x^i [\bar{F}(x)]^{\gamma_{r-u}+p-(j\beta)-1} f(x)\,dx\,. \quad (3.3)$$

Again by setting $z=[\bar{F}(x)]$ in (3.3) and simplifying the resulting equation, we get

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)]$$

$$= \frac{(\alpha\lambda^{1/\beta})^{i+j} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^s} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q} \frac{\Gamma\left(\frac{j}{\beta}+1\right)\Gamma\left(\frac{i}{\beta}+1\right)}{p!q!\Gamma\left(\frac{j}{\beta}+1-p\right)\Gamma\left(\frac{i}{\beta}+1-q\right)}$$

$$\times \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} B\left(\frac{k}{m+1}+(n-r)+u+\frac{p+q-\{(i+j)/\beta\}}{m+1},1\right)$$

$$\times \sum_{v=0}^{s-r-1}(-1)^v \binom{s-r-1}{v} B\left(\frac{k}{m+1}+(n-s)+v+\frac{p-(j/\beta)}{m+1},\ 1\right). \quad (3.4)$$

On using relation (2.4) in (3.4), we get

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)]$$

$$= \frac{(\alpha\lambda^{1/\beta})^{i+j} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^s} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q} \frac{\Gamma\left(\frac{j}{\beta}+1\right)\Gamma\left(\frac{i}{\beta}+1\right)}{p!q!\Gamma\left(\frac{j}{\beta}+1-p\right)\Gamma\left(\frac{i}{\beta}+1-q\right)}$$

$$\times B\left(\frac{k}{m+1}+n-r+\frac{p+q-\{(i+j)/\beta\}}{m+1},\ r\right)$$

$$\times B\left(\frac{k}{m+1}+n-s+\frac{p-(j/\beta)}{m+1},\ s-r\right). \quad (3.5)$$

Which after simplification yields

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^{p+q}\Gamma\left(\frac{j}{\beta}+1\right)\Gamma\left(\frac{i}{\beta}+1\right)}{p!q!\Gamma\left(\frac{j}{\beta}+1-p\right)\Gamma\left(\frac{i}{\beta}+1-q\right)}$$

$$\times \frac{(\alpha\lambda^{1/\beta})^{i+j}}{\prod_{a=1}^{r}\left(1+\frac{p+q-\{(i-j)/\beta\}}{\gamma_a}\right)\prod_{b=r+1}^{s}\left(1+\frac{p-(j/\beta)}{\gamma_a}\right)}. \quad (3.6)$$

Special cases

i) By setting $m=0$, $k=1$ in (3.5), we get product moments of order statistics from Marshall-Olkin log-logistic distribution as

$$E[X_{r:n}^{i}\, X_{s:n}^{j}]=\frac{(\alpha\lambda^{1/\beta})^{i+j}\, n!}{(n-s)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q}\,\frac{\Gamma\left(\frac{j}{\beta}+1\right)\Gamma\left(\frac{i}{\beta}+1\right)}{p!q!\Gamma\left(\frac{j}{\beta}+1-p\right)\Gamma\left(\frac{i}{\beta}+1-q\right)}$$

$$\times\frac{\Gamma[n-r+1+p+q-\{i+j)/\beta\}]\;\Gamma[n-s+1+p-(j/\beta)]}{\Gamma[n+1+p+q-\{(i-j)/\beta\}]\;\Gamma[n-r+1+p-(j/\beta)]}.$$

ii) If $m=-1$ in (3.6), we get product moments of $k-$th record values from Marshall-Olkin log-logistic distribution as

$$E[(Z_r^{(k)})^i(Z_s^{(k)})^j]=(\alpha\lambda^{1/\beta})^{i+j}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q}\,\frac{\Gamma\left(\frac{j}{\beta}+1\right)\Gamma\left(\frac{i}{\beta}+1\right)}{p!q!\Gamma\left(\frac{j}{\beta}+1-p\right)\Gamma\left(\frac{i}{\beta}+1-q\right)}$$

$$\times\frac{1}{\left(1+\frac{p+q-\{(i-j)/\beta\}}{k}\right)^{r}\left(1+\frac{p-(j/\beta)}{k}\right)^{s-r}}.$$

Making use of (1.6), we can derive recurrence relations for product moments of *gos* from (1.5).

Theorem 3.1 For the distribution as given in (1.5) and for $1\le r<s\le n$, $n\ge 2$ and $k=1,2,\ldots$

$$\left(1-\frac{j}{\beta\gamma_s}\right)E[X^i(r,n,m,k)X^j(s,n,m,k)]=E[X^i(r,n,m,k)X^j(s-1,n,m,k)]$$

$$+\frac{j\lambda\alpha^{\beta}}{\beta\gamma_s}E[X^i(r,n,m,k)X^{j-\beta}(s,n,m,k)]. \qquad (3.7)$$

Proof: From (1.3), we have

$$E[X^i(r,n,m,k)\,X^j(s,n,m,k)]=\frac{C_{s-1}}{(r-1)!(s-r-1)!}$$

$$\times\int_0^{\infty}x^i[\overline{F}(x)]^m f(x)\,g_m^{r-1}(F(x))\,G(x)\,dx, \quad (3.8)$$

where

$$G(x)=\int_x^{\infty}y^j[h_m(F(y))-h_m(F(x))]^{s-r-1}[\overline{F}(y)]^{\gamma_s-1}f(y)\,dy.$$

Solving the integral in $G(x)$ by parts and substituting the resulting expression in (3.8), we get

$$E[X^i(r,n,m,k)\,X^j(s,n,m,k)] = E[X^i(r,n,m,k)\,X^j(s-1,n,m,k)]$$
$$+\frac{j\,C_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^{j-1}[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))$$
$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s}\,dy\,dx, \tag{3.9}$$

Now on using (1.6) in (3.9), we have the result given in (3.7).

Remark 3.1 Putting $m=0$, $k=1$ in (3.7), we obtain recurrence relations for product moments of order statistics of the Marshall-Olkin log-logistic distribution in the form

$$\left(1-\frac{j}{\beta(n-s+1)}\right)E[X_{r:n}^i X_{s:n}^j] = E[X_{r:n}^i X_{s-1:n}^j] + \frac{j\lambda\alpha^\beta}{\beta(n-s+1)}E[X_{r:n}^i X_{s:n}^{j-\beta}].$$

Remark 3.2 Setting $m=-1$ and $k\ge 1$, in (3.7), we obtain the recurrence relations for product moments of $k-$th record values from Marshall-Olkin log-logistic distribution in the form

$$\left(1-\frac{j}{\beta k}\right)E[(X_{U(r)}^{(k)})^i(X_{U(s)}^{(k)})^j] = E[(X_{U(r)}^{(k)})^i(X_{U(s-1)}^{(k)})^j]$$
$$+\frac{j\lambda\alpha^\beta}{\beta k}E[(X_{U(r)}^{(k)})^i(X_{U(s)}^{(k)})^{j-\beta}].$$

Remark 3.3 At $j=0$ in (3.6), we have

$$E[X^i(r,n,m,k)] = \sum_{q=0}^{\infty}\frac{(-1)^q\,\Gamma\left(\frac{i}{\beta}+1\right)}{q!\,\Gamma\left(\frac{i}{\beta}+1-q\right)}\frac{(\alpha\lambda^{1/\beta})^i}{\prod_{a=1}^{r}\left(1+\frac{q-(i/\beta)}{\gamma_a}\right)},$$

as obtained in (2.6).

Remark 3.4 At $i=0$, Theorem 3.1 reduces to Theorem 2.1.

4 Characterization

Let $X(r,n,m,k)$, $r=1,2,\ldots,n$ be *gos* from a continuous population with *df* $F(x)$ and *pdf* $f(x)$, then the conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k)=x$, $1\le r<s\le n$, in view of (1.3) and (1.2), is

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y\,|\,x) = \frac{C_{s-1}}{(s-r-1)!\,C_{r-1}}$$
$$\times\frac{[(h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}}{[\bar{F}(x)]^{\gamma_{r+1}}}f(y),\quad x<y. \tag{4.1}$$

Theorem 4.1: Let X be a non negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$, then

$$E[X(s,n,m,k)\mid X(l,n,m,k)=x]=\sum_{p=0}^{\infty}\frac{(-\lambda\alpha^{\beta})^{p}\,\Gamma\left(\frac{1}{\beta}+1\right)}{p!\Gamma\left(\frac{1}{\beta}+1-p\right)(x^{\beta}+\lambda\alpha^{\beta})^{p-(1/\beta)}}$$

$$\times\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+p-(1/\beta)}\right), \tag{4.2}$$

if and only if

$$\bar{F}(x)=\frac{\lambda\alpha^{\beta}}{x^{\beta}+\lambda\alpha^{\beta}},\ x>0,\ \alpha,\ \beta,\lambda>0.$$

Proof: From (4.1), we have

$$E[X(s,n,m,k)\mid X(r,n,m,k)=x]=\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times\int_{x}^{\infty}y\left[1-\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{m+1}\right]^{s-r-1}\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_s-1}\frac{f(y)}{\bar{F}(x)}dy\,. \tag{4.3}$$

By setting $u=\frac{\bar{F}(y)}{\bar{F}(x)}=\left(\frac{x^{\beta}+\lambda\alpha^{\beta}}{y^{\beta}+\lambda\alpha^{\beta}}\right)$ from (1.5) in (4.3), we obtain

$$E[X(s,n,m,k)\mid X(r,n,m,k)=x]=\frac{(x^{\beta}+\lambda\alpha^{\beta})^{i/\beta}C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}$$

$$\times\sum_{p=0}^{\infty}\frac{(-1)^{p}\,\Gamma\left(\frac{1}{\beta}+1\right)}{p!\Gamma\left(\frac{1}{\beta}+1-p\right)}\left(\frac{\lambda\alpha^{\beta}}{x^{\beta}+\lambda\alpha^{\beta}}\right)^{p}\int_0^1 u^{\gamma_s+p-(1/\beta)-1}(1-u^{m+1})^{s-r-1}du\,. \tag{4.4}$$

Again by setting $t=u^{m+1}$ in (4.4) and simplifying the resulting expression, we derive the relation given in (4.2).

To prove sufficient part, we have from (4.1) and (4.2)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_{x}^{\infty}y[(\bar{F}(x))^{m+1}-(\bar{F}(y))^{m+1}]^{s-r-1}$$

$$\times[\bar{F}(y)]^{\gamma_s-1}f(y)dy=[\bar{F}(x)]^{\gamma_{r+1}}H_r(x), \tag{4.7}$$

where

$$H_r(x) = \sum_{p=0}^{\infty} \frac{(-\lambda\alpha^{\beta})^p \, \Gamma\left(\frac{1}{\beta}+1\right)}{p!\Gamma\left(\frac{1}{\beta}+1-p\right)(x^{\beta}+\lambda\alpha^{\beta})^{p-(1/\beta)}} \prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}+p-(1/\beta)}\right).$$

Differentiating (4.7) both sides with respect to x, we get

$$-\frac{C_{s-1}[\bar{F}(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}} \int_x^{\infty} y[(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-2}[\bar{F}(y)]^{\gamma_s - 1} f(y)dy$$

$$= H_r'(x)[\bar{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1}H_r(x)[\bar{F}(x)]^{\gamma_{r+1}-1} f(x)$$

or

$$-\gamma_{r+1}H_{r+1}(x)[F(x)]^{\gamma_{r+2}+m} f(x)$$

$$= H_r'(x)[F(x)]^{\gamma_{r+1}} - \gamma_{r+1}H_r(x)[F(x)]^{\gamma_{r+1}-1} f(x),$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{H_r'(x)}{\gamma_{r+1}[H_{r+1}(x) - H_r(x)]} = \frac{\beta}{x\left(1+\frac{\lambda\alpha^{\beta}}{x}\right)}$$

which proves that

$$\bar{F}(x) = \frac{\lambda\alpha^{\beta}}{x^{\beta}+\lambda\alpha^{\beta}}, \; x > 0, \; \alpha, \; \beta, \lambda > 0.$$

5. Application

Order statistics, record values and their moments are widely used in statistical inference. In this Section we suggest some application based on moments discussed in Section 2.

i) **Estimation:** The moments of order statistics and record values given in Section 2 can be used to obtain the best linear unbiased estimate of the parameters of the Marshall-Olkin log-logistic distribution.

ii) **Characterization:** The Marshall-Olkin log-logic distribution given in (1.5) can be characterization by using conditional expectation of generalized order statistics as Theorem 4.1.

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