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A note on λ-zeta function

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Abstract

In this paper, we introduce some properties of degenerate Bernoulli polynomials and construct degenerate Hurwitz's type zeta function which interpolates degenerate Bernoulli polynomials at negative integers.

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1. Introduction

It is well known that *Bernoulli polynomials* are given by the generating function to be

$$F(t,x)=\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\ \text{(see [1-10])}. \tag{1.1}$$

When $x=0$, $B_n=B_n(0)$ are called *Bernoulli numbers*.

For $s\in\mathbb{C}\setminus\{1\}$, the *Hurwitz's type zeta function* is defined as

$$\zeta(s,x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^s},\ (x\neq 0,-1,-2,\cdots). \tag{1.2}$$

Note that $\zeta(s,x)$ is analytic function in complex s-plane except for $s=1$.

Let $s=n\in\mathbb{N}$. Then we see that

$$\zeta(1-n,x)=-\frac{B_n(x)}{n},\ \text{(see [11])}. \tag{1.3}$$

For $s\in\mathbb{C}$ with $R(s)>0$, the *gamma function* is defined by improper integral to be

$$\int_0^{\infty}t^{s-1}e^{-t}dt=\Gamma(s),\ \text{(see [11])}. \tag{1.4}$$

From (1.4), we have

$$\Gamma(s+1)=s\Gamma(s)\text{ and }\Gamma(n+1)=n!,\ (n\in\mathbb{N}). \tag{1.5}$$

By (1.4), we get

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^{\infty}F(-t,x)t^{s-2}dt&=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{te^{-xt}}{1-e^{-t}}t^{s-2}dt\\&=\frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\int_0^{\infty}e^{-(n+x)t}t^{s-1}dt\\&=\frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{1}{(n+x)^s}\int_0^{\infty}e^{-y}y^{s-1}dy\\&=\sum_{n=0}^{\infty}\frac{1}{(n+x)^s}=\zeta(s,x).\end{aligned} \tag{1.6}$$

L. Carlitz introduced degenerate Bernoulli polynomials which are defined by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_n(x|\lambda)\frac{t^n}{n!},\ \text{(see [3])}. \tag{1.7}$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(0|\lambda)$ are called *degenerate Bernoulli numbers*.

Recently, Kim introduced *degenerate gamma function* which is given by

$$\Gamma(s|\lambda) = \int_0^\infty (1+\lambda t)^{-\frac{1}{\lambda}} t^{s-1} dt, \text{ (cf. [8])}, \tag{1.8}$$

where $s \in \mathbb{C}$ with $R(s) > 0$ and $\lambda \in (0, 1)$. From (1.8), we note that

$$\Gamma(s+1|\lambda) = s\Gamma\left(s \,\middle|\, \frac{\lambda}{1-\lambda}\right)\left(\frac{1}{1-\lambda}\right)^{s+1}, \tag{1.9}$$

and

$$\Gamma(n|\lambda) = \frac{(n-1)!}{(1-\lambda)(1-2\lambda)\cdots(1-n\lambda)}, \tag{1.10}$$

where $\lambda \in \left(0, \frac{1}{n}\right)$ and $n \in \mathbb{N}$.

In this paper, we study degenerate Hurwitz's type zeta function which interpolates at negative integers.

2. A note on λ-zeta function

In this section, we assume that $\lambda \in (0, 1)$. Let

$$F(t, x|\lambda) = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^\infty \beta_n(x|\lambda)\frac{t^n}{n!}. \tag{2.1}$$

Thus, by (2.1), we get

$$\sum_{n=0}^\infty \beta_n(1|\lambda)\frac{t^n}{n!} - \sum_{n=0}^\infty \beta_n(\lambda)\frac{t^n}{n!} = t. \tag{2.2}$$

From (2.2), we note that

$$\beta_0(\lambda) = 1, \ \beta_n(1|\lambda) - \beta_n(\lambda) = \delta_{1,n},$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Now, we observe that

$$\begin{aligned}\sum_{n=0}^\infty \beta_n(x|\lambda)\frac{t^n}{n!} &= \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \left(\sum_{n=0}^\infty \beta_n(\lambda)\frac{t^n}{n!}\right)\left(\sum_{m=0}^\infty (x|\lambda)_m \frac{t^m}{m!}\right) \\ &= \sum_{n=0}^\infty \left(\sum_{m=0}^n \binom{n}{m}(x|\lambda)_m \beta_m(\lambda)\right)\frac{t^n}{n!},\end{aligned} \tag{2.3}$$

where $(x|\lambda)_m = x(x-\lambda)(x-2\lambda)\cdots(x-(m-1)\lambda)$ and $(x|\lambda)_0 = 1$. By comparing the coefficients on the both sides of (2.3), we have

$$\beta_n(x|\lambda) = \sum_{m=0}^{n} \binom{n}{m} (x|\lambda)_m \beta_m(\lambda), \ (n \geq 0). \tag{2.4}$$

In the viewpoint of (1.6), we define *degenerate Hurwitz's type zeta function* which is given by

$$\zeta(s, x|\lambda) = \frac{1}{\Gamma(s|\lambda)} \int_0^\infty F(-t, x|-\lambda) t^{s-2} dt, \tag{2.5}$$

where $s \in \mathbb{C} \setminus \{1\}$ and $x \neq 0, -1, -2, \cdots$.

From (2.1) and (2.5), we have

$$\begin{aligned}
\zeta(s, x|\lambda) =& \frac{1}{\Gamma(s|\lambda)} \int_0^\infty F(-t, x|-\lambda) dt \\
=& \frac{1}{\Gamma(s|\lambda)} \int_0^\infty \frac{t}{1-(1+\lambda t)^{-\frac{1}{\lambda}}} (1+\lambda t)^{-\frac{x}{\lambda}} t^{s-2} dt \\
=& \frac{1}{\Gamma(s|\lambda)} \int_0^\infty \sum_{n=0}^{\infty} (1+\lambda t)^{-\frac{n+x}{\lambda}} t^{s-1} dt \\
=& \frac{1}{\Gamma(s|\lambda)} \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \int_0^\infty \left(1+\frac{\lambda}{n+x} t\right)^{-\frac{n+x}{\lambda}} t^{s-1} dt \\
=& \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \frac{\Gamma\left(s \,\middle|\, \frac{\lambda}{n+x}\right)}{\Gamma(s|\lambda)}.
\end{aligned} \tag{2.6}$$

Thus, by (2.6), we get

$$\zeta(s, x|\lambda) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \frac{\Gamma\left(s \,\middle|\, \frac{\lambda}{n+x}\right)}{\Gamma(s|\lambda)}, \tag{2.7}$$

where $s \in \mathbb{C} \setminus \{1\}$ and $x \neq 0, -1, -2, \ldots$.

Let $s = m \in \mathbb{N}$ and $\lambda \in \left(0, \frac{1}{m}\right)$. Then, by (1.10) and (2.7), we have

$$\begin{aligned}
\zeta(m, x|\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(n+x)^m} \frac{\Gamma\left(m \left| \frac{\lambda}{n+x}\right.\right)}{\Gamma(m|\lambda)} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+x)^m} \frac{(m-1)!}{\left(1-\frac{\lambda}{n+x}\right)\left(1-\frac{2\lambda}{n+x}\right)\cdots\left(1-\frac{m\lambda}{n+x}\right)} \\
&\times \frac{1}{(m-1)!}(1-\lambda)(1-2\lambda)\cdots(1-m\lambda) \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+x)^m} \frac{(1-\lambda)(1-2\lambda)\cdots(1-m\lambda)}{\left(1-\frac{\lambda}{n+x}\right)\left(1-\frac{2\lambda}{n+x}\right)\cdots\left(1-\frac{m\lambda}{n+x}\right)}.
\end{aligned} \tag{2.8}$$

Thus, (2.8), we get

$$\zeta(m, x|\lambda) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^m} \frac{(1-\lambda)(1-2\lambda)\cdots(1-m\lambda)}{\left(1-\frac{\lambda}{n+x}\right)\left(1-\frac{2\lambda}{n+x}\right)\cdots\left(1-\frac{m\lambda}{n+x}\right)}, \tag{2.9}$$

where $m \in \mathbb{N}$ and $\lambda \in \left(0, \frac{1}{m}\right)$. From (2.1) and (2.5), we can derive the following equation:

$$\begin{aligned}
\zeta(s, x|\lambda) &= \frac{1}{\Gamma(s|\lambda)} \int_0^{\infty} F(-t, x|-\lambda) t^{s-2} dt \\
&= \frac{1}{\Gamma(s|\lambda)} \int_0^{\infty} \frac{t(1+\lambda t)^{-\frac{x}{\lambda}}}{1-(1+\lambda t)^{-\frac{1}{\lambda}}} t^{s-2} dt \\
&= \frac{1}{\Gamma(s|\lambda)} \sum_{n=0}^{\infty} \beta_n(x|-\lambda)(-1)^n \frac{t^{n+s-2}}{n!}.
\end{aligned} \tag{2.10}$$

Let us take $s = -m$ $(m \in \mathbb{N})$. Then, by Laurent series, we get

$$\zeta(-m, x|\lambda) = 2\pi i \frac{\beta_{m+1}(x|-\lambda)}{(m+1)!}(-1)^{m+1} \frac{1}{\Gamma(-m|\lambda)}. \tag{2.11}$$

Now, we observe that

$$\Gamma(-m|\lambda) = \frac{2\pi i}{m!}(\lambda+1)(2\lambda+1)\cdots((m-1)\lambda+1)(-1)^m. \tag{2.12}$$

Therefore, by (2.11) and (2.12), we obtain the following equation:

$$\begin{aligned}
\zeta(-m, x|\lambda) &= \frac{2\pi i}{(m+1)!}\beta_{m+1}(x|-\lambda)(-1)^{m+1} \frac{m!}{2\pi i} \frac{(-1)^m}{(\lambda+1)(2\lambda+1)\cdots((m-1)\lambda+1)} \\
&= -\frac{\beta_{m+1}(x|-\lambda)}{m+1}.
\end{aligned} \tag{2.13}$$

Remark 2.1. By (2.13), we get

$$\begin{aligned}\lim_{\lambda\to 0}\zeta(-m,x|\lambda) &= \lim_{\lambda\to 0}\left(-\frac{\beta_{m+1}(x|-\lambda)}{m+1}\right) = -\frac{B_{m+1}(x)}{m+1}\\ &= \zeta(-m,x),\ \ (m\in\mathbb{N}).\end{aligned}$$

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