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Some Functors From The Category Of Fuzzy Groups

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Abstract

In the category of fuzzy groups objects are pairs (X, μ) where X is a group and $\mu : X \to [0, 1]$ is a function and a fuzzy group homomorphism from (X, μ) into (Y, η) is a pair (f, α) where $f : X \to Y$ is a homomorphism of groups and $\alpha : \mu(X) \to \eta(Y)$ is a function such that $\alpha\mu = \eta f$. As in any category, for each fuzzy group (X, μ) we have Covariant morphism functor $H^{(X,\mu)}$ and Contravariant morphism functor $H_{(X,\mu)}$ from the category of fuzzy groups into the category of sets. Some of the properties of these functors are studied.

Further unlike the categories of other algebraic structures, here we have three forgetful functors from the category of fuzzy groups namely

(i). $F_1 : \mathcal{F} \to \mathrm{A}$, where **A** is the category of fuzzy sets and F_1 is the one which forgets the group structure

(ii). $F_2 : \mathcal{F} \to \Im$, where $\Im$ is the category of sets and F_2 is the one which forgets both the group structure and the membership grade and

(iii). $F_3 : \mathcal{F} \to G$, where G is the category of groups and F_3 is the one which forgets the membership grade

It is proved that F_1 preserves coretractions, retractions, monomorphisms, epimorphisms. However it is not representative. F_2 preserves all the above morphisms and is representative. F_3 also preserves the above morphisms. Further it is representative and preserves kernels.

Keywords-Fuzzy groups, Categories, Coretraction, Rertraction, Categories, Monomorphism, Epimorphism, Covariant functor, Contravariant funtor, Morphism functors, Forgetful functor, Mono functor, Faithful functors, Representative functors.

1. Introduction and preliminaries

Among the various paradigmatic changes in science, in particular mathematics in this century, one such change concerns the concept of uncertainty [1]. An important point in the evolution of the modern concepts of uncertainty was the publication of seminal paper by Lotfi A. Zadeh in 1965[2]. Since then the development of fuzzy set theory and fuzzy algebraic structures has been dramatic. Various fuzzy algebraic structures have been studied in [3, 4, 5] and fuzzy topological spaces were studied in [6]. Fuzzy groups were introduced in [7, 8]. The existence and uniqueness of free fuzzy abelian groups of rank $\leq \aleph_0$ was proved in [9]. In particular the category of fuzzy groups has been studied in [10].

Following the notation in [11],

A Fuzzy set is a pair $(X, \mu) = \{(x, \mu(x)) / x \in X$ and $\mu : X \to [0,1]$ is a function$\}$.

If (X, μ) and (Y, η) are fuzzy sets then a fuzzy morphism from (X, μ) into (Y, η) is a pair (f, α) where $f : X \to Y$ and $\alpha : \mu(X) \to \eta(Y)$ are functions such that $\alpha\mu = \eta f$ [12, p. 76].

That is "*the image of the membership grade equals the membership grade of the image"*.

Fuzzy subgroups (we call them simply as fuzzy groups in this paper) have been defined in [7] and [8]. Thus in the above notation for fuzzy sets the definition of a fuzzy group will be as follows[9, p. 16].

Definition 1.1:

A fuzzy group is a pair $(X, \mu) = \{(x, \mu(x)) / x \in X$ and $\mu : X \to [0,1]$ is a function $\}$ where

(i) X is a group

(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}, x, y \in X$ and

(iii) $\mu(x^{-1}) = \mu(x)$, for all $x \in X$

The fuzzy group (X, μ) is called a fuzzy abelian group if X is abelian. Moreover in this case the binary operation is denoted by ' + ' and the identity element by ' 0 '. Thus (X, μ) is a fuzzy abelian group if

(i) X is an abelian group

(ii) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ and

iii) $\mu(-x) = \mu(x)$, for all $x \in X$

Definition 1.2:

Let (X, μ) and (Y, η) be fuzzy groups. Then a fuzzy group homomorphism from (X, μ) into (Y, η) is a pair (f, α) where $f : X \to Y$ is a homomorphism of groups and $\alpha : \mu(X) \to \eta(Y)$ is a function such that $\alpha\mu = \eta f$. That is f respects the binary operation and α respects the membership grade[9, p.17].

Monomorphisms (or) sections, epimorphisms and isomorphisms are defined in the natural way, as in any category [13, p.5] and[14, p.25]. Then the collection of all fuzzy groups is a category where the morphisms are the fuzzy group homomorphisms as defined above [10, p. 338].

Let $\mathcal{F}$ and $\Im$ denote the category of fuzzy groups and the category of sets respectively. Fix a fuzzy group $(X,\mu) \in \mathcal{F}$. For any fuzzy group $(Y,\eta) \in \mathcal{F}$, let $H^{(X,\mu)}(Y,\eta)$ be the set of all fuzzy group homomorphisms from $(X,\mu) \to (Y,\eta)$; that is $H^{(X,\mu)}(Y,\eta) = [(X,\mu),(Y,\eta)]_{\mathcal{F}}$. Further for each fuzzy group homomorphism $(f,\alpha):(Y,\eta) \to (Z,\theta)$, Let $H^{(X,\mu)}(f,\alpha):[(X,\mu),(Y,\eta)] \to [(X,\mu),(Z,\theta)]$ be the function defined as $[H^{(X,\mu)}(f,\alpha)](g,\beta) = (f,\alpha)(g,\beta)$ for each $(g,\beta) \in [(X,\mu),(Y,\eta)]$.

It follows that $H^{(X,\mu)}: \mathcal{F} \to \Im$ is a covariant functor.

For let $(f,\alpha):(Y,\eta) \to (Z,\theta)$ and $(g,\beta):(Z,\theta) \to (W,\Omega)$ be fuzzy group homomorphisms. Then for all $(h,\delta): (X,\mu) \to (Y,\eta)$

we have

$[H^{(X,\mu)}((g,\beta)(f,\alpha))](h,\delta) = [(g,\beta)(f,\alpha)](h,\delta)$

$= (g,\beta)[(f,\alpha)(h,\delta)]$

$= [H^{(X,\mu)}(g,\beta)]((f,\alpha)(h,\delta))$

$= [H^{(X,\mu)}(g,\beta) \circ H^{(X,\mu)}(f,\alpha)](h,\delta)$

and hence

$H^{(X,\mu)}[(g,\beta)(f,\alpha)] = H^{(X,\mu)}(g,\beta) \circ H^{(X,\mu)}(f,\alpha)$.

This shows that $H^{(X,\mu)}$ preserves composition of functions.

Moreover if $I_{(Y,\eta)}:(Y,\eta) \to (Y,\eta)$ denotes the identity homomorphism, then for all $(g,\beta):(X,\mu) \to (Y,\eta)$ we have $[H^{(X,\mu)} I_{(Y,\eta)}](g,\beta) = I_{(Y,\eta)}(g,\beta) = (g,\beta)$.

That is $H^{(X,\mu)} I_{(Y,\eta)}$ is the identity function in $[(X,\mu),(Y,\eta)] = H^{(X,\mu)}(Y,\eta)$.

In other words $H^{(X,\mu)} I_{(Y,\eta)} = I_{H^{(X,\mu)}(Y,\eta)}$.

Hence $H^{(X,\mu)}$ preserves identities so that $H^{(X,\mu)}$ is a Covariant functor.

Proposition 1.3:

Let $(f,\alpha):(Y,\eta) \to (Z,\theta)$ be a fuzzy group homomorphism. Then (f,α) is a monomorphism in $\mathcal{F}$ if and only if $H^{(X,\mu)}(f,\alpha)$ is a monomorphism in $\Im$ for all $(X,\mu) \in \mathcal{F}$.

Proof :

Let us assume that $(f,\alpha):(Y,\eta) \to (Z,\theta)$ is a monomorphism in $\mathcal{F}$.

Suppose $(g_1,\beta_1),(g_2,\beta_2) \in [(X,\mu) \to (Y,\eta)]$ such that $[H^{(X,\mu)}(f,\alpha)](g_1,\beta_1) = [H^{(X,\mu)}(f,\alpha)](g_2,\beta_2)$

Then $(f,\alpha)(g_1,\beta_1) = (f,\alpha)(g_2,\beta_2)$ so that $(g_1,\beta_1) = (g_2,\beta_2)$ by assumption.

That is $H^{(X,\mu)}(f,\alpha)$ is a monomorphism.

Conversely suppose $H^{(X,\mu)}(f,\alpha)$ is a monomorphism in $\Im$, for each $(X,\mu)\in$ ***F***. Let $(g_1,\beta_1),(g_2,\beta_2):[(X,\mu)\to(Y,\eta)]$ be such that $(f,\alpha)(g_1,\beta_1)=(f,\alpha)(g_2,\beta_2)$. Then $[H^{(X,\mu)}(f,\alpha)](g_1,\beta_1)=[H^{(X,\mu)}(f,\alpha)](g_2,\beta_2)$ so that $(g_1,\beta_1)=(g_2,\beta_2)$ by assumption. Thus (f,α) is a monomorphism in ***F***.

Corollary 1.4:
Let $(f,\alpha):(Y,\eta)\to(Z,\theta)$ be a coretraction in ***F***. Then for all $(X,\mu)\in$ ***F***. $H^{(X,\mu)}(f,\alpha):[(X,\mu),(Y,\eta)]\to[(X,\mu),(Z,\theta)]$ is a coretraction in ***F***.

Proof:
Suppose $(f,\alpha):(Y,\eta)\to(Z,\theta)$ is a coretraction in ***F***. Then (f,α) is a monomorphism in ***F*** (Since every coretraction is a monomorphism) and hence $H^{(X,\mu)}(f,\alpha)$ is a monomorphism in $\Im$ [proposition 1.3]. Therefore $H^{(X,\mu)}(f,\alpha)$ is a coretraction in $\Im$ [since in $\Im$ coretraction $\Leftrightarrow$ monomrphism].

Example 1.5:
The converse of the above proposition is not true. Let $Y=\{0,1,2\}=Z_3$ be the group of order 3 ; $\mu(0)=1$, $\mu(1)=s=\mu(2)$ where $0\le s<1$. $\eta(0)=1=\eta(1)=\eta(2)$.
$f:Z_3\to Z_3$ is the identity map and $\alpha:\mu(Z_3)\to\eta(Z_3)$ be defined as $\alpha\mu(x)=1$for all $x\in Z_3$. Then it is clear that $(f,\alpha):(Z_3,\mu)\to(Z_3,\eta)$ is a fuzzy group homomorphism. Moreover f is injective implies (f,α) is a monomorphism in ***F***. Hence $H^{(X,\mu)}(f,\alpha)$ is a monomorphism in $\Im$ for all (X,μ) [Proposition 1.3] so that $H^{(X,\mu)}(f,\alpha)$ is a coretraction in $\Im$ [Since in $\Im$ coretraction iff monomorphism].
However (f,α) is not a coretraction in ***F***, Since α is not an injective map [9]

Corollary 1.6:
$H(X,\mu)$: ***F*** $\to\Im$ is a mono functor

2. Contravariant morphism functor :
Let us fix a fuzzy group (X,μ) in ***F***. For each fuzzy group (Y,η) in ***F***, let $H_{(X,\mu)}(Y,\eta)=[(Y,\eta),(X,\mu)]$ ***F*** be the set of all fuzzy group homomorphism from (Y,η) in to (X,μ) and for each $(f,\alpha):(Y,\eta)\to(Z,\theta)$, let $H_{(X,\mu)}(f,\alpha):[(Z,\theta),(X,\mu)]\to[(Y,\eta)\to(X,\mu)]$ be the function which maps (g,β) *in to* $(g,\beta)(f,\alpha)$. Then we can verify that $H_{(X,\mu)}$: ***F*** $\to\Im$ is a contravariant functor.

Proposition 2.1:

Let $(f,\alpha):(Y,\eta)\to(Z,\theta)$ be a fuzzy group homomorphism. Then (f,α) is an epimorphism in $\mathcal{F}$ if and only if $H_{(X,\mu)}\ (f,\alpha):[(Z,\theta),(X,\mu)]\to[(Y,\eta)\to(X,\mu)]$ is a monomorphism in $\Im$.

Proof:
Let us first assume that $(f,\alpha):(Y,\eta)\to(Z,\theta))$ is an epimorphism. Suppose $(g_1,\beta_1),(g_2,\beta_2)\in[(Z,\theta)\to(X,\mu)]$ such that $[H_{(X,\mu)}\ (f,\alpha)](g_1,\beta_1)=[H_{(X,\mu)}(f,\alpha)](g_2,\beta_2)$. Then $(g_1,\beta_1)(f,\alpha)=(g_2,\beta_2)(f,\alpha)$ and hence $(g_1,\beta_1)=(g_2,\beta_2)$ [(f,α) is an epimorphism]. Therefore $H_{(X,\mu)}\ (f,\alpha)$ is a monomorphism in $\Im$.

Conversely, Suppose $H_{(X,\mu)}\ (f,\alpha)$ is a monomorphism in $\Im$ for all $(X,\mu)\in\Im$ and let $(g_1,\beta_1),(g_2,\beta_2):(Z,\theta)\to(X,\mu)$ be such that $(g_1,\beta_1)(f,\alpha)=(g_2,\beta_2)(f,\alpha)$. Then $[H_{(X,\mu)}\ (f,\alpha)](g_1,\beta_1)=[H_{(X,\mu)}(f,\alpha)](g_2,\beta_2)$ *so that* $(g_1,\beta_1)=(g_2,\beta_2)$ [Since $H_{(X,\mu)}\ (f,\alpha)$ is a monomorphism]. This shows that (f,α) is an epimorphism in $\mathcal{F}$.

Proposition 2.2:
The functors $H^{(X,\mu)}:\mathcal{F}\to\Im$ and $H_{(X,\mu)}:\mathcal{F}\to\Im$ preserves isomorphisms.

Proof :
The proof is Routine

Remark 2.3:
By proposition 1.3 $(f,\alpha):(Y,\eta)\to(Z,\theta)$ is a monomorphism if and only if $H^{(X,\mu)}(f,\alpha):[(X,\mu),(Y,\eta)]\to[(X,\mu),(Z,\theta)]$ is a monomorphism in $\Im$ for all $(X,\mu)\in\mathcal{F}$. However if (f,α) is an epimorphism, then $H^{(X,\mu)}\ (f,\alpha)$ need not be an epimorphism in $\mathcal{F}$ as the following example shows. Let $(Y,\eta)=(S_3,\eta)$ where $S_3=\{\rho_0,\rho_1,\rho_2,\tau_1,\tau_2,\tau_3\}$, $\rho_0=id$ and ρ_1,ρ_2 being the even permutations. $\eta(\rho_0)=\eta(\rho_1)=\eta(\rho_2)=1$ and $\eta(\tau_1)=\eta(\tau_2)=\eta(\tau_3)=s_1$ where $0\le s_1<1$.

Let $(Z,\theta)=(Z_2,\theta)$ where $Z_2=\{0,1\}$ is the additive group of integers modulo 2 and $\theta(0)=1$, $\theta(1)=t_1$, $0\le t_1<1$. Then (Y,η) and (Z,θ) are fuzzy groups. Consider the fuzzy group homomorphism $(f,\alpha):(S_3,\eta)\to(Z_2,\theta)$ where $f(\rho_0)=f(\rho_1)=f(\rho_2)=0$ and $f(\tau_1)=f(\tau_2)=f(\tau_3)=1$, $\alpha(1)=1$, $\alpha(s_1)=t_1$. Since $f:S_3\to Z_2$ is a surjective function $(f,\alpha):(S_3,\eta)\to(Z_2,\theta)$ is an epimorphism.

Let $X = Z_4$ be the additive group of integers modulo 4 and $\mu(0) = \mu(2) = 1$, $\mu(1) = \mu(3) = \gamma_1$. Then it is clear that (Z_4, μ) is a fuzzy group.
Let $(h, \gamma) : (Z_4, \mu) \to (Z_2, \theta)$ be the fuzzy group homomorphism defined by $h(0) = h(2) = 0$, $h(1) = h(3) = 1$ and $\gamma(1) = 1$, $\gamma(r_1) = t_1$.
Then there exists no fuzzy group homomorphism $(g, \beta) : (Z_4, \mu) \to (S_3, \eta)$ such that $(f, \alpha)(g, \beta) = (h, \gamma)$; that is such that $[H^{(X, \mu)}(f, \alpha)](g, \beta) = (h, \gamma)$.
For it is well known that the only group homomorphism $g : Z_4 \to S_3$ is the trivial homomorphism defined by $g(x) = \rho_0$ for all $x \in Z_4$. Thus (g, β) is the trivial fuzzy group homomorphism. Hence $H^{(X, \mu)}(f, \alpha)$ is not an epimorphism.

Remark 2.4:
Let $(f, \alpha) : (Y, \eta) \to (Z, \theta)$ be a fuzzy group homomorphism.. If (f, α) is an epimorphism in $\mathcal{F}$ then $H_{(X,\mu)}(f, \alpha) : [(Z, \theta), (X, \mu)] \to [(Y, \eta) \to (X, \mu)]$ where $H_{(X,\mu)}(g, \beta) = (g, \beta)(f, \alpha)$ need not be an epimorphism.
Let Z_2, Z_3, Z_6 be the groups of integers modulo 2, 3, 6 respectively. Let $(Y, \eta) = (Z_6, \eta)$ where $\eta(0) = \eta(3) = 1$ and $\eta(1) = \eta(2) = \eta(4) = \eta(5) = r$, $0 \le r < 1$. Let $(Z, \theta) = (Z_3, \theta)$ where $\theta(0) = 1$ and $\theta(1) = \theta(2) = s$, $0 \le s < 1$.
Then (Y, η) and (Z_6, θ) are fuzzy groups. Define $(f, \alpha) : (Z_6, \eta) \to (Z_3, \theta)$ by $f(0) = f(3) = 0$; $f(1) = f(4) = 1$; $f(2) = f(5) = 2$ and $\alpha(1) = 1$; $\alpha(r) = s$. Then we can verify that (f, α) is a fuzzy group homomorphism. Moreover since $f : Z_6 \to Z_3$ is a surjective function, (f, α) is an epimorphism in $\mathcal{F}$ [12, p. 78].
Consider the fuzzy group $(X, \mu) = (Z_2, \mu)$ where $\mu(0) = 1 = \mu(1)$. Define $(h, \gamma) : (Z_6, \eta) \to (Z_2, \mu)$ as $h(0) = h(2) = h(4) = 0$; $h(1) = h(3) = h(5) = 1$ and $\gamma(1) = \gamma(r) = 1$. Then (h, γ) is a fuzzy group homomorphism.
The only fuzzy group homomorphism $(g, \beta) : (Z_3, \theta) \to (Z_2, \mu)$ is the trivial one where $h(0) = h(1) = h(2) = 0$ and $\gamma(1) = \gamma(s) = 1$ and $(g, \beta)(f, \alpha) \ne (h, \gamma)$. Thus $H_{(X,\mu)}(f, \alpha)$ is not an epimorphism.

3. The Forgetful functors:

If the objects of a category **A** are sets with certain structures (like groups, R-modules, topological spaces) and the morphisms are structure preserving maps (like homomorphism between groups, R-homomorphisms, continuous maps), we define the forgetful functors F : **A** $\to \Im$ which assigns to each object its underlying set and to each morphism the corresponding set map. In this section we introduce three forgetful functors from the category of fuzzy groups and study their preservation properties.

Definition 3.1:

a) $F_1 : \mathcal{F} \to \mathbf{A}$ is the forgetful functor which assigns to each fuzzy group (X, μ) the underlying fuzzy set (X, μ) and to each fuzzy group homomorphism $(f, \alpha):(X, \mu) \to (Y, \eta)$ the corresponding fuzzy set morphism [5]

b) $F_2 : \mathcal{F} \to \Im$ is the forgetful functor which assigns to each fuzzy group (X, μ) the underlying set X and to each fuzzy group homomorphism $(f, \alpha):(X, \mu) \to (Y, \eta)$ the corresponding set map $f : X \to Y$.

c) $F_3 : \mathcal{F} \to$ G is the forgetful functor which assigns to each fuzzy group (X, μ) the underlying group X and to each fuzzy group homomorphism $(f, \alpha):(X, \mu) \to (Y, \eta)$ the corresponding homomorphism of groups $f : X \to Y$.

Proposition 3.2:

The forgetful functor $F_1 : \mathcal{F} \to \mathbf{A}$ preserves

i) Coretractions
ii) Monomorphisms
iii) Retractions
iv) epimorphisms
v) F_1 is not representative
vi) Preserves images.

Proof:

i) Let $(f, \alpha):(X, \mu) \to (Y, \eta)$ be a fuzzy group homomorphism which is a coretraction in $\mathcal{F}$. This implies that both $f : X \to Y$ and $\alpha : \mu(X) \to \eta(Y)$ are injective maps [12, p. 77]. Therefore (f, α) is a coretraction in **A** [5] and hence (i)

ii) (f, α) is a monomorphism in $\mathcal{F}$ implies that $f : X \to Y$ is an injective map[9] and therefore (f, α) is a monomorphism in **A** [12, p. 78] and hence (ii)

iii) $(f, \alpha):(X, \mu) \to (Y, \eta)$ is a retraction in $\mathcal{F}$ implies that there exists fuzzy group homomorphism $(g, \beta):(Y, \eta) \to (X, \mu)$ such that $(f, \alpha)(g, \beta) = I_{(Y, \eta)}$. Therefore (f, α) is a retraction in **A** [12, p. 77]

iv) (f, α) is an epimorphism in $\mathcal{F}$ implies that $f : X \to Y$ is an surjective map [9] and so Hence (f, α) is an epimorphism in **A**.

v) We have to produce a fuzzy set $(Y, \eta) \in$ **A** such that there is no fuzzy group $(X, \mu) \in \mathcal{F}$ such that the fuzzy set $F_1(X, \mu)$ is not isomorphic to (Y, η) in **A.** We recall that two fuzzy sets (Z_1, θ_1) and (Z_2, θ_2) are isomorphic if and only if there exists a fuzzy set homomorphism $(f, \alpha):(Z_1, \theta_1) \to (Z_2, \theta_2)$ such that both $f : Z_1 \to Z_2$ and $\alpha : \theta_1(Y_1) \to \theta_2(Y_2)$ are bijections.

Consider the fuzzy set (Y, η) where $Y = \{y_0 , y_1 , y_2 \}$ and $\eta (y_0) = 1 , \eta (y_2) = r , \eta (y_3) = s$ where $0 \le r < s < 1$.

If there exists a fuzzy group (X, μ) such that $F_1(X, \mu)$ is isomorphic to (Y, η) then there must be a fuzzy set homomorphism $(f, \alpha) : (X, \mu) \to (Y , \eta)$ in **A** such that $f : X \to Y$ and $\alpha : \mu(X) \to \eta(Y)$ are bijective.

Since $|Y| = 3$, X must be a group of order 3. The only group of order 3 is Z_3= { 0, 1, 2} (the group of integers modulo 3) and if (Z_3, μ) is a fuzzy group then either

i) $\mu (x) = 1$ for all $x \in Z_3$ OR ii) $\mu (0) = 1,\ \mu (1) = \mu(2) = t , 0 \le t < 1$.

Thus $| \mu (Z_3) | \le 2$. Since $|\eta(Y)| = 3$ there is no bijection from $\mu (Z_3)$ in to $\eta (Y)$.

In other words the fuzzy sets (Z_3, μ) and (Y, η) are not isomorphic fuzzy sets. Therefore F_1 is not representatives.

vi) We have proved that the category of fuzzy groups has (epimorphic) images and that if $(f, \alpha) : (X, \mu) \to (Y , \eta)$ is a given fuzzy group homomorphism then the image of (f, α) is given by $(i_{f(X)} , i_{\eta_1 f(X)}) : (f(X) , \eta_1) \to (Y, \eta)$ where $f(X) = \{ f(x) / x \in X \}$ and η_1 is the restriction of η to $f(X)$.

So if $(f(X), \eta_1)$ is the image of (f, α) in $\mathcal{F}$ then $F_1(f(X) , \eta_1) = (f(X) , \eta_1)$ is the image of the fuzzy set morphism $(f, \alpha) : (X, \mu) \to (Y , \eta)$ in **A** [5]

Proposition 3.3:

The forgetful functor F_2: F $\to \Im$ preserves

i) coretractions
ii) monomorphisms
iii) retractions
iv) epimorphisms
v) F_2 is representative
vi) F_2 preserves images

Proof:

We prove only (v). The proof for the rest are routine v) Let Y be any set.

Case 1:

Suppose $|Y| = n$ (a positive integer)

Consider the fuzzy group (Z_n , μ) where Z_n is the additive group of integers modulo n and $\mu(x) = 1$ for all $x \in Z_n$. Then (Z_n , μ) is a fuzzy group such that $F_2(Z_n , \mu) = Z_n$. Since $|Z_n| = |Y| = n$, there is a bijection $f : Z_n \to Y$ so that Z_n and Y are isomorphic in $\Im$.

Case 2:
Let $|Y| = \aleph_0$ (aleph naught).
Consider the fuzzy group (Z, μ) where Z is the additive group of integers and $\mu(x) = 1$ for all $x \in Z$.
Then $F_z(Z, \mu) = Z$. Now since $|Z| = \aleph_0 = |Y|$ there is a bijection $f : Z \to Y$ so that the sets Z and Y are isomorphic in $\Im$.

Case 3:
Let $|Y| = c$.
Then we consider the fuzzy group (R, μ) where R is the additive group of real numbers and $\mu(x) = 1$ for all $x \in R$.
Then the sets R and Y are isomorphic in $\Im$.

Case 4:
Let $|Y| \neq n, \aleph_{0,}$ C.
Then consider the free abelian group X on Y.
We have the fuzzy group (X, μ) where $\mu(x) = 1$ for all $x \in X$.
Since $|\mathrm{X}| = |\mathrm{Y}|$, X and Y are isomorphic in $\Im$.
Thus F_2 is representative.

Proposition 3.4:
The forgetful functor F_3 : F $\to$ G preserves

i) coretractions
ii) monomorphisms
iii) retractions
iv) epimorphisms
v) F_3 is representative
vi) preserves images
vii) preserves zero and the zero morphism
viii) preserves kernels.

Proof:
We prove only (v), (vii) and (viii) v) Let $\mathcal{G}$ be any group. Consider the fuzzy group (X, μ) where $\mu(x) = 1$ for all x$\in$ G.
Then $F_3(X, \mu) = X$ isomorphic to X in $\mathcal{G}$ implies that F_3 is representative.
vii) We know that (e, λ_0) where $\lambda_0(e) = 1$ and e is the singleton group is the zero object in F and $(\theta, \theta_0) : (X, \mu) \to (Y, \eta)$ is the zero fuzzy group homomorphism where $\theta(x) = e$ and $\theta_0(\mu(x)) = 1$ for all $x \in X$ [9].

Then $F_3(e\,,\lambda_o) = \; e$ is the zero object in $\mathcal{G}$ and $F_3(\theta\,,\theta_0) = \theta$ is the zero morphism in $\mathcal{G}$.
Hence F_3 preserves zero and the zeromorphism.
viii) Let $(f\,,\alpha):(X\,,\mu) \to (Y\,,\eta)$ be any fuzzy group homomorphism in $\mathcal{F}$. In [9] it has been proved that $\mathcal{F}$ has kernels and it is given by $\ker(f\,,\alpha) \;=(\; f^{-1}(e)\,,\mu'\,)\,,\mu' = \mu \,/\; f^{-1}(e)$ [μ restricted to $f^{-1}(e\,)$].
Then $F_3(\,\ker(f\,,\alpha)\,) \;=\; F_3(\; f^{-1}(e)\,,\mu'\,) = f^{-1}(e)$ which is the kernel of the group homomorphism $f : X \to Y$ in $\mathcal{G}$.
Thus F_3 preserves kernels.

Conclusion:
The study of the covariant and contravariant morphism functors $H^{(X\,,\mu)}$ and $H_{(X\,,\mu)}$ naturally leads to the study of projective and injective fuzzy groups in the category of fuzzy groups and their properties.

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