Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 5 (2015), pp. 3311-3317

http://www.ripublication.com

Equalizers and Intersections in the Category of Graphs

S. Buvaneswari*[1] and P.Alphonse Rajendran[2]

[1, 2]Department of Mathematics, Periyar Maniammai University
Vallam, Thanjavur - 613403, Tamil Nadu, India
**Email: buvaneswari@pmu.edu*

Abstract

If f: G $\rightarrow$ G_1 is morphism in the category of Abelian Groups (OR R-mod, vector spaces etc.) then Kerf ={ $x \in G / f(x)=0$ } is indeed the equalizer of *f* and the *zero* morphism 0. A generalization of this idea is that of an equalizer of any two morphisms in any arbitrary category $\mathscr{A}$. In this paper we prove the existence of equalizers of any two homomorphisms in the category $\mathcal{G}$ of graphs by actually constructing the same (up to isomorphism). Dually the coequalizer for morphisms f and g in $\mathcal{G}$ is defined as the equalizer for f and g in the dual category $\mathcal{G}^*$. This is in fact a generalization of a quotient by an equivalence relation. It is clear that $\mathcal{G}$ has coequalizers if and only if $\mathcal{G}^*$ has equalizers. We prove by an example that $\mathcal{G}$ does not have coequalizers. Finally we prove that $\mathcal{G}$ has finite intersections also.

1. Introduction

A graph G consists of a pair G = (V(G), E(G)) (also written as G = (V, E) whenever the context is clear) where V(G) is a finite set whose elements are called vertices and E(G) is a set of unordered pairs of distinct elements in V(G) whose members are called edges. The graphs as we have defined above are called simple graphs. Throughout our discussions all graphs are considered to be simple graphs [1, 2].

Let G and G_1 be graphs. A homomorphisms f: G $\rightarrow$ G_1 is a pair $f = (f^*, \tilde{f})$ where f^*: V(G) $\rightarrow$ V(G_1) and $\tilde{f}$: E(G) $\rightarrow$ E(G_1) are functions such that $\tilde{f}$ ((u, v)) = (f^*(u), f^*(v)) for all edges (u, v) $\in$ E(G). For convenience if (u, v) $\in$ E (G) then $\tilde{f}$ ((u, v)) is simply denoted as $\tilde{f}$ ((u, v)) [3].

Then we have the category of graphs say $\mathcal{G}$, where objects are graphs and morphisms are as defined above, where equality, compositions and the identity morphisms are defined in the natural way. It is also proved that two homomorphisms $f = (f^*, \tilde{f})$ and $g = (g^*, \tilde{g})$ of graphs are equal if and only if $f^* = g^*$ Lemma 1.6 [3].

2. Equalizers

Definition 2.1: Let $f, g : X \to Y$ be two given homomorphisms of graphs. Then a homomorphism $h : K \to X$ is said to be an equalizer for *f* and *g* if

i) $f\,h = g\,h$ and

ii) If $p : Z \to X$ is any graph homomorphism such that $f\,p = g\,p$ then there exists a unique homomorphism $q : Z \to K$ such that $h\,q = p$. (See Figure 2.1).

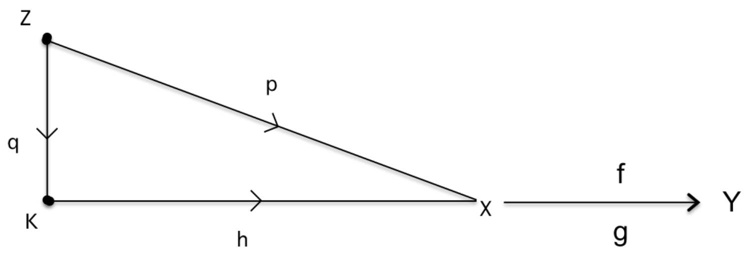

Figure 2.1

Proposition 2.2: The Category of graphs $\mathcal{G}$ has equalizers [4, 5, 6].

Proof: Let $f, g : X \to Y$ be given homomorphisms of graphs. Let *K* be the graph with vertex set $V(K) = \{x_0 \in X \,/\, f(x) = g(x)\}$ and for all $u, v \in E(K)$, the edge $(u, v) \in E(K)$ if and only if (u, v) belongs to *V(X)*. Let $i_K = (i^*_{V(K)}, \tilde{i}_{E(K)})$ be the inclusion homomorphism.

Claim: $i_K : K \to X$ is an equalizer for *f* and *g*. Now for all $x \in K$

$$
\begin{aligned}
(f\,i_K)^*(x) &= f^*\,{i_K}^*(x) \\
&= f^*(x) \\
&= g^*(x) \\
&= g^*\,{i_K}^*(x) \\
&= (g\,i_K)^*(x)
\end{aligned}
$$

So that $(f\,i_K)^* = (g\,i_K)^*$. Hence by Lemma 1.6 in [3].

$f\,i_K = g\,i_K$ Which is (i) of Definition 2.1 (See Figure 2.2).

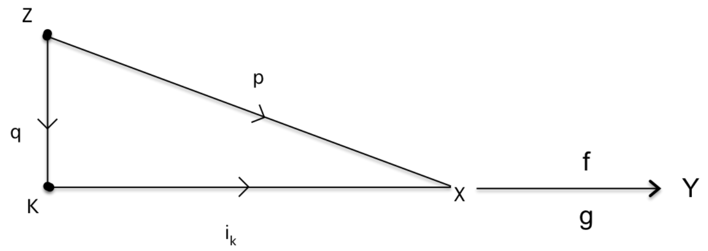

Figure 2.2

Suppose there exists a homomorphism of graphs $p : Z \to X$ such that $f\,p = g\,p$.
Then for all $\mathfrak{z} \in Z$, $f^*p^*(\mathfrak{z}) = g^*p^*(\mathfrak{z})$ so that by definition of K, $p^*(\mathfrak{z}) \in V(K)$. So define a homomorphism $q : Z \to K$ as follows. $q^*(\mathfrak{z}) = p^*(\mathfrak{z})$ for all $\mathfrak{z} \in Z$. Moreover if $(\mathfrak{z}_1, \mathfrak{z}_2) \in E(Z)$, then $(p^*(\mathfrak{z}_1), p^*(\mathfrak{z}_2)) \in E(X)$ (since p is a homomorphism) and hence $(p^*(\mathfrak{z}_1), p^*(\mathfrak{z}_2)) \in E(K)$ (by definition of $E(K)$). i.e. $(q^*(\mathfrak{z}_1), q^*(\mathfrak{z}_2)) \in E(K)$ so that $q : Z \to K$ is a homomorphism of graphs. Also by definition, for all $\mathfrak{z} \in Z$, $i_k^*\, q^*(\mathfrak{z}) = p^*(\mathfrak{z})$ and so $i_k\, q = p$.
Further q is unique, for if there exists $q_1 : Z \to K$ such that $i_k\, q_1 = p$. Then $q_1 = i_k q_1 = p = i_k q = q$, proving the uniqueness. Hence $\mathcal{G}$ has equalizers.

Remark 2.3: As in any category the following properties are true in $\mathcal{G}$ [7, 8].

i) If h is an equalizer for f and g then h is a monomorphism.

ii) Any two equalizers for f and g are isomorphic subobjects of X. Hence we can talk about 'the' equalizer of two given homomorphisms.

iii) $f = g$ if and only if 1_X is the equalizer for f and g.

Proposition 2.4: Let $f, g : X \to Y$ be homomorphism of graphs and $h: K \to X$ be the equalizer for f and g. If h is also an epimorphism then h is an isomorphism (See Figure 2.3)[4, 5].

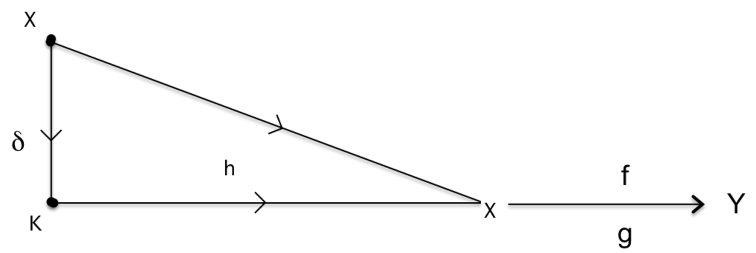

Figure 2.3

Proof: Given that h is an equalizer for f and g. Hence f is a monomorphism.
Moreover $fh=gh$ (by definition) and hence $f = g$ (since h is an epimorphism). Therefore by (iii) in Remark (2.3), 1_X is an equalizer for f and g. This shows that there exists a unique homomorphism $\delta : X \rightarrow K$ such that $h\delta = 1_X$. Therefore h is a retraction. Thus h is a monomorphism and also a retraction implies that h is an isomorphism (Proposition 5.1 in [3]).

3. Coequalizers

Definition 3.1: Let $f, g: X \rightarrow Y$ are two given homomorphism of graphs. Then a homomorphism $h: Y \rightarrow Z$ is said to be a coequalizer for f and g if

i) $hf = hg$ and
ii) $if\ h_1 f = h_1 g$ for some homomorphism $h_1 : Y \rightarrow Z_1$ then there exists a unique homomorphism $\gamma : Z \rightarrow Z_1$ such that $\gamma\, h = h_1$ (See Figure 3.1).

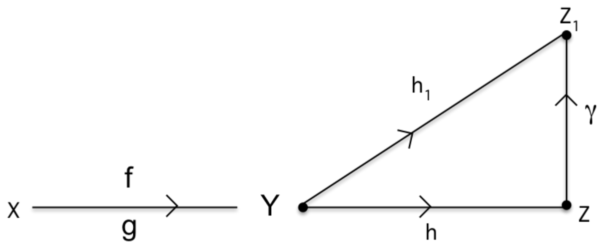

Figure 3.1

Remark 3.2: As for equalizers, we can prove the following statements for coequalizers [6].

1) If h is a coequalizer for f and g then h is an epimorphism.
2) Any two coequalizer for f and g are isomorphic graphs.
3) If h is a coequalizer for f and g and h is also a monomorphism then h is an isomorphism.
4) Every retraction is a coequalizer.

Definition 3.3: Let X, Y be arbitrary graphs in $\mathcal{G}$. If coequalizer for every pair of homomrphisms $f, g : X \rightarrow Y$ exists then $\mathcal{G}$ is said to have coequalizers.

Remark 3.4: We have proved that the category $\mathcal{G}$ has equalizers [6]. However this is not true in the case of coequalizers as the following example shows.

Example 3.5: Let X and Y be graphs where V(X) = {x}, E(X) = ϕ ;
V(Y) = { y_1, y_2 }, E(Y) = {(y_1, y_2)}and $f, g : X \rightarrow Y$ Where $f^*(x) = y_1\ and\ g^*(x) = y_2$ (See Figure 3.2).

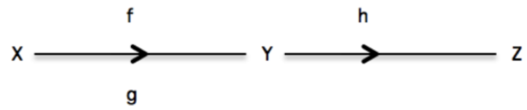

Figure 3.2

Suppose $h : Y \to Z$ is a coequalizer for f and g then $hf = hg$. Then $h^* f^*(x) = h^* g^*(x)$.
i.e. $h^*(y_1) = h^*(y_2)$...(1) But (y_1, y_2) is an edge in Y implies that
$\tilde{h}(y_1, y_2) = (h^*(y_1), h^*(y_2))$ is an edge in Z which is a contradiction by (1).
Therefore f and g do not have a coequalizer thus proving that $\mathcal{G}$ does not have coequalizers.

4. Intersection

Definition 4.1: Let $\{ u_i : A_i \to A \}$ be a family of subgraphs of A. A morphism $u : B \to A$ is called the intersection of the family if

i) for each $i \in I$ we can write $u = u_i v_i$ for some morphism $v_i : B \to A_i$ and
ii) if every morphism $C \to A$ which factors through each u_i factors uniquely through u (See Figure 4.1a&4.1b).

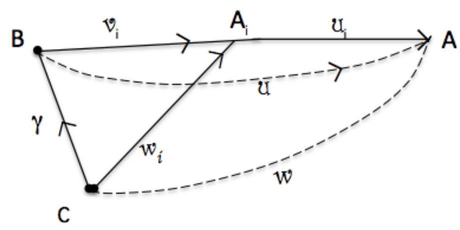

Figure 4.1a

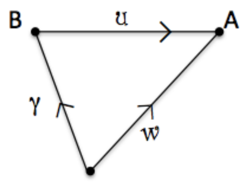

Figure 4.1b

Remark 4.2:

1) By definition of subgraphs, each u_i is a monomorphism and hence $u_i{}^*$ is injective.
2) The morphism u in the definition is also a monomorphism. For $u\gamma_1 = u\gamma_2$ for $\gamma_1, \gamma_2 : C \to B$, then $u_i v_i \gamma_1 = u_i v_i \gamma_2$. Hence $v_i \gamma_1 = v_i \gamma_2$ (since u_i is a monomorphism). Take $w_i = v_i \gamma_1 = v_i \gamma_2$ and $w = u_i v_i \gamma_1 = u_i v_i \gamma_2$. Then by uniqueness in the definition $\gamma_1 = \gamma_2$. So that u is a monomorphism.

3) Since $u = u_i v_i$ is a monomorphism each v_i is a monomorphism.
4) Any two intersections of a given family are isomorphism.

Proposition 4.3: The Category of graphs $\mathcal{G}$ has finite intersections [6].

Proof: Let $\{u_i : A_i \to A\}_{i=1\ to\ n}$ be a finite set of subgraphs of *A*. Then by the definition of subobjects, each $u_i : A_i \to A$ is a monomorphism and hence $u_i{}^*: V(A_i) \to V(A)$ is injective [3].

Consider the graph *B*, where $V(B) = \bigcap_{i=1}^{n} u_i^*(V(A_i)) \subseteq V(A)$.

If $x_1, x_2 \in V(B)$ then $x_1, x_2 \in u_i^*(V(A_i))$ for each *i = 1 to n.*
So define (x_1, x_2) is an edge in B if and only if (x_1, x_2) is an edge in A_i for all i.
Consider the graph *B*. we define a homomorphism $v_i: B \to A_i (i = 1\ to\ n)$ as follows: If $x \in V(B)$ then $x \in u_i^*(V(A_i))$ for each *i*. Hence there is a unique $y_i \in V(A_i)$ such that $x = u_i^*(y_i)$ .... (1). Define $v_i^*: V(B) \to V(A_i)$ by $v_i^*(x) = y_i$. Clearly $vi: B \to A_i$ is a homomorphism. Let $u: B \to A$ be defined as $u = u_i v_i$ .... (2)
Then for all $x \in V(B), u^*(x) = u_i^* v_i^*(x) = u_i^*(y_i) = x$ by (1)
Thus

$u^*(x) = x$ for all $x \in B$ ..... (3) [$i.e. u^* : V(B) \to V(A)$ is the inclusion map]
.

Claim 1: *B* is the intersection of the given family. Suppose there exists a morphism $w: C \to A$ such that $w = u_i w_i (i = 1\ to\ n)$ for some morphisms $w_i: C \to A_i$ (Refer Figures 4.1a & 4.1b).
Define $\gamma: C \to A$ as follows. $\gamma^*: V(C) \to V(A)$ *where* $\gamma^*(\mathfrak{z}) = w^*(\mathfrak{z})$ for all $\mathfrak{z} \in C$.
Since w is a homomorphism, it preserves edges in *C* and so does γ^* and hence γ is a homomorphism.
Moreover $u^*\gamma^*(\mathfrak{z}) = u^*(w^*(\mathfrak{z})) = w^*(\mathfrak{z})$ by... (3). Hence $u\gamma = w$ (by Lemma 1.6 in [3]).

Claim 2: Suppose there exists $\delta : C \to B$ such that $u\delta = w$. Then for all $\mathfrak{z} \in$V (*B*),
$\gamma^*(\mathfrak{z}) = w^*(\mathfrak{z})$ by definition
$= u^*\delta^*(\mathfrak{z})$ (by assumption)
$= \delta^*(\mathfrak{z})$ [since u^* is the inclusion]
and so $\gamma = \delta$ proving the uniqueness of γ. Thus $\mathcal{G}$ has finite intersections.

5. Conclusion

Hence the existence of equalizers of any two homomorphisms in the category $\mathcal{G}$ of graphs is proved. It is also proved by example that $\mathcal{G}$ does not have coequalizers and $\mathcal{G}$ has finite intersection.

Acknowledgement
The authors thank the anonymous referees.

REFERENCES

[1] Godsil, C and Royle, G., 2001 Algebraic Graph Theory, Springer-Verlag, New York NY.
[2] Balakrishnan, V.K., 1997 Schaum's Outline of Graph Theory: Including Hundreds of Solved Problems, Schaum's Outline Series.
[3] Buvaneswari, S and Alphonse Rajendran, P., 2015 "A study on Category of Graphs", *IOSR* Journal of Mathematics (IOSR-JM), 11(4), pp.38-46.
[4] Porselvi Agnes, A.R., 2012 A study on the Category of Fuzzy Sets and Fuzzy Structures, Ph.D., Dissertation, Manonmaniam Sundaranar University, India.
[5] Porselvi Agnes, A.R., Sivaraj, D and Tamizhchelvam, T. 2010 "On a new Category of Fuzzy Sets" Journal of Advanced Research in Pure Mathematics, 2(4), pp. 73-83.
[6] Vijayalakshmi, P and Alphonse Rajendran, P., 2015 "Intersections and Pullbacks, *IOSR* Journal of Mathematics (IOSR-JM), 11(2), pp. 61-69.
[7] Awodey, S., 2006 Category Theory, Clarendon Press; Oxford University Press Inc., New York, NY.
[8] Mitchell, B., 1965 Theory of categories**,** Academic Press, New York and London.