Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 5 (2015), pp. 3265-3274

http://www.ripublication.com/gjpam.htm

A note on higher order mixed-type special polynomials associated with Boole polynomials

Jongkyum Kwon
Department of Mathematics,
Kyungpook National University,
Daegu,702-701, Republic of Korea.
E-mail: mathkjk26@hanmail.net

Jin-Woo Park[1]
Department of Mathematics education,
Daegu University, Gyeongsan,
Kyungsangbukdo, 712-714, Republic of Korea.
E-mail: a0417001@knu.ac.kr

Abstract

In this paper, we consider various higher order special mixed-type polynomials which are related to Bernoulli, Euler, Daehee, Changhee associated with Boole polynomials. From those polynomials, we investigate some interesting properties of higher order special mixed-type polynomials related to Boole polynomials.

AMS Subject Classification: 11S80, 11B68, 05A30.
Keywords: higher order mixed-type special polynomials, higher order Boole polynomials.

1. Introduction

Let p be an odd prime number. $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \dfrac{1}{p}$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$.

[1]Corresponding author.

If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad ([4,9,12]). \tag{1.1}$$

and the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad ([1,3,7]). \tag{1.2}$$

From (1.1), we have

$$I_0(f_1) = I_0(f) + f'(0), \quad ([2\text{-}7,10]). \tag{1.3}$$

where $f_1(x) = f(x+1)$.

By using iterative method, we get

$$I_0(f_n) = I_0(f) + \sum_{i=0}^{n-1} f'(i). \tag{1.4}$$

where $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$.

From (1.2), we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \quad ([7,11]). \tag{1.5}$$

where $f_1(x) = f(x+1)$.

By using iterative method, we get

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{l=0}^{n-1} (-1)^{n-l-1} f(l), \tag{1.6}$$

where $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$.

As is known,the Boole polynomials of order r are defined by the generating function to be

$$\left(\frac{1}{1+(1+t)^{\lambda}}\right)^r (1+t)^x = \sum_{n=0}^{\infty} Bl_n^{(r)}(x|\lambda)\frac{t^n}{n!}, \quad ([13,14]) \tag{1.7}$$

The Bernoulli polynomials of order r are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \tag{1.8}$$

We recall that the Euler polynomials of order r are defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}, \tag{1.9}$$

The Daehee polynomials of order r are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}, \tag{1.10}$$

Finally, the Changhee polynomials of order r are defined by the generating function to be

$$\left(\frac{2}{t+2}\right)^r (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(r)}(x)\frac{t^n}{n!}, \tag{1.11}$$

The *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l. \tag{1.12}$$

Note that

$$(1+t)^x = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}C_l^{(r)}(x)D_{n-l}^{(r)}\right)\frac{t^n}{n!}. \tag{1.13}$$

and

$$(1+t)^x = \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}. \tag{1.14}$$

From (1.13) and (1.14), we have

$$(x)_n = \sum_{l=0}^{n}\binom{n}{l}C_l^{(r)}(x)D_{n-l}^{(r)}. \tag{1.15}$$

Special polynomials are very important in mathematics. Because of their remarkable properties, special polynomials have been used for centuries. For instance, since they have numerous applications in complex analysis and mathematical physics and astronomy, trigonometric functions have been studied for over a thousand years. Even the series expansions for sine and cosine were known to Madhava in the fourteen century. These series were rediscovered by Newton and Leibniz in the seventeen century. Since then, the theory of special polynomials has been continuously developed with contributions by a host of mathematicians, including Euler, Bernoulli, Legendre, Laplace, Gauss, Kummer, Eisenstein, Riemann, Jacobian, Chebyshev, Ramanujan, and so on.

In the past years,the development of new special polynomials and applications of special polynomials to new areas of mathematics have initiated a resurgence of interest in the numerical analysis and number theory,combinatorics,quantum field theory, and so on (see [1-14]). Moreover, in recent years, the various generalizations of the familiar special polynomials have been defined by using p-adic q-integral on $\mathbb{Z}_p$ and p-adic fermionic q-deformed integrals on $\mathbb{Z}_p$ introduced and investigated by Kim (see [2, 4, 5-8]).

In this paper, we consider various special higher order mixed-type polynomials which are related to Bernoulli, Euler, Daehee, Changhee associated with Boole polynomials of order r. From those polynomials, we investigate some interesting properties of special higher order mixed-type polynomials.

2. The higher order mixed-type special polynomials related to Boole polynomials

Let us consider the higher order Boole-Bernoulli mixed-type polynomials of order (r, s) as follows:

$$BlB_n^{(r,s)}(x|\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} B_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r), \quad (2.1)$$

where $Bl^{(r)}(x)$ is Boole polynomials of order r and $B_n^{(s)}(x)$ is Bernoulli polynomials of order s.

Then, we can find the generating function of $BlB_n(x|\lambda)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} BlB_n^{(r,s)}(x|\lambda)\frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} B_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x)\frac{t^n}{n!} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \left(\frac{t}{e^t - 1}\right)^s \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{(\lambda x_1 + \cdots + \lambda x_r + x)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \left(\frac{t}{e^t - 1}\right)^s \left(\frac{1}{1+(1+t)^{\lambda}}\right)^r (1+t)^x.
\end{aligned} \quad (2.2)$$

From(1.13), we get

$$\left(\frac{t}{e^t - 1}\right)^s \left(\frac{1}{1+(1+t)^{\lambda}}\right)^r (1+t)^x = \sum_{n=0}^{\infty} \left(B_n^{(s)} Bl_n^{(r)} \sum_{l=0}^{n} \binom{n}{l} C_l^{(r)}(x) D_{n-l}^{(r)} \right) \frac{t^n}{n!}. \quad (2.3)$$

Note that

$$B_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l}^{(s)} (\lambda x_1 + \cdots + \lambda x_r + x)^l, \quad (2.4)$$

From (2.1) and (2.4), we get

$$\begin{aligned}
BlB_n^{(r,s)}(x|\lambda) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} B_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} \lambda^l B_{n-l}^{(s)} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\frac{x}{\lambda} + x_1 + \cdots + x_r)^l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} \lambda^l B_{n-l}^{(s)} E_l^{(r)}(\frac{x}{\lambda}).
\end{aligned} \tag{2.5}$$

Therefore, by (2.5), we obtain the following theorem.

Theorem 2.1. For $n \in \mathbb{N}$, we have

$$BlB_n^{(r,s)}(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l} \lambda^l B_{n-l}^{(s)} E_l^{(r)}(\frac{x}{\lambda}).$$

Now, we consider the higher order Boole-Euler mixed-type polynomials of order (r, s) as follows:

$$BlE_n^{(r,s)}(x|\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} E_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r), \tag{2.6}$$

where $Bl^{(r)}(x)$ is Boole polynomials of order r and $E_n^{(s)}(x)$ is the Euler polynomials of order s.

Then, we can find the generating function of $BlE_n^{(r,s)}(x|\lambda)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} BlE_n^{(r,s)}(x|\lambda) \frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} E_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) \frac{t^n}{n!} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \left(\frac{2}{e^t + 1}\right)^s \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{(\lambda x_1 + \cdots + \lambda x_r + x)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \left(\frac{2}{e^t + 1}\right)^s \left(\frac{1}{1 + (1+t)^\lambda}\right)^r (1+t)^x.
\end{aligned} \tag{2.7}$$

From (1.13), we get

$$\left(\frac{2}{e^t + 1}\right)^s \left(\frac{1}{1 + (1+t)^\lambda}\right)^r (1+t)^x = \sum_{n=0}^{\infty} \left(E_n^{(s)} Bl_n^{(r)} \sum_{l=0}^{n} \binom{n}{l} C_l^{(r)}(x) D_{n-l}^{(r)} \right) \frac{t^n}{n!}. \tag{2.8}$$

Note that

$$E_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l}^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x)^l, \tag{2.9}$$

From (2.6) and (2.9), we get

$$\begin{aligned} BlE_n^{(r,s)}(x|\lambda) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} E_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \sum_{l=0}^{n} \binom{n}{l} \lambda^l E_{n-l}^{(s)} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\frac{x}{\lambda} + x_1 + \cdots + x_r)^l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \sum_{l=0}^{n} \binom{n}{l} \lambda^l E_{n-l}^{(s)} E_l^{(r)}(\frac{x}{\lambda}). \end{aligned} \tag{2.10}$$

Therefore, by (2.10), we obtain the following theorem.

Theorem 2.2. For $n \in \mathbb{N}$, we have

$$BlE_n^{(r,s)}(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l} \lambda^l E_{n-l}^{(s)} E_l^{(r)}(\frac{x}{\lambda}).$$

Let us consider the higher order Boole-Daehee mixed-type polynomials of order (r, s) as follows:

$$BlD_n^{(r,s)}(x|\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} D_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r). \tag{2.11}$$

where $Bl^{(r)}(x)$ is Boole polynomials of order r and $D_n^{(s)}(x)$ is the Daehee polynomials of order s.

Then, we can find the generating function of $BlD_n^{(r,s)}(x|\lambda)$ as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} BlD_n^{(r,s)}(x|\lambda) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} D_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) \frac{t^n}{n!} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \left(\frac{log(1+t)}{t}\right)^s \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_r + x)_l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \left(\frac{log(1+t)}{t}\right)^s \left(\frac{1}{1+(1+t)^{\lambda}}\right)^r (1+t)^x. \end{aligned} \tag{2.12}$$

Note that

$$D_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) = \sum_{l=0}^{n} \binom{n}{l} D_{n-l}^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x)_l, \tag{2.13}$$

and

$$\begin{aligned}
\left(\frac{log(1+t)^s}{t}\right)(1+t)^x &= \left(\frac{log(1+t)}{e^{log(1+t)}-1}\right)^s e^{xlog(1+t)} \\
&= \sum_{m=0}^{\infty} B_m^{(s)}(x) \frac{(log(1+t))^m}{m!} \\
&= \sum_{m=0}^{\infty} B_m^{(s)}(x) \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} B_m^{(s)}(x) S_1(n,m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.14}$$

From (2.11) and (2.13) and (2.14), we get

$$\begin{aligned}
BlD_n^{(r,s)}(x|\lambda) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} D_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} D_{n-l}^{(s)} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_r + x)_l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \sum_{l=0}^{n} \binom{n}{l} D_{n-l}^{(s)} Bl_l^{(r)}(x|\lambda).
\end{aligned} \tag{2.15}$$

and

$$BlD_n^{(r,s)}(x|\lambda) = \sum_{m=0}^{n} B_m^{(s)}(x) S_1(n,m) Bl_l^{(r)}(x|\lambda). \tag{2.16}$$

Therefore, by (2.15) and (2.16), we obtain the following theorem.

Theorem 2.3. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
BlD_n^{(r,s)}(x|\lambda) &= \sum_{l=0}^{n} \binom{n}{l} D_{n-l}^{(s)} Bl_l^{(r)}(x|\lambda) \\
&= \sum_{m=0}^{n} B_m^{(s)}(x) S_1(n,m) Bl_l^{(r)}(x|\lambda).
\end{aligned} \tag{2.17}$$

Now, we consider the higher order Boole-Changhee mixed-type polynomials of order (r, s) as follows:

$$BlCh_n^{(r,s)}(x|\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} Ch_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r). \tag{2.18}$$

where $Bl^{(r)}(x)$ is Boole polynomials of order r and $Ch_n^{(s)}(x)$ is the Changhee polynomials of order s.

Then, we can find the generating function of $BlCh_n^{(r,s)}(x|\lambda)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} BlCh_n^{(r,s)}(x|\lambda)\frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} Ch_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x)\frac{t^n}{n!} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \left(\frac{2}{t+2}\right)^s \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_r + x)_l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \left(\frac{2}{t+2}\right)^s \left(\frac{1}{1+(1+t)^\lambda}\right)^r (1+t)^x.
\end{aligned} \tag{2.19}$$

Note that

$$Ch_n^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x) = \sum_{l=0}^{n} \binom{n}{l} Ch_{n-l}^{(s)}(\lambda x_1 + \cdots + \lambda x_r + x)_l, \tag{2.20}$$

and

$$\begin{aligned}
\left(\frac{2}{t+2}\right)^s (1+t)^x &= \left(\frac{2}{e^{log(1+t)}+1}\right)^s e^{xlog(1+t)} \\
&= \sum_{m=0}^{\infty} E_m^{(s)}(x)\frac{(log(1+t))^m}{m!} \\
&= \sum_{m=0}^{\infty} E_m^{(s)}(x) \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} E_m^{(s)}(x) S_1(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.21}$$

From (2.18) and (2.20) and (2.21), we get

$$\begin{aligned}
&BlCh_n^{(r,s)}(x|\lambda)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} Ch_n^{(s)}(\lambda x_1+\cdots+\lambda x_r+x)\frac{t^n}{n!}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&=\sum_{l=0}^{n}\binom{n}{l}Ch_{n-l}^{(s)}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda x_1+\cdots+\lambda x_r+x)_l d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&=\sum_{l=0}^{n}\binom{n}{l}Ch_{n-l}^{(s)}Bl_l^{(r)}(x|\lambda).
\end{aligned}\tag{2.22}$$

and

$$BlD_n^{(r,s)}(x|\lambda)=\sum_{m=0}^{n}E_m^{(s)}(x)S_1(n,m)Bl_l^{(r)}(x|\lambda).\tag{2.23}$$

Therefore, by (2.22) and (2.23), we obtain the following theorem.

Theorem 2.4. For $n\in\mathbb{N}$, we have

$$\begin{aligned}
BlCh_n^{(r,s)}(x|\lambda)&=\sum_{l=0}^{n}\binom{n}{l}Ch_{n-l}^{(s)}Bl_l^{(r)}(x|\lambda)\\
&=\sum_{m=0}^{n}E_m^{(s)}(x)S_1(n,m)Bl_l^{(r)}(x|\lambda).
\end{aligned}\tag{2.24}$$

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