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On the zeros of a family of polynomials and an application in integer sequences

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Abstract

The paper deals with polynomials of the form

$$f(x) = x^m - \epsilon_1 a_1 x^{m-1} - \epsilon_2 a_2 x^{m-2} - \ldots - \epsilon_{m-1} a_{m-1} x - \epsilon_m a_m ,$$

where $\epsilon_i \in \{-1, 1\}$ for $i = 1, 2, 3, \ldots, m$. It is shown that for any positive integers $a_1, a_2, \ldots, a_m$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ and $m \in \mathbb{N}$ with $m \geq 2$, $f(x)$ has unique real zero outside the unit disk $|z| \leq 1$. It is also presented in this paper that the zeros of the polynomials can be applied in the study of integer sequences.

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1 Introduction

A polynomial $f(x)$ is of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_i \in \mathbb{R}$ $(i = 0, 1, 2, \ldots, n)$ are coefficients and x is the independent variable. If $n \neq 0$, then we say that the polynomial $f(x)$ is of order n.

Polynomials are studied in various fields of mathematics and lead to interesting questions. For example, if we are given a particular polynomial $f(x)$ having real coefficients, can we find any real zeros? If so, how many and where in the real number line can we locate them? Are they positive or negative? In dealing with problems involving roots of polynomials, we use the Fundamental Theorem of Algebra (FTA). The FTA tells us that any non-constant polynomials with complex coefficients have complex roots. It is formally stated as follows:

Theorem 1.1 (Fundamental Theorem of Algebra). *Given any positive integer* $n \geq 1$ *and any choice of complex numbers* $\alpha_0, \alpha_1, \ldots, \alpha_n$ *such that an* $\alpha_i \neq 0$, *the polynomial equation*

$$\alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n = 0$$

has at least one solution $z \in \mathbb{C}$.

For the proof of this remarkable statement, see Fine and Rosenberger [5]. It may be helpful also if we give a bound for the zeros to narrow down our search for zeros. Fortunately, there are many existing methods in the literature to answer this query. Two good examples are the Descartes' Rule of Signs [13] and the Budan-Fourier Theorem [3]:

Theorem 1.2 (Descartes' Rule of Signs). *Let* $f(x) = \alpha_0 x^n + \alpha_1 x^{n-1} + \cdots + \alpha_{n-1} x + \alpha_n$ *be a polynomial with real coefficients* α_i, $i = 0, 1, \ldots, n$, *and* α_n *and* α_0 *be nonzero. Let* v *be the number of changes of signs in its sequence of coefficients* $\alpha_0, \alpha_1, \ldots, \alpha_n$ *and p be the number of its real positive zeros, counted with their orders of multiplicity. Then there exists a nonnegative integer m such that*

$$p = v - 2m.$$

Theorem 1.3 (Budan-Fourier Theorem). *Let* $f(x)$ *be a non-constant polynomial of degree* n *with real coefficients. Let* $c \in \mathbb{R}$ *and* $v(c)$ *be the number of changes of signs in the sequence* $f(c), f'(c), f''(c), \ldots, f^{(n)}(c)$. *Then the number of zeros of* f *in the interval* $(a;\ b]$, *counted with their orders of multiplicity, is equal to*

$$v(a) - v(b) - 2m$$

for some $m \in \mathbb{R}$.

The complex zeros of any polynomial $f(x)$, however, usually cannot be avoided in the discussion of polynomials. So the zeros are placed on the complex plane and most of the time we also give a bound to narrow down our search for zeros. One good method for giving bounds is that of Lagrange and MacLaurin [16]:

Theorem 1.4 (Lagrange-Maclaurin Method). *Suppose* $\alpha_{n-k} < 0, k \geq 1$, *is the first of the negative coefficients of the polynomial* f *defined in Theorem 1.2. An upper bound* U *for the set of positive zeros of* f *may be given by*

$$U = 1 + \sqrt[k]{-\frac{B}{\alpha_0}},$$

where B *is the largest absolute value of the negative coefficients of the polynomial* $f(x)$, *and* $\sqrt[k]{-\frac{B}{\alpha_0}} = -\frac{B}{\alpha_0}$ *for k = 1.*

Recently, there are several methods that are developed, which are based on Theorem 1.2. For instance, in [12], Sagrallof and Mehlorn described a variant of the Descartes' method that isolates the real zeros of any real square-free polynomial through the so-called *coefficient oracles*. In [14], Nickalls presented a new bound for zeros of polynomials when all of them are real numbers. He described an upper bound having the property of being exact in the case where zeros are of $n - 1$ multiplicity. On the other hand, Dehmer and Mowshowitz [4] developed methods for establishing improved bounds on the moduli of the zeros of complex and real polynomials. Jain [6] also introduced an improved version of the Cauchy's classical bound for zeros of polynomials. Other related results for computing bounds for zeros of polynomials can be found in [2], [11], [15], [16], and references therein.

2 Main Results

The present paper is concerned with the existence of a unique real zero of certain families of polynomials outside the unit disk $|z| \leq 1$. This work is an extension of the following result by Wu and Zhang [17]:

Lemma 2.1. *Let $a_1, a_2, \dots, a_m$ be positive integers with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ and $m \in \mathbb{N}$ with $m \geq 2$. Then, the polynomial*

$$f(x) = x^m - a_1 x^{m-1} - a_2 x^{m-2} - \dots - a_{m-1} x - a_{m,} \tag{1}$$

(i) has exactly one positive real zero x^+ with $a_1 < x^+ < a_1 + 1$, and
(ii) has $m - 1$ zeros lying within the unit circle in the complex plane.

An example of a polynomial that satisfies the above lemma is the polynomial $x^2 - 2x - 1$, wherein $a_1 = 2$ and $a_2 = a_m = 1$. In this example, we observe that $a_1 = 2 \geq a_2 = 1 \geq 1$. The roots of the polynomial are $1 + \sqrt{2}$ and $1 - \sqrt{2}$. As one can see, the root $1 + \sqrt{2}$ satisfies conclusion (i) of the lemma, and the number $1 - \sqrt{2}$ is the only zero of the polynomial lying in the unit circle, thus satisfying conclusion (ii) of the lemma.

We now present our main result.

Theorem 2.2. *Let $a_1, a_2, \dots, a_m$ be positive integers with*

$$a_1 \geq a_2 \geq \cdots \geq a_m > 1 \tag{2}$$

and $m \in \mathbb{N} \backslash \{1\}$. Let the m^{th}-order polynomial f be defined by

$$f(x) = x^m - \epsilon_1 a_1 x^{m-1} - \epsilon_2 a_2 x^{m-2} - \dots - \epsilon_{m-1} a_{m-1} x - \epsilon_m a_m , \tag{3}$$

where $\epsilon_i \in \{-1, 1\}$ for $i = 1,2,3, \dots, m$. Then,

(i) *either*

(a) *if* $\epsilon_1 = -1$ *and the* ϵ_i*'s alternate in signs for* $i = 1, 2, \ldots, m$, $f(x)$ *has one negative real zero* x^- *with* $-(a_1+1) < x^- < -a_1$; *or*

(b) *if* $\epsilon_1 = 1$ *and* ϵ_i*'s alternate in signs for* $i = 1, 2, \ldots, m$, $f(x)$ *has one positive real zero* x^+ *with* $a_1 - 1 < x^+ < a_1$; *or*

(c) *if* $\epsilon_i = -1$ *for all* $i = 1, 2, \ldots, m$, $f(x)$ *has one negative real zero* x^- *with* $-a_1 < x^- < -(a_1 - 1)$;

and

(ii) *the other* $m - 1$ *zeros of* $f(x)$ *lie within the unit circle in the complex plane.*

Proof. We only prove (a) in (i), and (ii). The proofs for cases (b) and (c) are similar so we omit them.

Let $\epsilon_1 = -1$ and ϵ_i's alternate in signs for $i = 1, 2, \ldots, m$. Hence, $\epsilon_i = (-1)^i$ for $i = 1, 2, \ldots, m$. So we can write (3) in compact form as follows:

$$f(x) = x^m - \sum_{i=1}^{m} (-1)^i a_i x^{m-i}. \tag{4}$$

Now, for any positive integers $a_1 \geq a_2 \geq \cdots \geq a_m > 1$ and $m \in \mathbb{N}\backslash\{1\}$ we have

$$\begin{aligned} f(-a_1) &= (-a_1)^m - (-a_1)^m - \sum_{i=2}^{m} (-1)^i a_i x^{m-i} \\ &= (-1)^{m+1}(a_2 a_1^{m-2} + a_3 a_1^{m-3} + \cdots + a_{m-1}a_1 + a_m). \end{aligned}$$

Clearly, $f(-a_1) < 0$ if m is even and $f(-a_1) > 0$ otherwise. We also have

$$f\big(-(a_1+1)\big) = (-1)^m \left((a_1+m)^m - \sum_{i=1}^{m} a_i (a_1+1)^{m-i} \right). \tag{5}$$

Here, we have two possibilities for m. If m is even then by applying (2) and a formula for finite geometric series we can show that (5) is positive:

$$\begin{aligned} f\big(-(a_1+1)\big) &= (a_1+1)^m - \sum_{i=1}^{m} a_i (a_1+1)^{m-i} > (a_1+1)^m - a_1 \sum_{i=1}^{m} (a_1+1)^{m-i} \\ &= (a_1+1)^m - a_1 \cdot \frac{(a_1+1)^m - 1}{a_1} = 1 > 0. \end{aligned}$$

On the other hand, if m is odd then

$$\begin{aligned} f\big(-(a_1+1)\big) &= -(a_1+1)^m + \sum_{i=1}^{m} a_i (a_1+1)^{m-i} \\ &< -(a_1+1)^m + a_1 \cdot \frac{(a_1+1)^m - 1}{a_1} = -1 < 0. \end{aligned}$$

These imply that, in any case, there exists a negative real zero x^- of $f(x)$ with $-(a_1+1) < x^- < -a_1$. According to Descarte's rule of signs, the polynomial equation $f(x) = 0$ has at $\lfloor m/2 \rfloor$ negative real roots. Now we show that we have exactly one negative real root x^- in $(-(a_1+1), -a_1)$ and that there are no other negative real roots outside the interval. To do this, we use Budan-Fourier's theorem. To make things simpler, we first find a lower bound for the negative real root x^- of $f(x) = 0$ using the Lagrange-Maclaurin's method. We have already shown that $x^- > -(a_1+1)$. We will show that $-(a_1+1)$ is indeed the greatest lower bound for x^-. To prove this, we consider the transformed equation $g(x) = 0$ where $g(x) \equiv x^m f(-x)$, and denote the upper bound of its positive roots to be U_g. This means that we can obtain a lower bound for the negative real root of $f(x) = 0$, i.e. if x^- is the negative real root of $f(x) = 0$ then $x^- \geq -U_g$. We now write g as

$$g(x) = x^m\left((-x)^m - \sum_{i=1}^{m}(-1)^i a_i(-x)^{m-i}\right) = (-1)^m x^m\left(x^m - \sum_{i=1}^{m} a_i x^{m-i}\right).$$

If m is even then $g(x) = x^{2m} + (-a_1)x^{2m-1} - \sum_{i=2}^{m} a_i x^{2m-i}$. An upper bound for the positive root of $g(x) = 0$ is given by

$$U_g = 1 + \sqrt[m-1]{a_1} < 1 + a_1, \;\; m \geq 2,$$

where equality holds for $m = 2$. But, for $m = 2$, we have $f(x) = x^2 + a_1 x - a\_2$ whose roots are given by $x^{\pm} = (-a_1 \pm \sqrt{a_1^2 + 4a_2})/2$. Note that for $a_1 \geq a_2 > 1$ we have $x^- = (-a_1 + \sqrt{a_1^2 + 4a_2}\,)/2 < 0$. It can be verified easily that $x^- > -(a_1+1)$. Now if m is odd then $g(x) = -x^{2m} + \sum_{i=1}^{m} a_i x^{2m-i}$. Hence,

$$U_g = 1 - \sqrt[m]{a_1} < 1, \;\; m \geq 2.$$

It follows that, in any case, the negative real root $x^- > -(a_1+1)$. This implies that there is no other negative real root in $\left(-\infty, -(a_1+1)\right)$.

Now we claim that there is only one negative real root in $(-(a_1+1), 0)$. To verify this, we let $v(c)$ be the number of changes of signs in the sequence

$$f(c),\; f'(c),\; f''(c), \ldots, f^{(m)}(c),$$

where $c \in \{-(a_1+1), 0\}$. We first show that

$$v\big(-(a_1+1)\big) = m \;\text{ and }\; v(0) = m-1.$$

Denote the k^{th} derivative of $f(x)$ as $D^k(f(x))$: At $x = -(a_1+1)$ we have

$$D^k\big(f(x)\big)\Big|_{x=-(a_1+1)} = D^k\left(x^m - \sum_{i=1}^{m}(-1)^i a_i x^{m-i}\right)\Bigg|_{x=-(a_1+1)}. \quad (6)$$

We will show that, for a fixed positive integer m, the derivatives given by (6) alternate in signs for $k = 0,1,2,\dots,m$. We only prove the case when m is even since the case for m being odd is similar. We proceed using induction. For k = 0, we have $f\big(-(a_1+1)\big) > 0$ and for $k = 1$ we have

$$\begin{aligned} D^1\big(f(x)\big)\Big|_{x=-(a_1+1)} &= (-1)^{m-1}\left(m(a_1+1)^{m-1} - \sum_{i=1}^{m-1}(m-i)a_i(a_1+1)^{m-i-1}\right) \\ &\le -m(a_1+1)^{m-1} + ma_1\sum_{i=1}^{m-1}(a_1+1)^{m-i-1} \\ &= -m(a_1+1)^{m-1} + ma_1\left(\frac{(a_1+1)^{m-1}-1}{a_1}\right) = -m < 0. \end{aligned}$$

Next, we show that

$$D^n\big(f(x)\big)\Big|_{x=-(a_1+1)} \begin{cases} > 0 & \text{if } n \text{ is even} \\ < 0 & \text{if } n \text{ is odd} \end{cases},$$

for some natural number $n \le m$. First, suppose that n is even. Then,

$$\begin{aligned} D^n\big(f(x)\big)\Big|_{x=-(a_1+1)} &= \left(\frac{m!}{(m-n)!}x^{m-n} - \sum_{i=1}^{m-n}\frac{(m-i)!}{(m-n-i)!}(-1)^i a_i x^{m-n-i}\right)\Bigg|_{x=-(a_1+1)} \\ &= (-1)^{m-n}\left(\frac{m!}{(m-n)!}(a_1+1)^{m-n} - \sum_{i=1}^{m-n}\frac{(m-i)!}{(m-n-i)!}a_i(a_1+1)^{m-n-i}\right) \\ &\ge \frac{m!}{(m-n)!}(a_1+1)^{m-n} - \frac{m!}{(m-n)!}a_1\left(\frac{(a_1+1)^{m-n}-1}{a_1}\right) \\ &= \frac{m!}{(m-n)!} > 0. \end{aligned}$$

On the other hand, if n is odd then

$$\begin{aligned} D^n\big(f(x)\big)\Big|_{x=-(a_1+1)} &\le -\frac{m!}{(m-n)!}(a_1+1)^{m-n} + \frac{m!}{(m-n)!}a_1\left(\frac{(a_1+1)^{m-n}-1}{a_1}\right) \\ &= -\frac{m!}{(m-n)!} < 0. \end{aligned}$$

These show that $v\big(-(a_1+1)\big) = m$. On the other hand, it can be verified easily that $v(0) = m-1$, since for any positive integer m, $D^n\big(f(x)\big)\big|_{x=0}$ alternates in sign for $n = 0,1,2,\dots,m-1$ and $D^m\big(f(x)\big)\big|_{x=0} > 0$. Thus, by Budan-Fourier's theorem, the number of zeros of f in the interval $(-(a_1+1), 0]$, counted with their orders of multiplicity is equal to

$$v\big(-(a_1+1)\big) - v(0) - 2(0) = m - (m-1) - 0 = 1,$$

proving (a).

Now, we proceed to prove section (ii) of Theorem 2.2. Again we assume m to be even. A similar proof can be given for an odd m. From (a) of Theorem 2.2(i), there follows:

(2.7a) If $x \in \mathbb{R}$ such that $x < x^-$ then $f(x) > 0$, and

(2.7b) If $x \in \mathbb{R}$ such that $x^- < x < 0$, then $f(x) < 0$.

Let $h(x) = -(x+1)f(x)$. By applying (4) and re-indexing the series, we obtain the following:

$$h(x) = -x^{m+1} + \sum_{i=1}^{m}(-1)^i a_i x^{(m+1)-i} - x^m + \sum_{i=1}^{m}(-1)^i a_i x^{m-i}$$

$$= -x^{m+1} - (a_1+1)x^m + \sum_{i=1}^{m-1}(-1)^i (a_i - a_{i+1}) x^{m-i} + a_{m.} \quad (7)$$

Since $f(x)$ has exactly one negative real zero x^-, $h(x)$ has two negative real zeros, that is, x^- and -1. Observe that

"If $x \in \mathbb{R}$ such that $x < x^-$ then $h(x) > 0$," and

"If $x \in \mathbb{R}$ such that $x^- < x < 0$, then $h(x) < 0$." (8)

To complete the proof of (ii), it is sufficient to show that there are no zeros of f that lie on and outside of the unit circle.

Claim 1: The polynomial equation $f(x) = 0$ has no complex root z_1 with $-|z_1| < x^-$.

Proof of Claim 1. Assume the contrary that there exists such z_1. So, we have

$$f(z_1) = {z_1}^m - \sum_{i=1}^{m}(-1)^i a_i {z_1}^{m-i} = 0.$$

Then, by triangle inequality, we obtain

$$|z_1^m| \le \sum_{i=1}^{m} a_i |z_1^{m-i}|. \quad (9)$$

Note that

$$f(-|z_1|) = (-|z_1|)^m - \sum_{i=1}^{m}(-1)^i a_i (-|z_1|)^{m-i}.$$

$$= |z_1|^m - \sum_{i=1}^{m} a_i |z_1|^{m-i}.$$

Using (9) we get $f(-|z_1|) \le 0$. This contradicts (8).

Claim 2: $f(x) = 0$ has no complex root z_2 that satisfies $x^- < -|z_2| < -1$.

Proof of Claim 2. Assume the contrary, that is, assume that there exists a z_2 such that $f(z_2) = 0$. Then by (7) we obtain

$$h(z_2) = -z_2^{m+1} - (a_1+1)z_2^m + \left(\sum_{i=1}^{m-1}(-1)^i(a_i - a_{i+1})z_2^{m-i}\right) + a_m = 0.$$

This implies that

$$(a_1+1)|z_2^m| \leq |z_2^{m+1}| + \left(\sum_{i=1}^{m-1}(a_i - a_{i+1})z_2^{m-i}\right) + a_m$$

We note that $a_i - a_{i+1} \geq 0$ for all $i = 1, 2, \ldots, m$. Hence,

$$h(-|z_2|) = -(-|z_2|)^{m+1} - (a_1+1)(-|z_2|)^m + \left(\sum_{i=1}^{m-1}(-1)^i(a_{i+1} - a_i)(-|z_2|)^{m-i}\right) + a_m$$
$$= |z_2|^{m+1} - (a_1+1)|z_2|^m + \left(\sum_{i=1}^{m-1}(a_{i+1} - a_i)|z_2|^{m-i}\right) + a_m \geq 0.$$

This contradicts (8).

Claim 3: On the circles $|z_3| = -x^-$ and $|z_3| = 1$, $f(x)$ has the unique zero x^-.

Proof of Claim 3. If $f(z_3) = 0$; then

$$h(z_3) = -z_3^{m+1} - (a_1+1)z_3^m + \left(\sum_{i=1}^{m-1}(-1)^i(a_i - a_{i+1})z_3^{m-i}\right) + a_m = 0.$$

This implies that

$$(a_1+1)|z_3^m| \leq |z_3^{m+1}| + \left(\sum_{i=1}^{m-1}(a_i - a_{i+1})z_3^{m-i}\right) + a_m. \tag{10}$$

If $z_3 = x^-$ or $z_3 = -1$, then $h(z_3) = 0$, so (10) must be an equality. Therefore,

$$z_3^{m+1}, (a_1 - a_2)z_3^{m-1}, (a_2 - a_3)z_3^{m-2}, \ldots, (a_{m-1} - a_m)z_3^{m-i}, \text{ and } a_m$$

all lie on the same ray starting from the origin. Since a_m and $(a_i - a_{i+1}) \in \mathbb{R}^+$ for all $i = 1, 2, \ldots, m-1$, then so are z_3^{m+1}, z_3^{m-1}, z_3^{m-2}, …, z_3. Thus $f(z_3) \in \mathbb{R}^+$. On the circles $|z_3| = -x^-$ and $|z_3| = 1$, there are two conditions $z_3 = -1$ or $z_3 = x^-$. Since $f(-1) \neq 0$, then x^- is the unique zero of $f(x)$, proving our claim.

These three claims prove (ii).

Thus, $f(x)=x^m-\sum_{i=1}^{m}(-1)^i a_i x^{m-i}$ has exactly one negative real zero x^- satisfying the condition $-(a_1+1)<x^-<-a_1$ and its other zeros are found inside the unit circle.

■

To illustrate Theorem 2.2, we consider the polynomial x^2+3x-2. In this example, $a_1=3>2=a_2$, $\epsilon_1==-1$, and ϵ_1 and ϵ_2 alternate in signs. Computing for the zeros of the polynomial, we get $x=\frac{-3\pm\sqrt{17}}{2}$. One observes that there's a negative real zero, i.e., $x^-=\frac{-3-\sqrt{17}}{2}$ and this zero lies between $-4=-(a_1+1)$ and $-3=-a_1$. The other zero, which is $\frac{-3+\sqrt{17}}{2}$, lies inside the unit circle in complex plane.

Remark. One may notice from Theorem 2.2 the strict inequality $a_1>1$. The exclusion of 1 for the values of a_i's is due to the fact that the polynomial

$$f(x)=x^m+\sum_{i=1}^{m}(-1)^i x^{m-i}$$

a self-reciprocal polynomial (cf. [1], [7], [8], [9], [10]) so the derivative

$$f'(x)=mx^{m-1}+\sum_{i=1}^{m-1}(m-i)(-1)^i x^{m-i-1}$$

has all its zeros in the closed disk $|z|\leq 1$ and hence, by a result of Cohn [2], all its zeros lie on the unit circle. On the other hand, the results for (a) and (ii) still hold when a_i's = 1 for all $i=1,2,\dots,m$ and we leave the proof to the reader.

3 An application on integer sequences

One good application of our result is on the study of integer sequences. Consider for instance the m^{th}-order recursive sequence $\{u_n\}_{n=0}^{\infty}$ satisfying the recurrence equation defined by

$$u_n=-a_1u_{n-1}+a_2u_{n-2}-a_3u_{n-3}+\cdots+(-1)^m a_{n-m}u_m, \tag{11}$$

with initial values $u_i\in\mathbb{N}$ for $0\leq i<m$ and at least one of them is not zero. Clearly, the characteristic polynomial of (11) is given by (4). With this connection, we can easily obtain the following Lemma.

Lemma 3.1. *Let $m\geq 2$ and let $\{u_n\}_{n=0}^{\infty}$ be an integer sequence satisfying the recurrence formula (7). Then the closed formula of u_n is given by*

$$u_n=c\alpha^n+O(d^{-n})\qquad(n\to\infty), \tag{12}$$

where $c>0, d>1,$ and $\alpha\in-(-(a_1+1),-a_1)$ is the negative real zero of $f(x)$.

Proof. Let $\alpha, \alpha_1, \alpha_2, \dots, \alpha_t$ be the distinct roots of $f(x) = 0$; where $f(x)$ is the characteristic polynomial of (11), which is given by (4). From (a) of Theorem 2.2 we know that α is the simple root of $f(x) = 0$. Let r_j be the corresponding multiplicity of the root α_j for each $j = 1,2,3,\dots,t$. From the properties of m^{th}- order linear recursive sequences, u_n can be expressed as follows (cf. [17]):

$$u_n = c\alpha^n + \sum_{i=1}^{t} P_i(n)\alpha_i^n, \tag{13}$$

where $P_i(n) \in \mathbb{R}[n]$, $\deg P_i(n) = r_i - 1$, $r_1 + r_2 + \cdots + r_t = m - 1$, and $c \in \mathbb{R}$. From (ii) of Theorem 2.2 we have $|\alpha_i| < 1$ for $1 \le i \le t$. Since each α_i of the second term in (9) goes to 0 as $n \to \infty$, we can find constants $M, d \in \mathbb{R}$ with $d > 1$ for $n > n_0$ such that

$$\left|\sum_{i=1}^{t} P_i(n)\alpha_i^n\right| \le \sum_{i=1}^{t} |P_i(n)\alpha_i^n| \le Md^{-n},$$

completing the proof of the Lemma.

■

With the help of the previous Lemma, we can easily verify the following results.

Theorem 3.2. *Let $\{u_n\}$ be an m^{th}- order sequence defined by (11) with the restriction $a_1 \ge a_2 \ge \cdots \ge a_m > 1$. For any positive real number $\beta > 2$, there exists a positive integer n_1 such that*

$$\left\lfloor \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{u_k}\right)^{-1} \right\rfloor = u_n - u_{n-1}, \quad (n \ge n_1).$$

Letting $\beta \to +\infty$ in Theorem 3.2 we can immediately deduce the following.

Corollary 3.3. *Let $\{u_n\}$ be an m^{th}- order sequence defined by (11) with the restriction $a_1 \ge a_2 \ge \cdots \ge a_m > 1$. Then there exists a positive integer n_2 such that*

$$\left\lfloor \left(\sum_{k=n}^{+\infty} \frac{1}{u_k}\right)^{-1} \right\rfloor = u_n - u_{n-1}, \quad (n \ge n_2).$$

One may follow the proofs of Theorem 1 and Corollary 1 in [17] to prove the above results.

References

[1] Y. Choo, *On the Zeros of a Family of Self-Reciprocal Polynomials*, Int. Journal of Math. Analysis, **5**, 2011, no. 36, 1761-1766.

[2] A. Cohn, *Über die Anzahl der Wurzeln diner algebraischen Gleichung in einem Kreise*, Math. Z., **14**, 1922, 110-148.

[3] N. B. Conkwright, *An elementary proof of the Budan-Fourier theorem*, Amer. Math. Monthly, **50**, 1943, no. 10, 603-605.

[4] M. Dehmer and A. Mowshowitz, *Bounds on the Moduli of Polynomial Zeros*, pre-print.

[5] B. Fine and G. Rosenberger, *The Fundamental Theorem of Algebra*, Undergraduate Texts in Mathematics. Springer-Verlag New York, 1997.

[6] V. K. Jain, *On Cauchy's Bound for Zeros of a Polynomial*, The Mathematical Gazette, **30**, 2006, 95-100.

[7] S.H. Kim, *The zeros of certain family of self-reciprocal polynomials*, Bull. Korean Math. Soc., **44**, 2007, 461-473.

[8] P. Lakatos, *On zeros of reciprocal polynomials*, Publ. Math. Debrecen, **61**, 2002, no. 3-4, 645-661.

[9] P. Lakatos and L. Losonczi, *On zeros of reciprocal polynomials of odd degree*, J. Inequal. Pure Appl. Math., **4**, Art. 60 2003, no. 3, 8 pp.

[10] P. Lakatos and L. Losonczi, *Self-inversive polynomials whose zeros are on the unit circle*, Publ. Math. Debrecen, **65**, 2004, 409-420.

[11] J. M. McNamee and M. Olhovsky, *A Comparison of a Priori Bounds on (Real or Complex) Roots of Polynomials*, pre-print.

[12] K. Mehlorn and M. Sagrallof, *Computing Real Roots of Real Polynomials - An Efficient Method Based on Descartes' Rule of Signs and Newton Iteration*, Retrived from arXiv:1308.4088v1 [cs.SC] 19 Aug 2013.

[13] M. Mignotte, Mathematics for Computer Algebra, Springer-Verlag, New York, NY, 1992.

[14] R. W. D. Nickalls, *A new bound for polynomials when all the roots are real*, The Mathematical Gazette, **95**, 2011, no. 534, 520-526.

[15] S. M. Rump, *Ten methods to bound multiple roots of polynomials*, J. of Computation and Applied Mathematics, **156**, 2003, 403-432.

[16] P. S. Vigklas, *Upper bounds on the values of the positive roots of polynomials*, PhD Thesis, University of Thessaly, Volos, Greece, 2010.

[17] Z. Wu and H. Zhang, *On the reciprocal sums of higher-order sequences*, Adv. Diff. Equ., **2013**, 2013:189.