Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 5 (2015), pp. 3217-3228

http://www.ripublication.com

Optimal Investment Strategy for Asset and Liability Management Problems under the Ho-Lee Interest Rate Model and Quadratic Utility Function

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Abstract

Under the assumption that the risk-free interest rate dynamics follows the Ho-Lee interest rate model, we investigate the optimal investment strategy for portfolio selection with liability. Under the utility maximization criterion, the maximum principle is used to obtain the HJB equation for the value function, and we study the optimal investment strategies under quadratic utility. Finally, we obtain the closed-form solutions for the optimal investment strategies by applying the variable transformation approach.

Keywords: Ho-Lee model; asset and liability management; maximum principle; HJB equation; quadratic utility; optimal investment strategy

1. Introduction

Liability has a significant impact on asset allocation in many asset management institutions, such as in pension fund, bank, and insurance company. In recent years, studies on Asset - liability management have made some achievements. Sharpe and Tint [1] first present a single-period mean-variance asset-liability management model

*This research is supported by China West Normal University special funding for basic scientific research business expenses(no.14C004),Social Science Programming general Program of Nan Chong city(no.NC2013B027).The Foundation of China West Normal University (Grant No. 11A028,11A029), the Fundamental Research Funds of China West Normal University (Grant No. 13D016) and the Natural Science Foundation of Sichuan Provincial Education Department (Grant No. 14ZB0142).

and show that the optimal investment decision of a company should take its liabilities into account. Later, Kell and Muller [2] find that liabilities do have a significant effect on the efficient frontier. The role of liability has as well been studied by Leippold et al. [3] in a multiperiod setting by using the geometric approach and the embedding technique of Li and Ng [4]. Chiu and Li [5] generalize the mean-variance asset-liability management problem to a continuous-time setting and derive the analytical optimal portfolio strategy as well as the optimal initial funding ratio by using the stochastic LQ control. Xie and Li [6] consider the continuous-time mean-variance asset-liability management problem where the liability follows the Brownian motion with drift and derive the optimal strategy. Yi et al. [7] consider the impact of the uncertain exit time on the optimal asset-liability management strategy. Xie et al. [8] extend the model of Xie and Li [6] by assuming that the liability follows a geometric Brownian motion in a continuous-time incomplete market. Papi and Sbaraglia [9] studied the restrictive asset-liability management issues with the application of the principle of dynamic programming. These studies all make a contribution to enriching and developing the application of the portfolio theory in the asset - liability management to some extent , but the interest rates in the model are constant or determine the function of time. However, this assumption does not accord with the actual investment environment of investors and investment institutions . In this paper, the constant interest rate of asset - liability management issues has been extended to stochastic interest rate, therefore, on the basis of which optimal investment decision under quadratic utility function has been Studied.

As a matter of fact, in most of the real-world situations, stochastic interest rate is an important factor which the investors should take into consideration. In recent years, many scholars have introduced stochastic interest rates into portfolio selection problems and have achieved some research results. For instance, Deelstra et al. [10] applied martingale approach to portfolio selection problems with the Cox-Ingersoll-Ross model and obtained the optimal investment strategy for power utility maximization in explicit form. Korn and Kraft [11] concerned the Ho-Lee model and the Vasicek model and applied dynamic programming to derive the optimal portfolios. Gao [12] and Josa-Fombellida [13] introduced stochastic interest rates into the management of pension funds respectively. However, the literatures have mentioned before only study the issue of portfolio investment with stochastic interest rate while ignore liabilities. It is without doubt that in financial practices, the investment process is often accompanied by liabilities. The introduction of liabilities will make the continuous time case of optimal portfolio selection model more practical.

In this paper, we introduce the liability of [14] of into the market model of [11]. The interest rate is assumed to be subject to interest rate Ho-Lee model stochastic process. The introduction of liability and stochastic interest rate make the market model fitter for the reality on the one hand, and also make it more operational and targeted, which can provide scientific basis for investors and investment institutions. Under the utility maximization criterion, the maximum principle is used to obtain the HJB equation for the value function, and we study the optimal investment strategies under quadratic utility. Finally, we obtain the closed-form solutions for the optimal investment strategies by applying the variable transformation approach.

This paper is organized as follows: Section 2 presents the general problem framework and exposes the financial market structure. Section 3 applies dynamic programming principle to obtain a linear second-order partial different equation. In Section 4, we choose quadratic utility for our analysis and use variable change technique to obtain the optimal portfolios. In Section 5, we conclude the paper.

2. Problem Formulation

Throughout this paper we denote by $E(\cdot)$ the expectation operator, $[0,T]$ the fixed and finite time horizon of the investment. Assumed that $(W_1(t), W_2(t))$ is a 2-dimensional standard adapted and independent Brownian motion on a filtered complete probability space $(\Omega, \Im, (\Im_t)_{0\le t\le T}, p)$, where $(\Im_t)_{0\le t\le T}$ is generated by $(W_1(t), W_2(t))$.

We consider a financial market where three assets are traded continuously over $[0,T]$. One of the assets is a savings account with price $B(t)$ at time t, whose price process $B(t)$ satisfies the following differential equation

$$dB(t) = B(t)r(t)dt$$

Where $r(t)$ denotes the short rate which can be interpreted as the annualized as interest for the infinitesimal period $[t, t+dt]$, and $r(t)$ is supposed to be stochastic process.

In this paper a short rate is assumed to be driven by the stochastic differential equation

$dr(t) = a(t)dt + bdW_2(t)$

and as explicit examples we will consider the Ho-Lee model given by $a(t) = \tilde{a}(t) + b\zeta(t)$, where $b > 0$ are constants and $\tilde{a}(t), \zeta(t)$ are assumed to be deterministic and continuous function of the time t.

The second asset is a stock with the price $S(t)$ at time t, whose price process $S(t)$ is given by

$dS(t) = S(t)[u_1(t)dt + \sigma_1(t)dW_1(t) + \sigma_2(t)dW_2(t)]$

Where $u_1(t), \sigma_1(t), \sigma_2(t)$ are supposed to be deterministic and Borel-measurable bounded functions on $[0, T]$.

The third asset is a bond with maturity $T_1 > T$, whose price dynamics $P(t)$ follows

$dP(t) = P(t)[r(t)dt + \zeta(t)\sigma_3(t)dt + \sigma_3(t)dW_2(t)]$

Where $\zeta(t)$ is the deterministic function, we have $\sigma_3(t) = -b(T_1 - t)$ in the Ho-Lee model [11].

Suppose the investor has an initial wealth $w > 0$ and the initial liabilities $l(l \in ¡\ )$ at time $t = 0$, so investors were net initial wealth $x_0 = u - l > 0$ at time $t = 0$. The investor's accumulative liability at time t is denoted by $L(t)$, and $L(t)$ satisfies the following SDE:

$dL(t) = c(t)dt + d_1(t)dW_1(t) + d_2(t)dW_2(t)$

Where $d_1(t)$ is the volatility of stock prices caused by liabilities, $d_2(t)$ is the volatility of interest rates caused liabilities, $c(t), d_1(t), d_2(t)$ are assumed to be deterministic and continuous function of the time t.

Assume that the investor invests the market value of his wealth $\pi_1(t)$ and $\pi_2(t)$ into the stock and the bond at time t respectively, $t \in [0, T]$, and let $\pi(t) = (\pi_1(t), \pi_2(t))$. Clearly, the amount invested in the risk-free asset satisfies $X(t)-\pi_1(t)-\pi_2(t)$, in which $X(t)$ represents the net wealth of the investor at time t .Suppose that short-selling of stocks and borrowing at the interest rate of the bond is allowed and there is neither transaction cost nor consumption. The wealth process $X(t)$ corresponding to trading strategy $\pi(t)$ is subject to the following equation

$$dX(t)=(X(t)-\pi_1(t)-\pi_2(t))\frac{dB(t)}{B(t)}+\pi_1(t)\frac{dS(t)}{S(t)}+\pi_2(t)\frac{dP(t)}{P(t)}-dL(t)$$

i.e.,

$$\begin{aligned} dX(t)=&[r(t)X(t)+\pi_1(t)(u_1(t)-r(t))+\pi_2(t)\zeta(t)\sigma_3(t)-c(t)]dt \\ &+[\pi_1(t)\sigma_1(t)-d_1(t)]dW_1(t)+[\pi_1(t)\sigma_2(t)+\pi_2(t)\sigma_3(t)-d_2(t)]dW_2(t) \end{aligned}$$

The set of all admissible investment strategy $\pi(t)$ is denoted by $\Gamma=\{\pi(t):0\le t\le T\}$. Suppose that the investor's objective is to find a portfolio $\pi(t)$ such that the expected utility of terminal wealth is maximized. Mathematically, the portfolio selection for utility maximizing is formulated as

$$\max_{\pi(t)\in\Gamma} EU(X(t)) \tag{1}$$

where utility function $U(\cdot)$ satisfies the first-order derivative $U'(\cdot)>0$ and the second order derivative $U''(\cdot)<0$. The properties of convex function known the existence and uniqueness of the optimal investment strategy $\pi(t)=(\pi_1(t),\pi_2(t))$ investors expected utility of terminal wealth maximum value.

3. The Optimal Investment Strategy

The maximum principle is used to obtain the HJB equation for the value function, and we study the optimal investment strategies under quadratic utility. Finally, we obtain the closed-form solutions for the optimal investment strategies by applying the variable transformation approach.

We define the value function $H(t,r,x)$ as

$$H(t,r,x)=\max_{\pi(t)\in\Gamma}E[U(X(t))\,|\,X(t)=x,r(t)=r]$$

with terminal condition $H(T,r,x)=U(x)$.

In order to make the article more readability, we omit the symbol (t) in the following expressions.

According to dynamic programming principle, we obtain the following HJB equation

$$H_t+(rx-c)H_x+aH_r+\frac{1}{2}b^2H_{rr}+\sup_{\pi(t)\in\Gamma}F(\pi,t,x)=0 \quad (2)$$

Where $H_t,H_r,H_{rr},H_x,H_{xx},H_{rx}$ represent the partial derivatives of the value function $H(T,r,x)$ with respect to variables t,r,x respectively.

$$F(\pi,t,x)=\pi_1(u_1-r)H_x+\pi_2\zeta\sigma_3H_x+\frac{1}{2}(\pi_1\sigma_1-d_1)^2H_{xx}$$
$$+\frac{1}{2}(\pi_1\sigma_2+\pi_2\sigma_3-d_2)^2H_{xx}+b(\pi_1\sigma_2+\pi_2\sigma_3-d_2)H_{xr}$$

The optimal conditions satisfy

$$\begin{cases}\dfrac{\partial F(\pi,t,x)}{\partial\pi_1}=(u_1-r)H_x+(\pi_1\sigma_1-d_1)\sigma_1H_{xx}+(\pi_1\sigma_2+\pi_2\sigma_3-d_2)\sigma_2H_{xx}+b\sigma_2H_{xr}=0\\ \dfrac{\partial F(\pi,t,x)}{\partial\pi_2}=\zeta\sigma_3H_x+(\pi_1\sigma_2+\pi_2\sigma_3-d_2)\sigma_3H_{xx}+b\sigma_3H_{xr}=0\end{cases}$$

Solving the equations gets the optimal solution

$$\pi_1^*=\frac{\zeta\sigma_2-(u_1-r)}{\sigma_1^2}\frac{H_x}{H_{xx}}+\frac{d_1}{\sigma_1} \quad (3)$$

$$\pi_2^*=-[\frac{\zeta}{\sigma_3}+\frac{\sigma_2}{\sigma_3}\cdot\frac{\zeta\sigma_2-(u_1-r)}{\sigma_1^2}]\frac{H_x}{H_{xx}}-\frac{b}{\sigma_3}\frac{H_{xr}}{H_{xx}}+\frac{\sigma_1d_2-\sigma_2d_1}{\sigma_1\sigma_3} \quad (4)$$

Putting (3), (4) into HJB equation (2), we obtain

$$H_t+rxH_x+(K_2(t)-c)H_x+aH_r+\frac{1}{2}b^2H_{rr}+K_1(t)\frac{H_x^2}{H_{xx}}-b\zeta\frac{H_xH_{xr}}{H_{xx}}-\frac{1}{2}b^2\frac{H_{xr}^2}{H_{xx}}=0 \quad (5)$$

Where

$$K_1(t)=-\frac{1}{2}[\frac{\zeta\sigma_2-(u_1-r)}{\sigma_1}]^2-\frac{1}{2}\zeta^2\ ,\ K_2(t)=\frac{d_1}{\sigma_1}(u_1-r-\zeta\sigma_2)+\zeta d_2$$

The wealth process $X(t)$ corresponding to trading strategy π_1^*,π_2^* is subject to the following equation

$$dX(t)=[r(t)X(t)-[\zeta^2+(\frac{\zeta\sigma_2-(u_1-r)}{\sigma_1^2})^2]\frac{H_x}{H_{xx}}-\zeta b\frac{H_{xr}}{H_{xx}}+\frac{\zeta(\sigma_1 d_2-\sigma_2 d_1)}{\sigma_1}+\frac{d_1(u_1-r)}{\sigma_1}]dt$$
$$+[\pi_1(t)\sigma_1(t)-d_1(t)]dW_1(t)+[\pi_1(t)\sigma_2(t)+\pi_2(t)\sigma_3(t)-d_2(t)]dW_2(t) \quad (6)$$

4. Quadratic utility

Assume that the expression of quadratic utility is of the form $U(x)=x-wx^2, x<\frac{1}{2w}$ and $w>0$, and equations (4) are solved in such a utility function.

We assume that the value function has the following structure

$$H(t,r,x)=P(t,r)x-wQ(t,r)x^2+D(t,r) \quad (7)$$

with terminal condition $P(T,r)=1, Q(T,r)=1, D(T,r)=0$, where $P(t,r), Q(t,r), D(t,r)$ is the unknown function ,our goal is to find the analytical expression $P(t,r), Q(t,r)$ and $D(t,r)$ so that (7) is the equation (5) Solutions.

The partial derivatives are following

$$H_t=P_t x-wQ_t x^2+D_t \ , \ H_x=P-2wQx \ , \ H_{xr}=P_r-2wQ_r x$$

$$H_{xx}=-2wQ \ , \ H_r=P_r x-wQ_r x^2+D_r \ , \ H_{rr}=P_{rr}x-wQ_{rr}x^2+D_{rr}$$

Plugging $H_t, H_r, H_{rr}, H_x, H_{xx}, H_{xr}$ into (5) get

$$-xw[Q_t+(2r+2K_1(t))Q+\frac{1}{2}b^2Q_{rr}+(a-2b\zeta)Q_r-b^2\frac{Q_r^{\ 2}}{Q}]+[P_t+(r+2K_1(t))P$$
$$-(K_2(t)-c)2wQ+\frac{1}{2}b^2P_{rr}+(a-b\zeta)P_r-b\zeta\frac{PQ_r}{Q}-b^2\frac{P_rQ_r}{Q}]+[D_t+(K_2(t)-c)P$$
$$+aD_r+\frac{1}{2}b^2D_{rr}-\frac{K_1(t)}{2w}\frac{P^2}{Q}+\frac{b\zeta}{2w}\frac{PP_r}{Q}+\frac{b}{4w}^2\frac{P_r^2}{Q}]=0$$

We obtain $P(t,r), Q(t,r)$ and $D(t,r)$ satisfy the following second order partial differential equations respectively.

$$\begin{cases} Q_t+(2r+2K_1(t))Q+\frac{1}{2}b^2Q_{rr}+(a-2b\zeta)Q_r-b^2\frac{Q_r^{\ 2}}{Q}=0 \\ Q(T,r)=1 \end{cases} \quad (8)$$

$$\begin{cases} P_t+(r+2K_1(t))P-(K_2(t)-c)2wQ+\frac{1}{2}b^2P_{rr}+(a-b\zeta)P_r-b\zeta\frac{PQ_r}{Q}-b^2\frac{P_rQ_r}{Q}=0 \\ P(T,r)=1 \end{cases} \quad (9)$$

$$\begin{cases} D_t + (K_2(t)-c)P + aD_r + \frac{1}{2}b^2 D_{rr} - \frac{K_1(t)}{2w}\frac{P^2}{Q} + \frac{b\zeta}{2w}\frac{PP_r}{Q} + \frac{b}{4w}^2\frac{P_r^2}{Q} = 0 \\ D(T,r) = 0 \end{cases} \quad (10)$$

The solution of the equation (8)-(10) satisfies the following three theorems

Theorem1 Assume that the solution to (8) is $Q(t,r) = e^{A(t)+B(t)r}$,with terminal condition $A(T)=0,$ and $B(T)=0$, then

$$B(t) = 2(T-t), \quad A(t) = \int_t^T (a-2b\zeta)B(s) - \frac{1}{2}b^2 B(s) + 2K_1(s)ds$$

Proof Plugging $Q(t,r) = e^{A(t)+B(t)r}$ into (8) get

$$Q[(B'+2)r + A'(t) + (a-2b\zeta)B(t) - \frac{1}{2}b^2 B^2(t) + 2K_1(t)] = 0$$

Letting $B'+2=0$ and noting that $B(T)=0$, we have $B(t)=2(T-t)$. $A(t)$ satisfies the following equation

$$\begin{cases} A'(t) + (a-2b\zeta)B(t) - \frac{1}{2}b^2 B^2(t) + 2K_1(t) = 0 \\ A(T) = 0 \end{cases}$$

Its solution is

$$A(t) = \int_t^T (a-2b\zeta)B(s) - \frac{1}{2}b^2 B(s) + 2K_1(s)ds$$

Theorem2 Assume that the solution to (9) is $P(t,r) = F(t)e^{E(t)r}$, with terminal condition $F(T)=1$ and, $E(T)=0$ then

$$F(t) = e^{\int_t^T K_3(s)ds} - \int_t^T (K_2(s)-c)2we^{A(t)}e^{-\int_s^t K_3(z)dz}, E(t) = B(t)$$

Where

$$K_3(t) = 2K_1(t) - r - \frac{1}{2}b^2 B^2(t) + aB(t) - 2b\zeta D(t)B(t)$$

Proof Plugging $Q(t,r) = e^{A(t)+B(t)r}$ into (9) get, we have

$$e^{(E(t)-B(t))r}[F'(t)+F(t)E'(t)r+(r+2K_1(t))F(t)+(a-b\zeta)F(t)E(t)+\frac{1}{2}b^2F(t)E^2(t)$$
$$-b\zeta F(t)B(t)-b^2F(t)E(t)B(t)]=(K_2(t)-c)2we^{A(t)}$$

Noting that $E(t)=B(t)$ and theorem1, we have $E(t)=B(t)=2(T-t)$. So $F(t)$ satisfies

$$\begin{cases} F'(t)+[(2K_1(t)-r-\frac{1}{2}b^2B^2(t)+aB(t)-2b\zeta B(t)]F(t)=(K_2(t)-c)2we^{A(t)} \\ F(T)=1 \end{cases}$$

Solving the equation get

$$F(t)=e^{\int_t^T K_3(s)ds}-\int_t^T (K_2(s)-c)2we^{A(t)}e^{-\int_s^t K_3(z)dz}$$

Where

$$K_3(t)=2K_1(t)-r-\frac{1}{2}b^2B^2(t)+aB(t)-2b\zeta B(t)$$

Theorem3 Assume that the solution to (10) is $D(t,r)=G(t)e^{H(t)r}$, with terminal condition $G(T)=0$ and, $H(T)=0$ then

$$G(t)=-\int_t^T K_4(s)e^{-\int_s^t K_5(z)dz}ds$$

Where

$$K_4(s)=\frac{K_1(t)}{2w}\frac{F(t)^2}{e^{A(t)}}-\frac{b\zeta}{2w}\frac{E(t)F(t)^2}{e^{A(t)}}-\frac{b^2}{4w}\frac{E(t)^2F(t)^2}{e^{A(t)}}-(K_2(t)-c)F(t)$$

$$K_5(s)=2r-(aH(t)+\frac{1}{2}b^2B^2(t))$$

Proof Plugging $Q(t,r)=e^{A(t)+B(t)r}$, $P(t,r)=F(t)e^{E(t)r}$ and $D(t,r)=G(t)+H(t)r$ into(10)get, we have

$$e^{(H(t)-E(t))r}[G'(t)+G(t)H'(t)r+aG(t)H(t)+\frac{1}{2}b^2G(t)H^2(t)]$$
$$=\frac{K_1(t)}{2w}\frac{F(t)^2}{e^{A(t)}}-\frac{b\zeta}{2w}\frac{E(t)F(t)^2}{e^{A(t)}}-\frac{b^2}{4w}\frac{E(t)^2F(t)^2}{e^{A(t)}}-(K_2(t)-c)F(t)$$

Noting that $H(t)=E(t)$ and theorem 1, we have $E(t)=2(T-t)$. So $G(t)$ satisfies

$$\begin{cases} G'(t)-G(t)[2r-(aH(t)+\frac{1}{2}b^2B^2(t))]=\frac{K_1(t)}{2w}\frac{F(t)^2}{e^{A(t)}}-\frac{b\zeta}{2w}\frac{E(t)F(t)^2}{e^{A(t)}}-\frac{b^2}{4w}\frac{E(t)^2F(t)^2}{e^{A(t)}}-(K_2(t)-c)F(t) \\ G(T)=0 \end{cases}$$

Solving the equation get

$$G(t) = -\int_t^T K_4(s) e^{-\int_s^t K_5(z)dz} ds$$

Where

$$K_4(s) = \frac{K_1(t)}{2w}\frac{F(t)^2}{e^{A(t)}} - \frac{b\zeta}{2w}\frac{E(t)F(t)^2}{e^{A(t)}} - \frac{b^2}{4w}\frac{E(t)^2F(t)^2}{e^{A(t)}} - (K_2(t)-c)F(t)$$

$$K_5(s) = 2r - (aH(t) + \frac{1}{2}b^2B^2(t))$$

Theorem4 If the utility function is

$U(x) = x - wx^2, x < \frac{1}{2w}$ and $w > 0$,

the optimal portfolio of the problem (1) is given by

$$\pi_1^* = \frac{\zeta\sigma_2 - (u_1 - r)}{\sigma_1^2}x + [\frac{d_1}{\sigma_1} - \frac{\zeta\sigma_2 - (u_1-r)}{\sigma_1^2}\frac{P}{2wQ}] \quad (11)$$

$$\pi_2^* = -[\frac{\zeta}{\sigma_3} + \frac{\sigma_2}{\sigma_3}\cdot\frac{\zeta\sigma_2 - (u_1-r)}{\sigma_1^2} + \frac{b}{\sigma_3}B(t)]x + [\frac{\zeta}{\sigma_3} + \frac{\sigma_2}{\sigma_3}\cdot\frac{\zeta\sigma_2 - (u_1-r)}{\sigma_1^2} + \frac{b}{\sigma_3}B(t)]\frac{P}{2wQ} + \frac{\sigma_1 d_2 - \sigma_2 d_1}{\sigma_1\sigma_3} \quad (12)$$

and wealth optimal expectation

$$EX(t) = X(0)e^{\int_0^t K_6(s)ds} + \int_0^t K_7(s) e^{\int_0^t K_6(z)dz} ds$$

Where P, Q are determined by Theorem 1 and Theorem 2

$$K_6(t) = r(t) - (\zeta^2 + (\frac{\zeta\sigma_2 - (u_1-r)}{\sigma_1^2})^2) - \zeta bB(t)$$

$$K_7(t) = (\zeta^2 + (\frac{\zeta\sigma_2 - (u_1-r)}{\sigma_1^2})^2)\frac{P}{2wQ} + \zeta bB(t)\frac{P}{2wQ} + \frac{\zeta(\sigma_1 d_2 - \sigma_2 d_1)}{\sigma_1} + \frac{d_1(u_1-r)}{\sigma_1}$$

Proof By theorem 1 and theorem 3 have

$$\frac{H_x}{H_{xx}} = x - \frac{P}{2wQ}, \qquad \frac{H_{xr}}{H_{xx}} = B(t)x - \frac{B(t)P}{2wQ}$$

Plugging Theorem 1 to Theorem3 conclusions into (3) and (4) get, we have the optimal investment strategies under quadratic utility

$$\pi_1^* = \frac{\zeta\sigma_2 - (u_1 - r)}{\sigma_1^2}x + [\frac{d_1}{\sigma_1} - \frac{\zeta\sigma_2 - (u_1-r)}{\sigma_1^2}\frac{P}{2wQ}]$$

$$\pi_2^* = -[\frac{\zeta}{\sigma_3} + \frac{\sigma_2}{\sigma_3}\cdot\frac{\zeta\sigma_2 - (u_1-r)}{\sigma_1^2} + \frac{b}{\sigma_3}B(t)]x + [\frac{\zeta}{\sigma_3} + \frac{\sigma_2}{\sigma_3}\cdot\frac{\zeta\sigma_2 - (u_1-r)}{\sigma_1^2} + \frac{b}{\sigma_3}B(t)]\frac{P}{2wQ} + \frac{\sigma_1 d_2 - \sigma_2 d_1}{\sigma_1\sigma_3}$$

Under the optimal investment strategy, the expected value of the wealth process satisfies

$$dEX(t) = [[r(t) - (\zeta^2 + (\frac{\zeta\sigma_2 - (u_1 - r)}{\sigma_1^2})^2) - \zeta bB(t)]EX(t) + (\zeta^2 + (\frac{\zeta\sigma_2 - (u_1 - r)}{\sigma_1^2})^2)\frac{P}{2wQ}$$
$$+\zeta bB(t)\frac{P}{2wQ} + \frac{\zeta(\sigma_1 d_2 - \sigma_2 d_1)}{\sigma_1} + \frac{d_1(u_1 - r)}{\sigma_1}]dt$$

Solving the equation get

$$EX(t) = X(0)e^{\int_0^t K_6(s)ds} + \int_0^t K_7(s)e^{\int_s^t K_6(z)dz}ds$$

Where

$$K_6(t) = r(t) - (\zeta^2 + (\frac{\zeta\sigma_2 - (u_1 - r)^2}{\sigma_1^2}) - \zeta bB(t)$$

$$K_7(t) = (\zeta^2 + (\frac{\zeta\sigma_2 - (u_1 - r)}{\sigma_1^2})^2)\frac{P}{2wQ} + \zeta bB(t)\frac{P}{2wQ} + \frac{\zeta(\sigma_1 d_2 - \sigma_2 d_1)}{\sigma_1} + \frac{d_1(u_1 - r)}{\sigma_1}$$

5 . Conclusions

Under the assumption that the risk-free interest rate dynamics follows the Ho-Lee interest rate model, we investigate the optimal investment strategy for portfolio selection with liability. Under the utility maximization criterion, the maximum principle is used to obtain the HJB equation for the value function, and we study the optimal investment strategies under quadratic utility. Finally, we obtain the closed-form solutions for the optimal investment strategies.

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