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Spline Based Computational Techniques for Second, Third and Fourth Order Boundary Value Problems

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Abstract

The present paper provides a synthesis of the published research works on computational methods cored in splines to solve second, third and fourth order boundary value problems of physical significance. These problems are encountered in a variety of fields of engineering and applied mathematics. The research works published in referred journals during the last five years are accumulated here and aimed to fabricate a collection of important research articles in order to solve various boundary value problems with enhanced perceptive regarding the current scenario.

Keywords: Boundary Value Problems (BVPs), Cubic Spline, Quartic Spline, Quintic Spline, etc.

1. Introduction

Differential equations are mathematically deliberated from numerous different perspectives in various fields of pure and applied Mathematics. These disciplines are concerned with the properties of differential equations of various types. These differential equations, emerging in a wide variety of application areas are sculpted by the boundary value problems (BVPs). Due to enormous applications of BVPs, faster and accurate solution of these has been always a very mesmerizing subject matter for researchers. During the last five decades, many researchers have been developed various numerical techniques to solve various order of boundary value problems. Due to the great usefulness and remarkable convergence properties of spline functions, the numerical solution of boundary value problems using splines has been growing interest for researchers in developing and using highly accurate numerical methods.

The present communication deliberates the topical works paid out in same area. By accumulating a range of research articles, we will develop an insight into the methods based on application of spline functions for the solution of various order boundary value problems of physical importance. The scope of this paper is limited to the articles associated to the solution of second, third and fourth orders BVPs using numerical methods based on various splines.

The rest of this article is divided into four sections. In next section 2 talks about the applications of these boundary value problems in diverse fields. Section 3 presents a summary of these spline techniques for solving second, third and fourth order boundary value problems from a range of research articles in chronological order. At last, section 4 presents major conclusions.

2. Boundary Value Problems Arise in Diverse Disciplines

Boundary value problems for ordinary differential equations arise in various fields of Mathematics, Physical and Engineering Sciences and encounter a system of differential equations of various orders with different types of boundary conditions. Many problems are formulated mathematically as BVPs for second order differential equations such as in heat transfer, optimal control, deflection in cables and plates, vibration of springs, electric circuits and in a number of other scientific applications. A variety of problems are converted in boundary value problems for third order differential equations such as in physical oceanography, in the frame work of variational inequality theory, in the deflection of a curved beam, a three-layer beam, electromagnetic waves or gravity driven flows and so on. Generally, fourth order boundary value problems take place in the mathematical modelling of viscoelastic and inelastic flows, deformation of beams and plates deflection theory and so on.

3. Spline Solution of Lower Order Boundary Value Problems

Following is the collection of some important published work, which illustrates the chronological advancement of various splines during last five years to solve different classes of boundary value problems.

Spline Solution of Second order Boundary Value Problems

Authors in [1], concentrated on the numerical solution of special class of fractional boundary value problems of order two. The method of solution is based on a spline analysis in addition to conjugating collocation combined with shooting method. A theoretical analysis about the existence and uniqueness of exact solution for the present class is proven. Examples involving Bagley–Torvik equation subject to boundary conditions are also presented. Numerical results illustrate the accuracy of the present method and seem encouraging for solution of proposed second order fractional boundary value problems.

Authors in [2], deal with the method to develop a numerical algorithm using non-polynomial quintic spline functions for the solution of the following second order two-point boundary value problem associated with heat transfers

$$y^{(2)} + f(x)y = g(x) \tag{1}$$

subject to boundary conditions given by

$$y^{(1)}(a) - A_1 = y^{(1)}(b) - A_2 = 0 \tag{2}$$

where A_i, i = 1, 2 are finite real constants and the functions f, g $\in C[a, b]$. Using non-polynomial quintic spline functions of class $C^4 \in [a, b]$, second and fourth order boundary formula are obtained. They presented examples to demonstrate the comparative performance of the non-polynomial quintic algorithm over the non-polynomial quadratic spline method used in [3]. Maximum observed errors are computed in absolute values for examples that evidence the better approximation by the present method in the sense of accuracy and computational work.

Authors in [4], looked for a method based on cubic B-spline collocation with finite element approach to solve second-order non-linear eigen value problem that is well-known Troesch's problem given by

$$u'' = \lambda \sinh \lambda u\,,$$

$$u(0) = 0, u(1) = 1, \tag{3}$$

where u = u(x), $0 \leq x \leq 1$.Here, authors applied the use of shape functions defined by

$$\varphi(x) = \sum_{i=-3}^{n-1} a_i\, \varphi_i(x) \tag{4}$$

satisfying the above boundary conditions, where a_i's are unknown real coefficients and $\varphi_i(x)$ are cubic B-spline functions. Substituting the solution given in the shape function, the above differential equation (3) can be modeled as a system of equations and also written in the matrix form. This nonlinear system of equations can be solved using the computer algebra system Maple. Authors discussed the lack of conventional methods (finite differences method) and B-spline collocation method over a uniform mesh to solve the above Troesch's problem. Authors also talked about the drawbacks and inconveniences of using these methods. Moreover for large value of eigenvalue λ, a piecewise-uniform shishkin-type mesh was used. Adaptive collocation approach was implemented over a non-uniform mesh for special cases of proposed problem. They used the B-spline collocation method to solve the problem for different values of the parameter λ. The solution was obtained for different nodal points and tabular results were shown. Result was compared with other existing numerical results and found more accurate with respect to a wide range of cases. Authors justified the competency of present proposal with the advantage of using computers to find numerical solutions of (3) for different cases and extended to more general classes of problems. The convergence analysis for different values of the parameter was done to illustrate the effectiveness of the current scheme.

Authors in [5] used the three point difference method based on non-polynomial spline for finding the solution of the following non-linear second order ordinary differential equation:

$$y''(\mathrm{x}) = \mathrm{F}\ (\mathrm{x}, y,\ y'), \mathrm{y}(0) = y_a\ , \mathrm{y}(1) = y_b;\ 0 \leq\ \mathrm{x}\ \leq\ 1 \tag{5}$$

where $\mathrm{F}(\mathrm{x}, y, y')$ is twice continuously differentiable in the region. Here, authors present an effective framework by making use of the difference equations in addition with the Taylor's expansion for the solution of linear and nonlinear singular second order ODE. Authors recommended the usage of two parameter alternating group explicit (TAGE) and Newton-TAGE method used in {[6] & [7]} to familiarize more accurate method for the solution of nonlinear singular second order ODE. To demonstrate the efficiency and accuracy of the proposed numerical scheme, numerical examples for linear and nonlinear cases were solved. Supremacy of the method was proven over SOR and Newton-SOR method in terms of error, execution time and CPU performance. Results show the practical applicability of the presented method. Method is fourth and second order accurate according the selection of parameters and can also be proposed for multi-dimensional cases.

Authors in [8], employed extended cubic uniform B-spline technique to solve the following class of singular boundary value problems

$$y''(x) + \frac{k}{x}\ p(x)y'(x) + r(x)y(x) = f(x), 0\ < x\ < 1 \tag{6}$$

with boundary conditions

$$y'(0) =\ 0, y(1) =\ \beta \tag{7}$$

This technique has superiority over ordinary B-spline that it introduces one supplementary free parameter. Many examples were solved by applying the extended cubic uniform B-spline. Authors considered the same examples solved by different methods as that of references {[9], [10] & [11]}. The obtained numerical results for each problem are presented in tabular forms and compared with the exact solutions and the methods used in {[9], [10] & [11]}. The accuracy of the method was also tested by calculating the error norms. It is observed that present method gives better result than available in literature and with the flexibility of extensions; the approximations of the solution can be done by adjusting the free parameter.

Authors in [12] presented an efficient numerical method based on quartic (fourth-degree) B-spline basis function. The scheme deals with the following class of second-order singular boundary value problems:

$$x^{-k}\,(x^k\,y')' = f(x,y)\ , 0\ < x\ < 1;$$

$$y'(0) = 0, y(1) = \beta \qquad (8)$$

Using Gauss–Jordan Elimination Method to the linear system, coefficients, C_i, $i = -2, -1, \ldots, n+1$, are obtained through optimization. Error analysis is discussed. Several examples have been constructed, illustrating the competence of the method. Authors compared their results with the results produced by higher order finite difference method [13], B-spline method [14], Cubic spline method [15] and also result was compared with the exact solution. However, it was observed that their approach produce better numerical solutions.

Authors in [16], discussed the single sweep alternating group explicit (SWAGE) and Newton-SWAGE iteration methods to solve the non-linear second order differential equation subject to given natural boundary conditions along with third order cubic spline numerical method on a geometric mesh. The convergence of the SWAGE iteration method is discussed in detail. They compared the results of proposed SWAGE and Newton-SWAGE iterative methods with the corresponding TAGE and Newton-TAGE iterative methods to demonstrate computationally the efficiency of the proposed method. This method includes very less number of iterations, as the SWAGE method requires only one sweep to solve the problem. For uniform mesh, the order of the method was found nearly four. This approach has added advantage that it is applicable to both singular and non-singular problems. Since this method is competitive in its principle and gives a required accuracy to solve the one-dimensional problems, this encourages the authors to solve multi-dimensional problems too.

Authors in [17], developed a numerical method by using non-polynomial spline method for solving following second order two point boundary value problems

$$y''(x) + p(x)y' + q(x)y = r(x) \qquad (9)$$

with boundary conditions

$$y(x_0) = \gamma_1\ , y(x_n) = \gamma_2 \qquad (10)$$

To obtain the result, authors discretized the anticipated problem at the knots and used finite difference method to approximate first derivative. Examples were considered for the numerical illustration of the method developed for different values of at subintervals and calculated in Maple 13. The obtained numerical results were compared with some other existing method and exact solution. Results were found satisfactory and converge to the exact solution when h goes to zero.

Authors in [18], features the development of a numerical technique that is based on quintic non-polynomial spline for the solution of non- linear second order boundary value problems. They employed the present technique to solve some problems arising in population dynamics, first order equation in chemical reactor, heat transfer and convection diffusion problems. Sixth order finite difference method and hyperbolic spline finite difference methods are used to achieve the desired accuracy. It provides continuous approximations for y and dy/dx and gives convergent solution to the problems related with logistic equation in population dynamics and chemical reactor theory. Proposed method is sixth order accurate and can be extended to nonlinear elliptic problems. The supremacy of the two parameter hyperbolic spline finite difference method over single parameter and classical finite difference method has been demonstrated by implementing on examples. The graphical sketch of numerical results demonstrates high precision of the proposed method for solving non- linear second order boundary value problems.

Spline Solution of Third order Boundary Value Problems

Authors in [19], applied the quartic non-polynomial spline function by making use of second and fourth order convergent methods to find the solution of following linear third order two-point boundary value problem.

$$y^{(3)} + f(x) = g(x)\,, x \in [a,b] \tag{11}$$

subject to the boundary conditions

$$y(a) = k_{1,}\, y^{(1)}(a) = k_2, y^{(1)}(b) = k_3, \tag{12}$$

where k_i, i= 1, 2, 3 are finite real constants and the functions f (x) and g(x) are continuous on the interval [a,b]. Using the continuity conditions of quartic non-polynomial spline, they developed the scheme given by

$$-S_{i-2} + 3\,S_{i-1} - 3\,S_i + S_{i+1} = h^3[\,\alpha\,(T_{i-2} + T_{i+1}) + \beta\,(T_{i-1} + T_i)],$$
$$i = 1, 2..\, n-1 \tag{13}$$

They also discussed convergence analysis and shown that this procedure is effective and accurate for linear third order BVPs with approximations converging rapidly to accurate solutions. This scheme has been revealed on test problems. Numerical examples were solved to illustrate practical usefulness of this approach. MATLAB 7 is used to reduce cost of computations. The numerical results using the second and fourth-order method are summarized and compared to represent an improvement over existing methods [20, 21, 22, 23]. The solution acquired by the anticipated method approximates the analytical solution very closely with low computational cost to find the solution of two-point boundary value problems.

Authors in [24] came up with a standard method based on non-polynomial quintic spline to solve general third-order non-linear boundary value problems arise in the study of draining and coating flows. The feature of the new approach is that it gives family of fourth and second-order methods by running the code once and also skips the multiplications. To demonstrate the efficiency of the proposed method, the result is compared with existence and numerical method given in (X. Q. Li and M. G. Cui [25]) and showed their algorithm performs better. Practical efficacy of the approach was revealed by illustrating the algorithm based on quintic spline.

Authors in [26] built up a new numerical method using non-polynomial quintic spline functions for the solution to certain boundary value problems involving the third-order ordinary differential equation associated with draining and coating flows of the form:

$$y''' = f(x,y), x \in [a,b] \tag{14}$$

$$y(a) - A_1 = y^{(1)}(a) - A_2 = y^{(1)}(b) - A_3 = 0$$

$$\mathrm{y(a)} = \mathrm{A_0}, \mathrm{y}''(a) = \mathrm{B_0} \quad , \mathrm{y(b)} = A_1, \mathrm{y}''(b) = B_1 \tag{15}$$

Approximate solutions obtained by developed algorithms were compared with the solutions obtained by non-polynomial quartic spline method in [27] and the polynomial quintic spline in [28]. Authors demonstrated this scheme on various numerical examples associated with draining and coating flows and results look like qualitatively compatible with experimental data [27, 28].

A non-polynomial quintic spline function has built up in [29] to find the solution of system of linear and non-linear boundary value problems of third order, associated with odd-order obstacle problems that take place in physical oceanography. Derivations for equations at each end of the range of integration for direct computation of splines are established and truncation error is also obtained. The convergence analysis is given and the class of methods is shown to be second and fourth order convergent. Authors claimed that these types of problems may be extended to support the variational inequality theory. Examples were solved to test the robustness of the method which states that present method can be used to grasp the better approximation than those presented by other methods.

Authors in [30], design a method by using parametric quintic spline function for the solution of third-order boundary value problems of the form

$$u''' = f(x,u), x \in [a,]\, b \tag{16}$$

subject to the boundary conditions

$$u(a) = k_1, u'(a) = k_2, u(b) = k_3$$

$$\mathrm{y(a)} = \mathrm{A_0}, \mathrm{y}''(a) = B_0, \mathrm{y(b)} = A_1, \mathrm{y}''(b) = B_1 \tag{17}$$

Authors here, built-up the end equations of the splines and discussed the truncation error. In their scheme, the accuracy for investigated class of methods was found to be second, fourth and sixth-order. Several numerical tests were included to demonstrate the realistic applicability of the proposed method. The results obtained by the method were compared with the some other existing spline method and concluded that their method is much better with a view to accuracy and utilization.

Spline Solution of Fourth order Boundary Value Problems

Authors in [31], presented a simple finite element method in which quintic B-splines is used as basis functions with Galerkin approach for the solution of the following fourth order boundary value problems with two different cases of boundary conditions given below

$$a_0(x)y^{(4)} + a_1(x)y^{(3)} + a_2(x)y'' + a_3(x)\, y' + a_4(x)y = b(x);\ a < x < b \tag{18}$$

subject to boundary conditions

$$y(a) = A_0, y(b) = B_0, y'(a) = A_1, y'(b) = B_1 \text{ or}$$

$$y(a) = A_0, y(b) = B_0, y''(a) = A_2, y''(b) = B_2 \tag{19}$$

where A_0, A_1, A_2, B_0, B_1, B_2 are finite real constants and $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$ and $b(x)$ are all continuous functions defined on the interval [a, b]. The essential idea in this method is that basis functions are redefined into a new set, which vanish at the boundary where the dirichlet type of boundary conditions are specified. "When a differential equation is approximated by quintic B-splines, the method yields sixth order accurate results" [32], this fact was used by authors to justify their method. The proposed approach was tested on the various examples of fourth order linear and nonlinear boundary value problems and encouraging results have been obtained with the exact solutions accessible in the literature.

Authors in [33] implemented a scheme using non-polynomial septic spline to find the numerical solution of the fourth-order two point boundary value problems occurring in a plate deflection theory The scheme is based on the development of direct methods of second, fourth and sixth order. The numerical results are found to be in good covenant with the exact solutions. This approach has the advantage that it provides an approximation not only for y(x) at the nodal points but also at every point in the range of integration. The presented numerical results were compared with those of Rashidinia and Aziz [34]. It has been observed that on decreasing the step-size, maximum observed error $\| E \|$ is approximately reduces by a factor. Result shows that the presented methods perform better than other collocation, finite-difference and spline methods and thus represent an improvement over existing methods.

Authors in [35], concerned to solve nonlinear fourth-order boundary value problems by developing numerical algorithms using a non-polynomial quintic spline function. The authors adopted the ideas presented in [36, 37 and 38] for smooth approximations of nonlinear fourth-order problem of the form

$$y^{(4)}(x) = f(x, y), a < x < b\ , a, b, x \in R \tag{20}$$

subject to boundary conditions:

$$y(a) = A_0, y''(a) = B_0 \quad , y(b) = A_1, y''(b) = B_1$$

$$y(a) = A_0, y''(a) = B_0 \quad , y(b) = A_1, y''(b) = B_1 \tag{21}$$

where A_i, B_i, i = 0, 1 are real finite constants subject to boundary conditions. The novelty of this method is that it connects spline values at mid knots and their corresponding values of the fourth-order derivatives. In constructing a family of fourth and second-order methods, it runs the code only once even as saving on the some algebraic multiplications. Similar approach with the finite difference method specified in [39] was used, in order to compare their pertaining approximate solutions. The method shows dominance over the method [39] as this can be generalized to provide continuous approximation not only to$y(x)$, but also to $y', y''$and higher-order derivatives at every point of the range of integration.

Authors in [40], study the system of fourth order boundary value problem arising in the field of elasticity, structural analysis, transportation science, economics and optimization. They developed an exponential quintic spline scheme at mid knots for the solution of following fourth order two point bvp given by

$$u^{(4)}(x) = \begin{cases} f(x), a \le x \le c, \\ f(x) + g(x)u(x) + r, c \le x \le d \\ f(x), d \le x \le b \end{cases} \tag{22}$$

along with the following boundary conditions:

$$u(a) = u(b) = A_1, u(c) = u(d) = A_2; \; u''(a) = u''(b) = B_1, u''(c) = u''(d) = B_2 \tag{23}$$

where f(x) and g(x) are continuous functions on [a, b] and [c, d] respectively and A_1, A_2, B_1 and B_2 are finite real constants. Suggested method gives rise to a class of methods of order two, four and six. Authors demonstrated this scheme on numerical examples and compared with the existing literature and found in good agreement with results. Authors wish to carry out the proposed schemes for future study that arise in science and engineering field of other type of unilateral and contact problems

Authors in [41], proposed a scheme to solve fourth order two point boundary value problems using cubic B-splines as basis functions. Authors considered the following fourth order two point boundary value problems with three different cases of boundary conditions

$$a_0(x)y^{(4)}(x) + a_1(x)\, y'''(x) + a_2(x)y''(x) + a_3(x)\, y'(x) + a_4(x)y(x) = b(x) \tag{24}$$

with the three different boundary conditions
$y(c) = A_0\,, y(d) = C_0,\; y'(c) = A_1,\; y'(d) = C_1$

or

$$y(c) = A_0\,, y(d) = C_0,\ y''(c) = A_2,\ y''(d) = C_2 \tag{25}$$

or

$$y(c) = A_0\,, y(d) = C_0,\ y'(c) + \sigma_1 y(c) = A_3,\ y'(d) + \sigma_2 y(d) = C_3$$

They reclassified the basis functions with the above boundary conditions into a new set of basis functions. Authors used the finite element method along with Galerkin method and cubic B-splines were used as basis functions. To test the intended method, the proposed method has been applied to solve several linear and nonlinear fourth order boundary value problems. The acquired numerical results were found to be in great concurrence with exact solutions accessible in the literature.

Authors in [42], proposed a method using the hyperbolic B-splines of order three to solve the following linear fourth order boundary value problem

$$y^{(4)}(\Theta) + f(\Theta)y(\Theta) = g(\Theta) \tag{26}$$

with the boundary conditions

$$y(a) = a_0\,,\ y'(a) = a_1, y(b) = b_0, y'(b) = b_1 \tag{27}$$

Taylor series method was used to put up an approximate of $y^{(4)}(\theta_k)$. Maximum absolute error and order of convergence is established for various examples. Numerical results certify the order of convergence foreseen by the examination. Technique seems more feasible for the problems for hyperbolic exact solution. Authors wish to extend the same method to solve higher order boundary value problems.

4. Conclusions

As Boundary value problems manifest themselves in many branches of Science and Engineering, their numerical solution has been an important area of research. A range of research articles is considered here to have a glance on the relevant works paid out in this area during latest five years. The survey finds that various types of splines have been used recently for solving second, third and fourth order boundary value problems. The results obtained in these papers clearly indicate that these spline methods produce more accurate numerical results than finite difference and some other methods available in literature. So, this survey paper summarizes various numerical techniques based on splines for boundary value problems, which are important for the development of new research in the field of numerical analysis and beneficial for new researchers.

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