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Interior and Closure of Fuzzy open sets in a Fuzzy Topological TM-System

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Abstract

Recently in 2010, Tamilarasi and Megalai introduced a new class of algebras known as TM-algebras. In this papaer, we discuss the notion of interior and closure of fuzzy open sets in a fuzzy topological TM-system.

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1. Introduction

The notion of a fuzzy set provides a natural framework for generalizing many of the concepts of general topology. The theory of fuzzy topological spaces is developed by Chang [7], Wong [10], Lowen [8] and others. We have studied the notion of Fuzzy Topological subsystem on a TM-algebra [1], On L-Fuzzy Topological TM-system [2], On L-Fuzzy Topological TM-subsystem [3], Fuzzy supratopological TM-system [4], Fuzzy and L-Fuzzy compactness in a Fuzzy Topological TM-system [5]. Azad K.K. introduced the concepts of Fuzzy Semicontinuity, Fuzzy Almost Continuity and Fuzzy Weakly Continuity [6]. Warren R.H. introduced the concepts of Neighbourhoods, Bases and Continuity in Fuzzy Topological Spaces [9]. In this paper, we discuss the notion of Interior and Closure of fuzzy open sets in a fuzzy topological TM-system and investigate some of their properties.

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2. Preliminaries

In this section we recall some basic definitions that are required in the sequel.

Definition 2.1. For any non-empty set X, $\mu : X \to [0, 1]$ is called a fuzzy set of X.

Definition 2.2. A TM-Algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x * 0 = x$
2. $(x * y) * (x * z) = z * y$ for all $x, y, z \in X$.

Definition 2.3. Fuzzy Topological TM-System
Let X be a TM-Algebra. X is said to be a Fuzzy Topological TM-System if there is a family T of fuzzy sets in X which satisfies the following conditions

1. $\phi, X \in T$
2. If $A, B \in T$ then $A \cap B \in T$
3. If $A_i \in T$ for each $i \in I$ then $\cup_I A_i \in T$ where I is an indexing set.

Any element in T is called a T-open fuzzy set in the TM-Algebra X.

3. Interior and Closure of Fuzzy Open Sets in a Fuzzy Topological TM-System

Definition 3.1. Fuzzy Interior TM-system
Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy T-open set μ^* in (X, T). The fuzzy Interior TM-system of μ^* is the union of all fuzzy open sets contained in μ^* and it is denoted by $(\mu^*)^\circ$. That is $(\mu^*)^\circ = \cup \left\{\mu : \mu \subseteq \mu^*, \mu \in T\right\}$.

Example 3.2. Let $\mathbb{Z}$ be the set of all integers and let $n\mathbb{Z} = \{nx : x \in \mathbb{Z}\}, n \in \mathbb{Z}$. Then $(\mathbb{Z}, -, 0)$ and $(n\mathbb{Z}, -, 0)$ are TM-algebras where '-' is the usual subtraction. Let the fuzzy subsets $\mu_i : \mathbb{Z} \to [0, 1]$, $\mu_i(n) = \dfrac{n}{10^r}$, r denotes the number of digits in n, $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ be given by

$$\mu_1(x) = \begin{cases} \dfrac{x}{10} & if\ x = 1, 2, \ldots, 9 \\ 0 & \text{otherwise} \end{cases} \qquad \mu_2(x) = \begin{cases} \dfrac{x}{10^2} & if\ x = 10, 11, \ldots, 99 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_3(x) = \begin{cases} \dfrac{x}{10^3} & if\ x = 100, 101, \ldots, 999 \\ 0 & \text{otherwise} \end{cases} \qquad \mu_4(x) = \begin{cases} \dfrac{x}{10^4} & if\ x = 1000, 1001, \ldots, 9999 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_5(x) = \begin{cases} \frac{x}{10^5} & if\ x = 10000, 10001, \ldots, 99999 \\ 0 & \text{otherwise} \end{cases} \quad \mu_6(x) = \begin{cases} \frac{x}{10^6} & if\ x = 100000, \ldots, 999999 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_7(x) = \begin{cases} \frac{x}{10^7} & if\ x = 1000000, \ldots, 9999999 \\ 0 & \text{otherwise} \end{cases} \quad \mu_8(x) = \begin{cases} \frac{x}{10^8} & if\ x = 10000000, \ldots, 99999999 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_9(x) = \begin{cases} \frac{x}{10^9} & if\ x = 100000000, \ldots, 999999999 \\ 0 & \text{otherwise} \end{cases}$$

Then the collection $T = \{\phi, X, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, \mu_9\}$ is a Fuzzy Topology on X. Hence (X, T) is a fuzzy topological TM-system. Let $\mu^\star = \mu_6$ $(\mu_6)^\circ = \cup \{\mu_7, \mu_8, \mu_9\} = \mu_7$.

Definition 3.3. Fuzzy Closure TM-system
Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy T-open set μ^* in (X, T). The fuzzy Closure TM-system of μ^* is the intersection of all fuzzy closed sets containing μ^* and it is denoted by $\overline{(\mu^*)}$.
That is $\overline{(\mu^*)} = \cap \left\{\mu' : \mu' \supseteq \mu^*, \mu' \text{ is } T - closed\right\}$.

Example 3.4. Consider the TM-algebra and fuzzy sets in example 3.2. Let $\mu^* = \mu_2$. $\overline{(\mu_2)} = \cap \left\{\mu'_1, \mu'_2, \mu'_3, \ldots, \mu'_9\right\} = \mu'_1$.

Theorem 3.5. Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy T-open set μ^* in (X, T). Then 1-$\overline{(\mu^*)} = (1 - \mu^*)^\circ$.

Proof. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy open set μ^* in (X, T).

$$\begin{aligned} 1 - \overline{(\mu^*)} &= 1 - \cap \left\{\mu' : \mu' \supseteq \mu^*, \mu' \text{ is T-closed}\right\} \\ &= \cup \left\{1 - \mu' : \mu' \supseteq \mu^*, \mu' \text{ is T} - \text{closed}\right\} \\ &= \cup \left\{\mu : 1 - \mu \supseteq \mu^*, \mu \in T\right\} \\ &= \cup \left\{\mu : \mu \supseteq 1 - \mu^*, \mu \in T\right\} \\ &= \cup \left\{\mu : \mu \subseteq 1 - \mu^*, \mu \in T\right\} \\ &= (1 - \mu^*)^\circ. \end{aligned}$$

Hence the Proof. ■

Theorem 3.6. Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy open set μ^* in (X, T). Then $1 - (\mu^*)^\circ = \overline{(1 - \mu^*)}$.

Proof. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy T-open set μ^*

in (X, T).

$$\begin{aligned} 1-(\mu^*)^\circ &= 1-\cup\left\{\mu : \mu \subseteq \mu^*, \mu \in T\right\} \\ &= \cap\left\{1-\mu : \mu \subseteq \mu^*, \mu \in T\right\} \\ &= \cap\left\{\mu' : \mu' \supseteq 1-\mu^*, \mu' \text{ is T} - \text{closed}\right\} \\ &= \overline{(1-\mu^*)}. \end{aligned}$$

Hence the Proof. ■

Theorem 3.7. Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. Consider the fuzzy T-open sets $\mu_i, \mu_j, i \neq j$ in (X, T). i)If $\mu_i \subseteq \mu_j, i \neq j$ then $\overline{\mu_i} \subseteq \overline{\mu}_j$ and $\mu_i^\circ \subseteq \mu_j^\circ$. ii) $\overline{\overline{\mu}} = \overline{\mu}$ and $(\mu^\circ)^\circ = \mu^\circ$.

Proof. This proof is obvious and directly follows from our definition of fuzzy interior TM-system and fuzzy closure TM-system. ■

Theorem 3.8. Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. The union of closure of two fuzzy sets in a fuzzy topological TM-system is equal to the closure of union of the two fuzzy sets.

Proof. Let (X, T) be a fuzzy topological TM-system. Let $\mu_i, \mu_j, i \neq j$ be any two fuzzy T-open sets.
Then $\overline{\mu_i} \cup \overline{\mu_j}$ is fuzzy T-closed and $\overline{\mu_i} \cup \overline{\mu_j} \supseteq \mu_i \cup \mu_j$ it follows from fuzzy closure TM-system definition, we have

$$\overline{\mu_i} \cup \overline{\mu_j} \supseteq \overline{\mu_i \cup \mu_j} \tag{3.1}$$

We know that

$$\mu_i \cup \mu_j \supseteq \mu_i \Rightarrow \overline{\mu_i \cup \mu_j} \supseteq \overline{\mu_i}$$

Similarly

$$\overline{\mu_i \cup \mu_j} \supseteq \overline{\mu_j} \Rightarrow \overline{\mu_i \cup \mu_j} \supseteq \overline{\mu_i} \cup \overline{\mu}_j \tag{3.2}$$

From 3.1 and 3.2 we get $\overline{\mu_i \cup \mu_j} = \overline{\mu_i} \cup \overline{\mu}_j$. ■

Theorem 3.9. Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. The intersection of closure of two fuzzy sets in a fuzzy topological TM-system contains the closure of the intersection of the two fuzzy sets.

Proof. Let (X, T) be a fuzzy topological TM-system. Let $\mu_i, \mu_j, i \neq j$ be any two fuzzy T-open sets. Then $\overline{\mu_i} \cap \overline{\mu_j}$ is T-closed set. We take $\mu_k = \overline{\mu_i \cap \mu_j}, i \neq j$ and $\mu_l = \overline{\mu_i} \cap \overline{\mu_j}, i \neq j$. We know that $\overline{\mu_i \cap \mu_j} \subseteq \overline{\mu_i} \cap \overline{\mu_j}, i \neq j$.
From theorem 3.7,

$$\begin{aligned} &\mu_k \subseteq \mu_l \Rightarrow \overline{\mu_k} \subseteq \overline{\mu_l} \\ &\Rightarrow \overline{\overline{\mu_i \cap \mu_j}} \subseteq \overline{\overline{\mu_i} \cap \overline{\mu_j}}, i \neq j \end{aligned}$$

$$\Rightarrow \overline{\mu_i \cap \mu_j} \subseteq \overline{\mu_i} \cap \overline{\mu_j}, i \neq j \,(\text{since } \overline{\overline{\mu}} = \overline{\mu})$$
$$\Rightarrow \overline{\mu_i} \cap \overline{\mu_j} \supseteq \overline{\mu_i \cap \mu_j}, i \neq j.$$

■

Theorem 3.10. Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. The union of interior of two fuzzy sets in a fuzzy topological TM-system is contained in the interior of union of two fuzzy sets.

Proof. Let (X, T) be a fuzzy topological TM-system. Let $\mu_i, \mu_j, i \neq j$ be any two fuzzy T-open sets. Take $\mu_k = (\mu_i)^\circ \cup (\mu_j)^\circ, i \neq j$, $\mu_l = (\mu_i \cup \mu_j)^\circ, i \neq j$.
We know that

$$(\mu_i)^\circ \cup (\mu_j)^\circ \subseteq (\mu_i \cup \mu_j)^\circ, i \neq j.$$

From theorem 3.7,

$$\mu_k \subseteq \mu_l \Rightarrow \mu_k^\circ \subseteq \mu_l^\circ$$
$$\Rightarrow ((\mu_i)^\circ \cup (\mu_j)^\circ)^\circ \subseteq ((\mu_i \cup \mu_j)^\circ)^\circ, i \neq j$$
$$\Rightarrow (\mu_i)^\circ \cup (\mu_j)^\circ \subseteq (\mu_i \cup \mu_j)^\circ, i \neq j \text{ (since } (\mu^\circ)^\circ = \mu^\circ).$$

■

Theorem 3.11. Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. The intersection of interior of two fuzzy sets in a fuzzy topological TM-system is equal to the interior of the intersection of two fuzzy sets.

Proof. Let (X, T) be a fuzzy topological TM-system. Let $\mu_i, \mu_j, i \neq j$ be any two fuzzy T-open sets. Then it follows from fuzzy interior TM-system definition, $(\mu_i)^\circ \cap (\mu_j)^\circ \subseteq \mu_i \cap \mu_j, i \neq j$. Therefore we have

$$(\mu_i)^\circ \cap (\mu_j)^\circ \subseteq (\mu_i \cap \mu_j)^\circ, i \neq j \tag{3.3}$$

We know that

$$\mu_i \cap \mu_j \subseteq \mu_i \Rightarrow (\mu_i \cap \mu_j)^\circ \subseteq {\mu_i}^\circ$$

Similarly

$$(\mu_i \cap \mu_j)^\circ \subseteq {\mu_j}^\circ \Rightarrow (\mu_i \cap \mu_j)^\circ \subseteq {\mu_i}^\circ \cap {\mu_j}^\circ \tag{3.4}$$

From 3.3 and 3.4 we get

$${\mu_i}^\circ \cap {\mu_j}^\circ = (\mu_i \cap \mu_j)^\circ$$

■

4. Semiopen and Semiclosed of Fuzzy open sets in a Fuzzy Topological TM-system

Definition 4.1. Fuzzy semiopen TM-system
Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy T-open set μ^* in (X, T). Then μ^* is called as fuzzy semiopen TM-system if there exists $\mu \in T$ such that $\mu \subseteq \mu^* \subseteq \overline{\mu}$.

Example 4.2. Consider the TM-Algebra and fuzzy sets as in the example 3.2. Let $\mu^*(x) = \mu_2(x)$. Consider $\mu_3(x)$ and $\overline{\mu_3(x)} = \mu'_1$. Therefore $\mu_3 \subseteq \mu_2 \subseteq \overline{\mu_3}$.

Definition 4.3. Fuzzy semiclosed TM-system
Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy T-open set μ^* in (X, T). Then μ^* is called as fuzzy semiclosed TM-system if there exists $\mu' \in T - closed$ such that $(\mu')^\circ \subseteq \mu^* \subseteq \mu'$.

Example 4.4. Consider the set $X = \{0, 1, 2, 3\}$ with the following cayley table

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X, *)$ is a TM-algebra. Let the fuzzy subsets $\mu_i : X \to [0, 1]$, $i = 1, 2, 3, 4, 5, 6, 7$ be given by

$$\mu_1(x) = \begin{cases} .5 & \text{if } x = 0 \\ .2 & \text{if } x = 1, 2 \\ .3 & \text{if } x = 3 \end{cases} \quad \mu_2(x) = \begin{cases} .5 & \text{if } x = 0 \\ .3 & \text{if } x = 1, 2 \\ .4 & \text{if } x = 3 \end{cases} \quad \mu_3(x) = \begin{cases} .4 & \text{if } x = 0, 3 \\ .3 & \text{if } x = 1, 2 \end{cases}$$

$$\mu_4(x) = \begin{cases} .9 & \text{if } x = 0 \\ .7 & \text{if } x = 1, 2 \\ .8 & \text{if } x = 3 \end{cases} \quad \mu_5(x) = \begin{cases} .8 & \text{if } x = 0 \\ .6 & \text{if } x = 1, 2 \\ .7 & \text{if } x = 3 \end{cases} \quad \mu_6(x) = \begin{cases} .3 & \text{if } x = 0 \\ .1 & \text{if } x = 1, 2 \\ .2 & \text{if } x = 3 \end{cases}$$

$$\mu_7(x) = \begin{cases} .3 & \text{if } x = 0 \\ 0 & \text{if } x = 1, 2 \\ .2 & \text{if } x = 3 \end{cases}$$

Then the collection $T = \{\phi, X, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7\}$ is a fuzzy topology on X. Hence (X, T) is a fuzzy topological TM-system. Let $\mu^*(x) = \mu_3(x)$. Consider $\mu'_2(x)$ and $(\mu'_2(x))^\circ = \mu'_5$. Therefore $(\mu'_2)^\circ \subseteq \mu_3 \subseteq \mu'_2$.

Theorem 4.5. Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy T-open set μ^* in (X, T). If μ^* is fuzzy semiclosed TM-system then $(\mu^*)'$ is fuzzy semiopen TM-system.

Proof. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy T-open set μ^* in (X, T). Given μ^* is fuzzy semiclosed TM-system then by definition,

$$(\mu')^\circ \subseteq \mu^* \subseteq \mu'$$
$$\Rightarrow (1-\mu)^\circ \subseteq \mu^* \subseteq 1-\mu \text{ [since } \mu' = 1-\mu]$$
$$\Rightarrow 1-\overline{\mu} \subseteq \mu^* \subseteq 1-\mu \text{ [By theorem 3.5].}$$

Taking complements in all the sets, We get

$$\Rightarrow (1-\overline{\mu})' \supseteq (\mu^*)' \supseteq (1-\mu)'$$
$$\Rightarrow \overline{\mu} \supseteq (\mu^*)' \supseteq \mu$$
$$\Rightarrow \mu \subseteq (\mu^*)' \subseteq \overline{\mu}$$
$$\Rightarrow (\mu^*)' \text{ is fuzzy semiopen TM-system.}$$

■

Theorem 4.6. Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy T-open set μ^* in (X, T). If $(\mu^*)'$ is fuzzy semiopen TM-system then μ^* is fuzzy semiclosed TM-system.

Proof. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy T-open set μ^* in (X, T). Given $(\mu^*)'$ is fuzzy semiopen TM-system then by definition, We have $\mu \subseteq (\mu^*)' \subseteq \overline{\mu}$.
Taking complements in all the sets, We get

$$(\mu)' \supseteq ((\mu^*)')' \supseteq (\overline{\mu})'$$
$$\Rightarrow \mu' \supseteq \mu^* \supseteq 1-\overline{\mu}$$
$$\Rightarrow \mu' \supseteq \mu^* \supseteq 1-1+(1-\mu)^\circ \text{ [By theorem 3.5]}$$
$$\Rightarrow \mu' \supseteq \mu^* \supseteq (\mu')^\circ$$
$$\Rightarrow (\mu')^\circ \subseteq \mu^* \subseteq \mu'$$
$$\Rightarrow \mu^* \text{ is fuzzy semiclosed TM-system.}$$

■

Theorem 4.7. Let $(X, *)$ be a TM-Algebra. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy T-open set μ^* in (X, T). μ^* is fuzzy semiclosed TM-system iff $(\overline{\mu})^\circ \subseteq \mu^*$.

Proof. Let (X, T) be a fuzzy topological TM-system. Consider a fuzzy T-open set μ^* in (X, T). Suppoose μ^* is fuzzy semiclosed TM-system. Then by definition, We have

$$(\mu')^\circ \subseteq \mu^* \subseteq \mu'$$
$$\Rightarrow (\mu')^\circ \subseteq \mu^* \subseteq \overline{\mu^*} \subseteq \mu'$$

$\Rightarrow (\mu')^\circ$ is the largest fuzzy open set contained in μ', we have

$$(\overline{\mu^*})^\circ \subseteq (\mu')^\circ \subseteq \mu^*$$
$$\Rightarrow (\overline{\mu^*})^\circ \subseteq \mu^*$$

Conversely, $(\overline{\mu^*})^\circ \subseteq \mu^*$. Let $\overline{\mu^*} = \mu'$ we get, $(\mu')^\circ \subseteq \mu^* \Rightarrow (\mu')^\circ \subseteq \mu^* \subseteq \mu'$. Hence μ^* is fuzzy semiclosed TM-system. ■

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