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Symmetric identities for Carlitz's (h, q)-tangent numbers and polynomials associated with p-adic integral on $\mathbb{Z}_p$

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Abstract

Our aim in this paper is to discover special symmetric properties for Carlitz's (h, q)-tangent polynomials.

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1. Introduction

Many mathematicians have studied some identities of symmetry for q-extension of Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, Tangent numbers and polynomials (see [1, 2, 3, 4, 6, 7, 8]). Recently, Y. Hu investigated several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field (see [2]). D. Kim *et al.*[3] derived some identities of symmetry for Carlitz's q-Euler numbers and polynomials in complex field. J. Y. Kang and C. S. Ryoo studied some identities of symmetry for q-Genocchi polynomials (see [1]). In this paper, we obtaine some identities of symmetry for Carlitz's (h, q)-tangent polynomials associated with p-adic integral on $\mathbb{Z}_p$.

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number

$q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q} \text{ (cf. [1, 2, 3, 4])} .$$

Hence, $\lim_{q \to 1} [x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. Let

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}.$$

For $g \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \text{ see [5].} \tag{1.1}$$

2. Symmetric identities for Carlitz's (h, q)-tangent numbers and polynomials

Our primary goal of this section is to obtain symmetric identities for Carlitz's (h, q)-tangent numbers $T_{n,q}^{(h)}$ and polynomials $T_{n,q}^{(h)}(x)$.

By using the similar method of [4, 7], expect for obvious modifications, we are going to obtain the main results of Carlitz's (h, q)-tangent numbers and polynomials.

For $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, (h, q)-tangent polynomials $T_{n,q}^{(h)}(x)$ are defined by

$$T_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} q^{hy} [x+2y]_q^n d\mu_{-1}(y), \text{ see [9].} \tag{2.1}$$

When $x = 0$, $T_{n,q}^{(h)}(0) = T_{n,q}^{(h)}$ is called the n-th (h, q)-tangent numbers.

Let w_1 and w_2 be odd numbers. Then we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{w_1 h y} e^{\left[w_2 x + \frac{2w_2}{w_1} j + 2y\right]_{q^{w_1}} [w_1]_q t} d\mu_{-1}(y) \\
&= \lim_{N \to \infty} \sum_{y=0}^{w_2 p^N - 1} q^{w_1 h y} e^{[w_1 w_2 x + 2w_2 j + 2w_1 y]_q t} (-1)^y \\
&= \lim_{N \to \infty} \sum_{i=0}^{w_2-1} \sum_{y=0}^{p^N-1} q^{w_1 h (i + w_2 y)} e^{[w_1 w_2 x + 2w_2 j + 2w_1 (i + w_2 y)]_q t} (-1)^{i + w_2 y}
\end{aligned} \tag{2.2}$$

From (2.2), we can derive the following equation (2.3):

$$\begin{aligned}
&\sum_{j=0}^{w_1-1}(-1)^j q^{hw_2j}\int_{\mathbb{Z}_p} q^{w_1hy} e^{\left[w_2x+\frac{2w_2}{w_1}j+2y\right]_{q^{w_1}}[w_1]_q t} d\mu_{-1}(y) \\
&= \lim_{N\to\infty}\sum_{j=0}^{w_1-1}\sum_{i=0}^{w_2-1}\sum_{y=0}^{p^N-1}(-1)^{i+j}q^{w_2hj}q^{w_1hi}q^{w_1w_2hy} \\
&\qquad\times e^{[w_1w_2x+2w_2j+2w_1i+2w_1w_2y]_q t}(-1)^y
\end{aligned} \tag{2.3}$$

By the same method as (2.3), we have

$$\begin{aligned}
&\sum_{j=0}^{w_2-1}(-1)^j q^{hw_1j}\int_{\mathbb{Z}_p} q^{w_2hy} e^{\left[w_1x+\frac{2w_1}{w_2}j+2y\right]_{q^{w_2}}[w_2]_q t} d\mu_{-1}(y) \\
&= \lim_{N\to\infty}\sum_{j=0}^{w_2-1}\sum_{i=0}^{w_1-1}\sum_{y=0}^{p^N-1}(-1)^{i+j}q^{w_1hj}q^{w_2hi}q^{w_1w_2hy} \\
&\qquad\times e^{[w_1w_2x+2w_1j+2w_2i+2w_1w_2y]_q t}(-1)^y
\end{aligned} \tag{2.4}$$

Therefore, by (2.3) and (2.4), we have the following theorem.

Theorem 2.1. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned}
&\sum_{j=0}^{w_1-1}(-1)^j q^{hw_2j}\int_{\mathbb{Z}_p} q^{w_1hy} e^{\left[w_2x+\frac{2w_2}{w_1}j+2y\right]_{q^{w_1}}[w_1]_q t} d\mu_{-1}(y) \\
&= \sum_{j=0}^{w_2-1}(-1)^j q^{hw_1j}\int_{\mathbb{Z}_p} q^{w_2hy} e^{\left[w_1x+\frac{2w_1}{w_2}j+2y\right]_{q^{w_2}}[w_2]_q t} d\mu_{-1}(y).
\end{aligned} \tag{2.5}$$

By substituting Taylor series of e^{xt} into (2.5) and after calculations, we obtain the following corollary.

Corollary 2.2. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned}
&[w_1]_q^n\sum_{j=0}^{w_1-1}(-1)^j q^{hw_2j}\int_{\mathbb{Z}_p} q^{w_1hy}\left[w_2x+\frac{2w_2}{w_1}j+2y\right]_{q^{w_1}}^n d\mu_{-1}(y) \\
&= [w_2]_q^n\sum_{j=0}^{w_2-1}(-1)^j q^{hw_1j}\int_{\mathbb{Z}_p} q^{w_2hy}\left[w_1x+\frac{2w_1}{w_2}j+2y\right]_{q^{w_2}}^n d\mu_{-1}(y).
\end{aligned} \tag{2.6}$$

By (2.1) and Corollary 2, we have the following theorem.

Theorem 2.3. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned}
&[w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j q^{hw_2 j} T_{n,q^{w_1}}^{(h)} \left(w_2 x + \frac{2w_2}{w_1} j \right) \\
&= [w_2]_q^n \sum_{j=0}^{w_2-1} (-1)^j q^{hw_1 j} T_{n,q^{w_2}}^{(h)} \left(w_1 x + \frac{2w_1}{w_2} j \right).
\end{aligned}$$

By (2.6), we can derive the following equation (2.7):

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{w_1 hy} \left[w_2 x + \frac{2w_2}{w_1} j + 2y \right]_{q^{w_1}}^n d\mu_{-1}(y) \\
&= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [2j]_{q^{w_2}}^i q^{w_2(n-i)j} \int_{\mathbb{Z}_p} q^{w_1 hy} [w_2 x + 2y]_{q^{w_1}}^{n-i} d\mu_{-1}(y) \\
&= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [2j]_{q^{w_2}}^i q^{w_2(n-i)j} T_{n-i,q^{w_1}}^{(h)} (w_2 x).
\end{aligned} \tag{2.7}$$

By (2.7), and Theorem 2.3, we have

$$\begin{aligned}
&[w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j q^{hw_2 j} \int_{\mathbb{Z}_p} q^{w_1 hy} \left[w_2 x + \frac{2w_2}{w_1} j + 2y \right]_{q^{w_1}}^n d\mu_{-1}(y) \\
&= \sum_{j=0}^{w_1-1} (-1)^j q^{hw_2 j} \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} [2j]_{q^{w_2}}^i q^{w_2(n-i)j} T_{n-i,q^{w_1}}^{(h)} (w_2 x) \\
&= \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} T_{n-i,q^{w_1}}^{(h)} (w_2 x) \sum_{j=0}^{w_1-1} (-1)^j q^{w_2(n-i+h)j} [2j]_{q^{w_2}}^i \\
&= \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} T_{n-i,q^{w_1}}^{(h)} (w_2 x)\, \mathcal{S}_{n,i}^{(h)}(w_1, q^{w_2}),
\end{aligned} \tag{2.8}$$

where

$$\mathcal{S}_{n,i}^{(h)}(w_1, q) = \sum_{j=0}^{w_1-1} (-1)^j q^{(n-i+h)j} [2j]_q^i.$$

By the same method as (2.8), we get

$$\begin{aligned}
&[w_2]_q^n \sum_{j=0}^{w_2-1} (-1)^j q^{hw_1 j} \int_{\mathbb{Z}_p} q^{w_2 hy} \left[w_1 x + \frac{2w_1}{w_2} j + 2y \right]_{q^{w_2}}^n d\mu_{-1}(y) \\
&= \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} T_{n-i,q^{w_2},\zeta^{w_2}}^{(h)} (w_1 x)\, \mathcal{S}_{n,i}^{(h)}(w_2, q^{w_1}).
\end{aligned} \tag{2.9}$$

By (2.8) and (2.9), we have the following theorem.

Theorem 2.4. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i}\mathcal{S}_{n,i}^{(h)}(w_1, q^{w_2})T_{n-i,q^{w_1}}^{(h)}(w_2x)\\&=\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i}\mathcal{S}_{n,i}^{(h)}(w_2, q^{w_1})T_{n-i,q^{w_2}}^{(h)}(w_1x)\,.\end{aligned}$$

By Theorem 2.4, we obtain the interesting symmetric identity for Carlitz's (h, q)-tangent numbers.

Corollary 2.5. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i}T_{n-i,q^{w_1}}^{(h)}\mathcal{S}_{n,i}^{(h)}(w_1, q^{w_2})\\&=\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i}T_{n-i,q^{w_2}}^{(h)}\mathcal{S}_{n,i}^{(h)}(w_2, q^{w_1}).\end{aligned}$$

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