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A Study on Qualitative Properties of Stochastic Difference Equations and Stability

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Abstract:

In this paper, we introduce a method based on the Poisson distribution to show existence and qualitative properties of solutions for the problem
$c\Delta^{\alpha}u(n) = Au(n+1), n \in \mathbb{N}_0;\ u(0) = u_0 \in X.$
Using operator-theoretical conditions on A. We show how several properties for fractional differences, including their own definition, are connected with the continuous case by means of sampling using the Poisson distribution.

Keywords: Stochastic Difference Equations, Poisson distribution, Fractional differences

I. Introduction

The study of existence and qualitative properties of discrete solutions for fractional difference equations has drawn a great deal of interest [24]. Holm further developed and applied the tools of discrete fractional calculus to the arena of fractional difference equations. In spite of the significant increase of research in this area, there are still many open questions regarding fractional difference equations [8,11]. In particular, the study of fractional difference equations with unbounded linear operators and their stability properties remains an open problem. These abstract fractional models, with unbounded operators, are closely connected with numerical methods. We propose a novel method to deal with this problem based on the sampling of fractional differential equations by means of the Poisson distribution [5,9]. We will use it to prove the existence of a unique solution to the initial value problem.

II. Mathematical Model using Poisson Distribution

Mathematical understanding of the linear equation is meant as a preliminary critical step for the subsequent analysis of full nonlinear models. The approach followed here is purely operator theoretic and has as main ingredient the use of the Poisson distribution:

$$p_n(t) = e^{-t}\frac{t^n}{n!}, \qquad n \in \mathbb{N}_0, \qquad t \geq 0.$$

The method relies in to take advantage of the properties of this distribution when it is applied to continuous phenomena. More precisely, given a continuous evolution $u(t), t \in [0,\infty]$ we can discretize it by means of that we will call the Poisson transformation

$$u(n) = \int_0^\infty p_n(t)u(t)dt, \qquad n \in \mathbb{N}_0.$$

In this paper, we will show that when this procedure is applied to fractional models defined on the time scale $\mathbb{R}_+$, these transformations are well behaved and t perfectly in the discrete fractional concepts [13]. In other words, our approach is as follows: Suppose that a solution of the fractional Cauchy problem

$$D^\alpha u(t) = Au(t), \;\; t \geq 0, \qquad 0 < \alpha \leq 1,$$

Exists. It happens, for instance, if A is the generator of a C_0-semi group or A is sectorial and references therein. Then, by sampling each side of the above equation by means of the Poisson distribution, we obtain that $u(n)$ defined by a solution of

$$\Delta^\alpha u(n) = Au(n+1), \qquad n \in \mathbb{N},$$

Where D^α and Δ^α denote the fractional operators on $\mathbb{R}_+$ and $\mathbb{N}_0$, respectively, in the sense of Riemann-Liouville [11]. It is remarkable that by this mechanism we recover the concept of fractional nabla sum and difference operator introduced by Atici and Eloe [6], which has been used recently and independently of the method used here by other authors in order to obtain several qualitative properties of fractional difference equations, notably concerning stability properties. We take advantage of this important connection to derive several sufficient conditions for stability in case of unbounded operators A. Among others, in this paper we prove the following practical criteria in Hilbert spaces: Let A be the generator of a C_0-semi group on a Hilbert space H such that $\{\mu \in \mathbb{C} : Re(\mu) > 0\} \subset \rho(A)$ and satisfies.

$$\sup_{Re(\mu) > 0} ||(\mu - A)^{-1}|| < \infty,$$

then, the solution of the fractional difference equation of order $\alpha \in (0,1)$

$$c\Delta^\alpha u(n) = Au(n+1), n \in \mathbb{N},$$

Exists and is stable for all initial conditions $u_0 \in D(A)$.

The outline of this paper is as follows: We give some preliminary background in notation and definitions. The remarkable fact is that we use here a particular choice of the definition introduced by Atici and Eloe in for the nabla operator [6]. This choice that has been used by other authors is proved to be the right notion in the sense that the following notable relation holds

$$\int_0^\infty p_{n+1}(t)D_t^\alpha u(t)dt = \Delta^\alpha u(n), \qquad n \in \mathbb{N}_0$$

Where D_t^α denotes the Riemann-Liouville fractional derivative on $\mathbb{R}_+$ and $u(n)$ is defined. Then, we can connect the Delta operator (i.e. the Riemann-Liouville fractional difference) in the right hand side with the Caputo-like fractional difference

by means of the identity [1]
$c\Delta^{\alpha}u(n) = c\Delta^{\alpha}u(n) - k^{1-\alpha}(n+1)u(0), \qquad n \in \mathbb{N}_0.$

III. Forward Difference Equation

For a real number a, we denote
$\mathbb{N}_a = \{a, a+1, a+2, \ldots..\}$
and we write $\mathbb{N}_1 \equiv \mathbb{N}$. Let X be a complex Banach space. We denote by $s(\mathbb{N}_a; X)$ the vectorial space consisting of all vector-valued sequences $f: \mathbb{N}_a \longrightarrow X$.
The forward Euler operator Δ_a: $s(\mathbb{N}_a; X) \longrightarrow s(\mathbb{N}_a; X)$ is defined by
$\Delta_a f(t) := f(t+1) - f(t), \qquad t \in \mathbb{N}_a.$
For $m \in \mathbb{N}_2$, we define recursively Δ_a^m: $s(\mathbb{N}_a; X) \longrightarrow s(\mathbb{N}_a; X)$ by
$\Delta_a^m := \Delta_a^{m-1} \circ \Delta_a,$
and is called the m-th order forward difference equation. For instance, for any $f \in s(\mathbb{N}_a; X)$, we have

$$\Delta_0^m f(n) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-1} f(n+j), \qquad n \in \mathbb{N}_0$$

In particular, we obtain
$(\Delta_0^1 f)(n) = f(n+1) - f(n), \qquad n \in \mathbb{N}_0$
We also denote $\Delta_a^0 \equiv I_a$, where $I_a: s(\mathbb{N}_a; X) \longrightarrow s(\mathbb{N}_a; X)$ is the identity operator, and $\Delta \equiv \Delta_0^1$.
We define

$$k^{\alpha}(j) := \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)\Gamma(j+1)}, \qquad j \in \mathbb{N}_0$$

Fractional Sum:

Let $\alpha > 0$. For any given positive real number a, the α-th fractional sum of a function f is

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t-s+1)^{\overline{\alpha-1}} f(s),$$

Where $t \in \mathbb{N}_a$ *and* $t^{\overline{\alpha}} := \frac{\Gamma(t+\alpha)}{\Gamma(t)}$
In particular, in case $a = 0$ we denote

$$\Delta^{-\alpha} f(n) = \nabla_0^{-\alpha} f(n) = \sum_{k=0}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)\Gamma(n-k+1)} f(k) = \sum_{k=0}^{n} k^{\alpha}(n-k) f(k), \qquad n \in \mathbb{N}_0$$

III. A Method Based on the Poisson Distribution

For each $n \in \mathbb{N}_0$, we recall that the Poisson distribution is defined by

$$p_n(t) := e^{-t}\frac{t^n}{n!}, \qquad t \geq 0.$$

As expected, $p_n(t) \geq 0$ and

$$\int_0^\infty p_n(t)\,dt = 1, \qquad n \in \mathbb{N}_0$$

The Poisson distribution arises in connection with Poisson processes. In this section we will realize their application to abstract difference equations [12]. The method itself uses an idea of discretization of the derivative in time used.

First, we recall some concepts. Let $s: \mathbb{R}_+ \to \beta(X)$ be strongly continuous, that is, for all $x \in X$ the map $t \to S(t)x$ is continuous on $\mathbb{R}_+$. we say that a family of bounded and linear operators $\{S(t)\}t \geq 0$ is exponentially bounded if there exists real numbers $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\left|\left|S(t)\right|\right| \leq Me^{\omega t}, \qquad t \geq 0.$$

We say that $\{S(t)\}t \geq 0$ is bounded if $\omega = 0$. Note that if $\{S(t)\}t \geq 0$ is exponentially bounded then the Laplace transform

$$S(\lambda)x := \int_0^\infty e^{-\lambda t}\, S(t)x dt, \qquad x \in X$$

Exists for all $Re(\lambda) > \omega$.

We recall that the Z-transform of a vector-valued sequence $f \in s(\mathbb{N}_0; X)$, is defined by

$$\tilde{f}(z) := \sum_{j=0}^{\infty} z^{-j} f(j)$$

Where z is a complex number. Note that convergence of the series is given for |z| > R with R sufficiently large.

An interesting connection between the vector-valued Z-transform and the vector-valued Laplace transform can be given by means of the Poisson distribution.

Example

For $\alpha > 0$ define

$$g_\alpha(t) := \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Note the semi group property:

$$g_{\alpha+\beta} = g_\alpha * g_\beta, \qquad \alpha, \beta > 0$$

We have the following interesting property of sampling

$$g_\alpha(n) := \int_0^\infty p_n(t) g_\alpha(t) dt = \int_0^\infty e^{-t} \frac{t^{n+\alpha+1}}{\Gamma(\alpha) n!} dt = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} = k^\alpha(n),$$

For all $n \in \mathbb{N}_0$. By the preceding theorem we obtain

$$\tilde{k}^\alpha(z) = \frac{z^\alpha}{(z-1)^\alpha}$$

For all $|z| > 1$ and

$$k^{\alpha+\beta}(n) = \left(k^\alpha * k^\beta\right)(n), \qquad \alpha, \beta > 0.$$

In particular, for $\alpha, \beta > 0$ we deduce the identities

$$\Delta^{-\alpha}\left(\Delta^{-\beta} u\right)(n) = \Delta^{-(\alpha+\beta)} u(n) = \Delta^{-\beta}(\Delta^{-\alpha} u)(n), \qquad \forall n \in \mathbb{N}_0.$$

Indeed,

$$\Delta^{-\alpha}\left(\Delta^{-\beta} u\right) = \Delta\left(k^\beta * u\right) = k^\alpha\left(k^\beta * u\right) = \left(k^\alpha * k^\beta\right) * u = k^{\alpha+\beta} * u = \Delta^{\alpha+\beta} u,$$

and interchanging the role of α and β we obtain. We finally remark that, for $\alpha, \beta > 0$, we get the identity

$$\Delta^{-\alpha}k^{\beta} = k^{\alpha} * k^{\beta} = k^{\alpha+\beta}$$

The next property connecting the continuous and discrete convolution will be very useful in the treatment of abstract difference equations.

IV. Numerical Example

A MATLAB program is composed according to the conditions. The program takes the system parameters $r_i, A, B_j, C_i, D_{ij}, i = 1,2, \ldots\ldots, K, j = 1,2, \ldots, K,$ and grid parameters $N_i, i = 1,2, \ldots, K,$ as inputs, and check the feasibility of the linear matrix inequalities. To get an idea how close the conditions are to the necessary and sufficient conditions, the following system is tested,

$$\dot{x}(t) = Ax(t) + B_1 y_1(t - r_1) + B_2 y_2(t - r_2),$$
$$y_1(t) = C_1 x(t),$$
$$y_2(t) = C_2 x(t) + D_{21} y_1(t - r_1) + D_{22} y_2(t - r_2),$$

$$A = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 \\ -0.5 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -2 & 0 \\ -1 & -1.45 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C_1 = (0\ 1\ 0\ 0\ 0\ 0),$$
$$C_2 = \begin{pmatrix} 0.2\ 0\ 1\ 0\ 0\ 0 \\ 0\ 0\ 0\ 1\ 0\ 0 \end{pmatrix},$$
$$D_{21} = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix}, D_{22} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix},$$

The system has two delay channels. From the special structure of the system, it is not difficult to show by solving the characteristic equations that the system is exponentially stable if and only if $r_i \in (0, r_i\, max)$, i = 1; 2, where

$$r_1 max = 2\pi,$$
$$r_2 max = 4.7388.$$

Other methods, such as the one covered in Cebotarev and Meiman (1949), may also be used to obtain this conclusion.

For three given ratios $\sqrt{5}, 1/\sqrt{2}$ and $1/\sqrt{5}$ of r_1/r_2, the maximum r_2 that satisfies the conditions are computed by using MATLAB program with a bisection process, and the results for difference N_1 and N_2 are listed in the following tables

$r_1/r_2 = \sqrt{5}$				
N_1	2	3	4	Analytical
N_2	1	1	2	
$r_2 max$	2.8028	2.8087	2.8094	2.8099

$r_1/r_2 = 1/\sqrt{2}$				
N_1	1	1	2	Analytical
N_2	1	2	3	
$r_2 max$	4.6850	4.7354	4.7381	4.7388

$r_1/r_2 = 1/\sqrt{5}$				
N_1	1	2	2	Analytical
N_2	2	3	4	
$r_2 max$	4.7354	4.7381	4.7386	4.7388

It can be seen from the tables that the results approach the analytical results very quickly. The grid has been chosen so that h_1/h_2 is not too far from 1 in order to minimize the need for a large W in to reduce conservatism.
It should be pointed out that the distribution of non smooth points of $U_{ij}(\xi)$ in this case is rather simple. Preliminary study indicates that it is much more difficult to obtain a stability bound that is close to the analytical limit using this discretization method if $U_{ij}(\xi)$ has more complicated distribution of non smooth points.

V. Conclusion

This paper we considered the coupled differential-difference equations with multiple delay channels with single delay in each channel is a more reasonable representation of a large class of practical systems since it more faithfully represents the problem size.

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