Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 5 (2015), pp. 3073-3080

http://www.ripublication.com

t-Graceful Labeling (t ≥ 2) and 1 modulo t Graceful Labeling (t ≥ 3) of Even Cycles C_n with Parallel P_3 Chords

V. Srividya[1], R. Govindarajan[2] and S. Manikandan[3]

[1]*Associate Professor, Department of Mathematics*
Bharathi Women's College, Chennai – 108
mylaporevidya@gmail.com
[2]*Associate Professor and HOD of Mathematics*
D.G. Vaishnav College, Chennai – 106
govindarajan.babu57@gmail.com
[3]*Assistant Professor, Department of Mathematics*
R.V. Government Arts College, Chengalpattu – 1
manikandansankaranarayanan69@gmail.com

Abstract

In this paper we have proved that every even cycle Cn (n ≥6) with parallel P_3 chords have t-graceful labeling for t ≥ 2 and one modulo t-graceful labeling for t ≥ 3.

Keywords: Graceful labeling, t-graceful labeling, 1 modulo t graceful labeling, cycles with parallel P_3 chords.

Subject Classification: 05C78.

1. Introduction

A graph labeling is an assignment of integers to the vertices (or) edges or both subject to certain conditions. Labeled graphs serve as useful models in a broad range of applications such as coding theory, X-ray crystallography, circuit design and communication network addressing. Labeling in graph theory gained popularity after the mid sixties of 20^{th} century. Most graph labeling methods trace their origin to the β-valuation introduced by Rosa [6] in 1967. The β-labelings were later renamed as graceful by Golomb [2]. In the intervenning years variations in graceful labelings came into existence.

By a graph G = (V, E) we mean a finite undirected graph without loops (or) multiple edges where |V(G)| = p, |E(G)| = q. A graceful labeling of a graph G with q edges is an injection $f : V(G) \to \{0, 1, 2, \ldots, q\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$ the resulting edge labels are distinct and are $\{1, 2, 3, \ldots, q\}$. A graph G which admits graceful labeling is called a graceful graph. Several classes of graphs have been shown to be graceful. Study on gracefulness of cycle related graphs initiated from Rosa's result [6] that cycle C_n is graceful iff $n \equiv 0$ (or) 3 (mod 4).
A t-graceful labeling of a graph G = (V, E) with q edges is an injection $f : V(G) \to \{0, 1, 2, \ldots, q+t-1\}$ such that the set of edge labels induced by the absolute value of the difference of the labels of the adjacent vertices is $\{t, t+1, t+2, \ldots, q+t-1\}$. This is a generalization of graceful labeling which was introduced by Slater [10] in 1982 and also by Maheo and Thuillier [4] in 1982. They have shown that cycle C_n is t-graceful iff either n = 0 or 1 (mod 4) with t-even and $t \leq \frac{(n-1)}{2}$ (or) n = 3 (mod 4) with t-odd and $t \leq \frac{n^2-1}{2}$. Seoud and Elsakhawi [8] have proved that paths and ladders are arbitrarily graceful which shows that they are t-graceful for all t.
Another variation of graceful labeling was one modulo t graceful labeling which was introduced by Ramachandran and Sekar [5] in 2013. For a positive integer t, a graph G with q edges is said to be one modulo t-graceful if there is an injection ϕ from V(G) to $\{0, 1, t, t+1, 2t, 2t+1, \ldots, t(q-1), t(q-1)+1\}$ such that ϕ induces a bijection ϕ^* from E(G) to $\{1, t+1, 2t+1, \ldots, t(q-1)+1\}$ where $\phi^*(uv) = |\phi(u) - \phi(v)|$ for each edge uv. Some of the graphs that they have proved to be one modulo t-graceful for all t are paths, star, caterpillars. In 2002 Sekar [7] proved that paths P_m, cycles C_n if n = 0 (mod 4), $K_{m,n}$, caterpillars, crown $C_n \odot K_1$, cycles C_{2n} with path P_m attached at each vertex of the cycle are one modulo-three graceful. He conjecture that every one modulo-three graceful graph is graceful. For a detailed survey on graceful labeling graphs, refer to the dynamic survey of graph labeling by Gallian [1]. In this paper we prove that every even cycle C_n ($n \geq 6$) with parallel P_3 chords is t-graceful for $t \geq 2$ and one modulo t-graceful for $t \geq 3$.

2. Main Result

In this section we prove two theorems on variation of graceful labeling for even cycles C_n with parallel P_3 chords. Here C_n denote the cycle of length n and P_3 denotes a path of order 3. The basic definitions of some of the required concepts are reviewed in this section.

Definition 2.1.
A chord of a cycle is an edge joining two otherwise non-adjacent vertices of a cycle.

Definition 2.2.
A graph G is called a cycle with parallel P_3 chords [9] if G is obtained from the cycle $C_n : v_0 v_1 \ldots v_{n-1} v_0$ ($n \geq 6$) by adding disjoint paths P_3's between the pair of vertices v_1

v_{n-1},..., v_2 v_{n-2},... v_α v_β where $\alpha = \left[\frac{n}{2}\right] - 1$, $\beta = \left[\frac{n}{2}\right] + 2$ if n is odd (or) $\beta = \left[\frac{n}{2}\right] + 1$ if n is even. Then by definition G has N = [3n–2]/2 vertices and M = 2n–2 edges.

Theorem 2.1.
Every even cycle C_n ($n \geq 6$) with parallel P_3 chords is t-graceful for all $t \geq 2$.

Proof.
If t = 1, it is graceful and it was proved by Sethuraman and Elumalai [9]. Hence $t \geq 2$ is considered. Let G be an even cycle C_n with parallel P_3 chords G has N = $\frac{3n-2}{2}$ vertices and M = 2n–2 edges. Let v_0 v_1 v_2... v_{n-1} be a Hamiltonian path in G as shown in the figure 1. Labeling of the vertices is considered for the following 2 cases.

Case 1. If $n \equiv 0 \pmod 4$

Define f : $V(G) \rightarrow \{0, 1, 2, ..., M+t-1\}$ as follows
$f(v_0) = 0$
$f(v_1) = 2n+t-3$
$f(v_2) = 1$
$f(v_{p+2i}) = f(v_p) - i$, $1 \leq i \leq 2$, p = 3, 9, 15,..., N–8
$f(v_{q+2}) = f(v_q) - 2$, q = 1, N–4
$= f(v_q) - 3$, q = 7, 13, 19,..., N–10
$f(v_{2i}) = i+1$, $2 \leq i \leq \frac{N-5}{2}$
$f(v_{N-3}) = f(v_{N-2}) - (t+1)$
$f(v_{N-1}) = f(v_{N-2}) - t$
Thus the entire N vertices are labeled and clearly f is an injection from $V(G) \rightarrow \{0, 1, 2, ..., M+t-1\}$. Let E_1, E_2, E_3, E_4 denote the sets of edges of G given by
$E_1 = \{v_0v_1, v_1v_2, v_0v_3, v_2v_3, v_1v_6\}$
$E_2 = \{v_3v_4, v_4v_5, ..., v_{N-5}v_{N-4}\}$
$E_3 = \{v_{N-4}v_{N-3}, v_{N-4}v_{N-1}, v_{N-3}v_{N-2}, v_{N-2}v_{N-1}\}$
$E_4 = E(G) - (E_1 \cup E_2 \cup E_3)$
The induced edge labels for the edges in the above sets are given below as
$E_1' = \{M+t-1, M+t-2, ..., M+t-5\}$
$E_2' = \{M+t-6, M+t-7, ..., M+t-10, M+t-13, M+t-14, ..., M+t-18, M+t-21, ..., t+6, t+5\}$
$E_3' = \{t+3, t+2, t+1, t\}$
$E_4' = \{t, t+1, ..., q+t-1\} - (E_1' \cup E_2' \cup E_3')$
Clearly $E_1' \cup E_2' \cup E_3' \cup E_4' = \{t, t+1, t+2, ..., M+t-1\}$. Hence the graph G admits t-graceful labeling for all $t \geq 2$. An illustration of the labeling defined in the above theorem is given in Figure 1.

Case 2. If $n \equiv 2 \pmod 4$

Define f : $V(G) \rightarrow \{0, 1, 2,..., M+t-1\}$ as follows
$f(v_0) = 0$
$f(v_1) = 2n+t-3$
$f(v_2) = 1$
$f(v_{p+2i}) = f(v_p) - i, 1 \le i \le 2, p = 3, 9, 15,..., N-5$
$f(v_{q+2}) = f(v_q) - 3, q = 7, 13, 19,..., N-7$
$= f(v_q) - 2, q = 1$
$f(v_{2i}) = i+1, 2 \le i \le \frac{N-2}{2}$
$f(v_{N-1}) = f(v_{N-2}) + t$
From the above definition it is clearly seen that f is an injection from the vertex set of G to $\{0, 1, 2,..., M+t-1\}$. Let E_1, E_2, E_3, E_4 denote the sets of edges of G given by
$E_1 = \{v_0v_1, v_1v_2, v_0v_3, v_2v_3, v_1v_6\}$
$E_2 = \{v_3v_4, v_4v_5,..., v_{N-3}v_{N-2}\}$
$E_3 = \{v_{N-2}v_{N-1}, v_{N-1}v_{N-4}\}$
$E_4 = E(G) - (E_1 \cup E_2 \cup E_3)$
The edge labels induced by the absolute values of the difference of the label of adjacent vertices in the above sets are given below as
$E_1' = \{M+t-1, M+t-2,..., M+t-5\}$
$E_2' = \{M+t-6, M+t-7,..., M+t-10, M+t-13, M+t-14,..., M+t-18, M+t-21,..., t+3, t+2\}$
$E_3' = \{t, t+1\}$
$E_4' = \{t, t+1, t+2,..., M+t-1\} - (E_1' \cup E_2' \cup E_3')$
Clearly $E_1' \cup E_2' \cup E_3' \cup E_4' = \{t, t+1,..., M+t-1\}$. From the above vertex labeling the edge
labels are distinct and graph G admits t-graceful labeling.
An illustration is given in Figure 1.

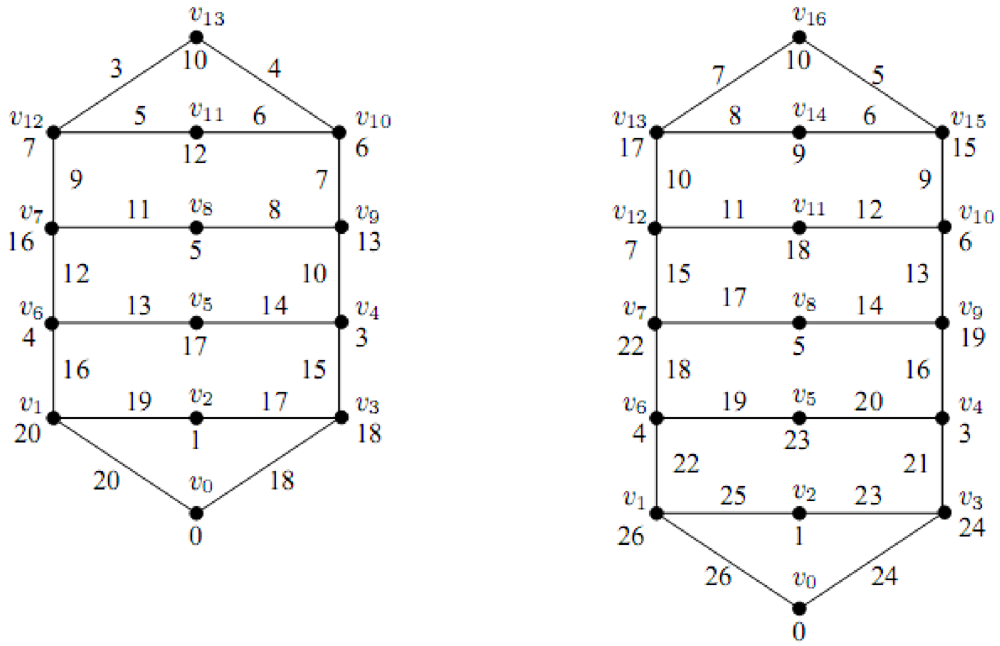

Figure 1. 3-graceful labeling of C_{10} and 5-graceful labeling of C_{12} with parallel P_3 chords

Theorem 2.2.
Every even cycle C_n ($n \geq 6$) with parallel P_3 chords is 1 modulo t-graceful for all $t \geq 3$.

Proof.
If t = 1, it is a graceful labeling and proved by Sethuraman and Elumalai [9]. If t = 2 the labeling is an odd graceful labeling which was proved in our paper [3]. Let G be an even cycle C_n with parallel P_3 chords. By definition G has $N = \frac{3n-2}{2}$ vertices and $M = 2n-2$ edges. Let $v_0\ v_1\ v_2 \ldots v_{N-1}$ be a Hamiltonian path in G. Labeling of the vertices is obtained for two cases given below

Case 1. If $n \equiv 0 \pmod 4$

Define the vertex labeling $\phi : V(G) \rightarrow \{0, 1, t, 1+t, 2t, 2t+1, \ldots, t(M-1), t(M-1)+1\}$ as follows

$\phi(v_0) = 0$
$\phi(v_1) = (2n-3)t+1$
$\phi(v_2) = t$
$\phi(v_3) = (2n-5)t+1$

$\phi(v_{2i+2}) = t(i+2),\ 1 \le i \le \left\lceil \frac{N-7}{2} \right\rceil$

$= t(i+3)$, if $i = \left\lceil \frac{N-5}{2} \right\rceil$

$\phi(v_{2i+p}) = \phi(v_p) - ti,\ 1 \le i \le 2,\ p = 3, 9, 15,..., N-8$

$\phi(v_{q+2}) = \phi(v_q) - 3t$, if $q = 7, 13, 19,..., N-10$

$= \phi(v_q) - 2t$, if $q = N-4$

$\phi(v_{N-1}) = \phi(v_{N-2}) - 1$

From the above vertex labeling it is seen that all the vertices have distinct labels and clearly ϕ is an injection from $V(G) \to \{0, 1, t, t+1, 2t, 2t+1,..., t(M-1), t(M-1)+1\}$. Let A, B, C, D denote the sets of edges of G given by

$A = \{v_0v_1, v_1v_2, v_0v_3, v_2v_3, v_1v_6\}$

$B = \{v_3v_4, v_4v_5,..., v_{N-5}v_{N-4}\}$

$C = \{v_{N-4}v_{N-3}, v_{N-4}v_{N-1}, v_{N-3}v_{N-2}, v_{N-2}v_{N-1}\}$

$D = E(G) - (A \cup B \cup C)$

Let A′, B′, C′, D′ denote the edge labels induced by the edges in the above sets as given below

$A' = \{(M-1)t+1, (M-2)t+1,..., (M-5)t+1\}$

$B' = \{(M-6)t+1, (M-7)t+1,..., (M-10)t+1, (M-13)t+1, (M-14)t+1,..., (M-18)t+1, (M-21)t+1,..., 6t+1, 5t+1\}$

$C' = \{3t+1, 2t+1, t+1, 1\}$

$D' = \{1, t+1, 2t+1,..., t(M-1)+1\} - (A' \cup B' \cup C')$

and $A' \cup B' \cup C' \cup D' = \{1, t+1, 2t+1,..., t(M-1) + 1\}$. Hence the above graph is one modulo t-graceful for $t \ge 3$. An illustration is given in Figure 2.

Case 2. If $n \equiv 2 \bmod 4$

Define the vertex labeling $\phi : V(G) \to \{0, 1, t, 1+t, 2t,..., t(M-1)+1\}$ as follows

$\phi(v_0) = 0$

$\phi(v_1) = (2n-3)t+1$

$\phi(v_2) = t$

$\phi(v_3) = (2n-5)t+1$

$\phi(v_{2i+2}) = t(i+2),\ 1 \le i \le \left\lceil \frac{N-4}{2} \right\rceil$

$\phi(v_{2i+p}) = \phi(v_p) - ti,\ 1 \le i \le 2,\ p = 3, 9, 15,..., N-5$

$\phi(v_{q+2}) = \phi(v_q) - 3t,\ q = 7, 13, 19,..., N-7$

$\phi(v_{N-1}) = \phi(v_{N-2})+1$

From the above vertex labeling it is clearly seen that all the vertices have been given distinct labels and ϕ is an injection from the vertex set of G to $\{0, 1, t, t+1, t+1,..., t(M-1), t(M-1)+1\}$. Let A, B, C, D denote the set of edges of G given by

$A = \{v_0v_1, v_1v_2, v_0v_3, v_2v_3, v_1v_6\}$

$B = \{v_3v_4, v_4v_5, \ldots, v_{N-4}v_{N-3}, v_{N-3}v_{N-2}\}$
$C = \{v_{N-1}v_{N-2}, v_{N-1}v_{N-4}\}$
$D = E(G) - (A \cup B \cup C)$

The edge labels induced by the absolute difference of the labels of adjacent vertices in the above sets are given below in the sets A′, B′, C′, D′ as
$A' = \{(M-1)t+1, (M-2)t+1, \ldots, (M-5)t+1\}$
$B' = \{(M-6)t+1, (M-7)t+1, \ldots, (M-10)t+1, (M-13)t+1, (M-14)t+1, \ldots, (M-18)t+1, (M-21)t+1, \ldots, 3t+1, 2t+1\}$
$C' = \{1, t+1\}$
$D' = \{1, t+1, 2t+1, 3t+1, \ldots, (M-1)t+1\} - (A' \cup B' \cup C')$
and $A' \cup B' \cup C' \cup D' = \{1, t+1, 2t+1, \ldots, (M-1)t + 1\}$.

Hence the above graph is one modulo t-graceful for t ≥ 3.
An illustration is given in Figure 2.

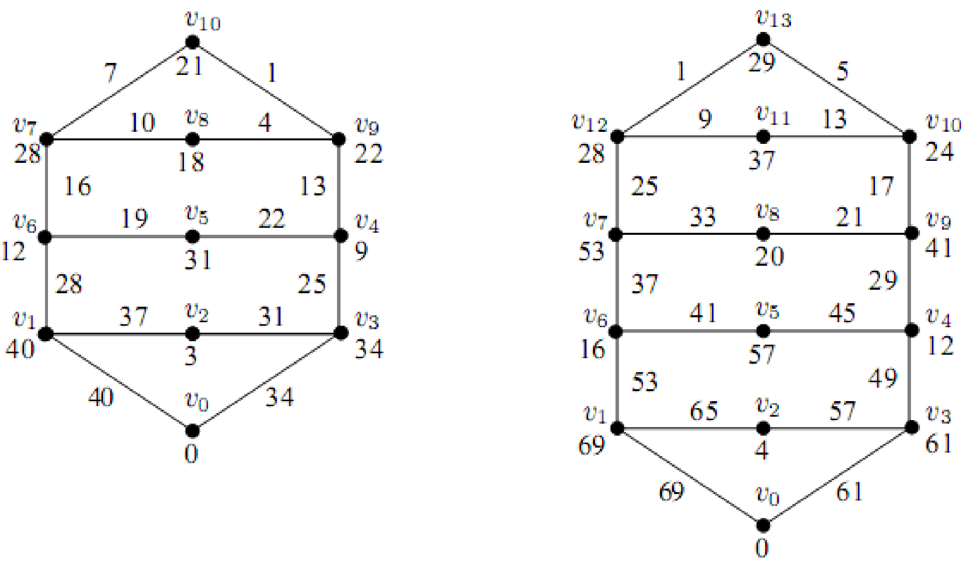

Figure 2. One modulo 3-graceful labeling of C_8 and one modulo 4-graceful labeling of C_{10} with parallel P_3 chords

3. Conclusion

In this paper we have applied two labelings on even cycle C_n (n ≥ 6) with parallel P_3 chords to show that they are t-graceful for t ≥ 2 and one modulo t graceful for t ≥ 3.

References

[1] J.A. Gallian, A dynamic survey of graph labeling, Electronics Journal of Combinatorics, 17 (2014), #DS6.

[2] S.W. Golomb, How to number a graph in graph theory and computing, R.C. Read ex., Academic Press, New York (1972), 23–37.

[3] R. Govindarajan and V. Srividya, Odd graceful labeling of cycle with parallel P_k chords, Annals of Pure and Applied Mathematics, 8(2) (2014), 123–129.

[4] M. Maheo and H. Thuillier, On d-graceful graphs, Ars. Combin., 13 (1982), 181–192.

[5] V. Ramachandran and C. Sekar, One modulo N gracefulness of acyclic graphs, Ultra Scientist, 25(3)A (2013), 417–424.

[6] A. Rosa, On certain valuation of the vertices of a graph, Theory of graphs, Proceedings of the Symposium, Rome, Gordon and Breach, New York (1967), 349–355.

[7] C. Sekar, Studies in graph theory, Ph.D. Thesis, Madurai Kamaraj University (2002).

[8] M.A. Seoud and E.a. Elsahawi, On variations of graceful labeling, Ars. Combin., 87 (2008), 127–138.

[9] G. Sethuraman and A. Elumalai, Gracefulness of a cycle with parallel P_k chords, Australian Journal of Combinatorics, 32 (2005), 205–211.

[10] P.J. Slater, On k-graceful graphs, Proceedings of 13th S.E. Conference on Combinatorics, Graph Theory and Computing (1982), 53–57.