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Generalized perfect-fluid scalar-tensor theory field equations in a reformulated total Lagrangian formalism

Remy Magloire Etoua
*Department of Mathematics,
National Advanced School Polytechnic,
University of Yaounde I,
P.O.Box : 812 Yaounde, Cameroon.*

Raoul Domingo Ayissi
*Department of Mathematics,
Faculty of Science,
University of Yaounde I,
P.O.Box : 812, Yaounde, Cameroon.
E-mail: raoulayissi@yahoo.fr*

Abstract

The mathematical theory on Bianchi models is revisited: all details leading to the classification are displayed. A reformulation of Lagrangian formalism is proposed. The entire process leading to field equations in a generalized scalar-tensor theory is explained. Exact solutions of the hyperextended scalar-tensor theory in the empty Bianchi type I model are investigated.

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1. Introduction

The basic two elements used to define a cosmology are a geometry; mathematically represented by a metric, and a matter content; mathematically described by a Lagrangian. In this paper, we will be interested in a geometry described by the Bianchi cosmological homogeneous models for which the expansion of Universe is not the same according to direction of observation: they are consequently anisotropic on the contrary to the

Freedman-Lemaitre-Robertson-Walker(FLRW) models where the expansion is the same for any direction. Concerning the matter content, we consider the presence of scalar fields in the Universe, together with a perfect fluid. A scalar field is a function which associates at each spatial-temporal point a number. A good example can be the temperature of a room: for each point of a room, one can associate a quantity T defining its temperature. Another example is the gravitational potential ϕ exterior to a mass M. Those fields are abundant in particle physics although they have not yet been detected and it seems thus natural to take them in consideration in cosmology, while the links between those two branches of physics are more and more narrow. The physical interest of scalar fields and Bianchi models will be displayed in this paper through the mathematical study of Bianchi models and the related scalar field equations. We can then ask two questions in connection with the above geometrical and physical description.

The first question is to understand why our Universe is described by model of type FLRW whose spatial symmetry is maximal. In fact, there is no reason for the expansion to be exactly the same in all directions. Do we have to accept the perfection of FLRW models or do we have to abandon it in aid of an approximately perfect Universe? To this question essentially correspond two currents: the first one claims the existence of a quantum principle as a theory of initial conditions that should select among the set of possible models, those of type FLRW; the second one postulates the existence of a primordial Universe less symmetric than a FLRW model, but dynamically evolving toward the latter. We also adopt the latter point of view.

The second question concerns the scalar fields properties theoretically present in our Universe. In fact, there exists an infinite number of possible scalar-tensor theories. That's why, it is necessary to eliminate those leading to absurd physical results or, on the contrary, to locate those leading to physically interesting behaviors for our Universe. It is for this second question that we deeply study in this paper the mathematical theory on Bianchi cosmological models. This study will help us and others to a better understanding of the following problem: which properties need to have scalar-tensor theories so that those models possess asymptotically the dynamic characteristics of our actual Universe or bring a answer to some problems as those of the cosmological constant and isotropization?

Several authors have studied for a long time Bianchi models and various scalar-tensor theories exist. Our contribution is based on the fact that, we deeply clarify step by step the whole mathematical theory on Bianchi models, making in detail all calculations leading to their classification, to the obtainment of scalar curvature using the complicated Cartan method and the models metric. We then reformulate the "total Lagrangian formalism" and we obtain field equations for a generalized perfect-fluid scalar-tensor theory in class A Bianchi models. We give then an application for exact solutions.

The paper is organized as follows:

In section 2, we display the classification of Bianchi models and give the mathematical and geometrical tools.

In section 3, we reformulate the Lagrangian formalism, we give and we solve field equations for a generalized scalar-tensor theory.

2. The Bianchi Models

Greek indexes α, β, γ, ... range from 1 to 3. We adopt the Einstein summation convention:

$$A^{\alpha}B_{\alpha} = \sum_{\alpha} A^{\alpha}B_{\alpha}.$$

2.1. Classification of spatially homogeneous Bianchi models

A Lie algebra is specified by a basis $x_1, \ldots, x_n$ of its associated space vectors. Since the Lie product $\left[x_i, x_j\right]$ of two elements x_i and x_j of the basis also belongs to the Lie algebra, one can write:

$$\left[x_i, x_j\right] = C^k_{ij} x_k, \tag{2.1}$$

where the C^k_{ij} are the structure constants of the Lie algebra.

Isometries of a Lie algebra are generated by **killing vectors** ξ verifying the **killing equations**:

$$\xi_{a;b} + \xi_{b;a} = 0. \tag{2.2}$$

Every killing vector generates an isometry and the set of all those killing vectors forms the Lie algebra of the corresponding group of isometries. Consequently, the search of isometries of a Riemannian manifold is simply equivalent to the search of solutions to killing equations. For example, if we consider a Minkwoskian space for which the quadridimensional infinitesimal length element writes:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2,$$

the killing equations provide 10 independent killing vectors that generate in fact the Poincare group corresponding to the 4 generators of spatial-temporal translations, 3 generators of $3-$ dimensional rotations and 3 generators of Lorentz homogeneous transformations. We then obtain a group G_{10} acting on a Riemannian manifold M_4 representing a manifold with a maximal number of symmetries. This implies that the Minkwoski space-time has a constant Riemannian curvature. In general, a group of isometries G_r with r parameters acting on a $n-$ dimensional Riemannian manifold M_n is such that:

$$n \leq r \leq \frac{n\,(n+1)}{2}. \tag{2.3}$$

M_n is said to have maximal symmetry if

$$r = \frac{n\,(n+1)}{2}. \tag{2.4}$$

To impose the spatial homogeneity, we need a group of isometries acting transitively on the spatial sections ($n = 3$) of the space-time. For $n = 4$, the possibilities are, according to (3) : G_3, G_4, G_5 and G_6. But the group G_6 of maximal symmetry corresponds to the isotropic and homogeneous model of the FLRW class. The group G_5 is not convenient by the *Fubini theorem* which states that: a $n-$ dimensional Riemannian manifold with $n > 2$, without a constant Riemannian curvature has at most a group of isometries with $\frac{n(n+1)}{2} - 1$ parameters. The group G_4 can be treated; apart for the Kantowski-Sachs model; like G_3, because G_4 always admits, apart for one case, a subgroup with 3 parameters acting simply and transitively on spatial hyper-surfaces.

It follows that, apart the Kantowski-Sachs model, the classification of all homogeneous universe models reduces to the classification of spatial isometries with 3 parameters or equivalently the $3-$ dimensional real Lie algebras.

2.2. Classification of $3-$ dimensional real Lie algebras

Let ξ_λ, $\lambda = 1, 2, 3$ be a basis of the Lie algebra such that:

$$\left[\xi_\lambda, \xi_\mu\right] = C^\nu_{\lambda\mu}\xi_\nu.$$

The commutation operators [,] being antisymmetric and verifying the Jacobi identities, one has:

$$C^\nu_{(\lambda\mu)} = 0, \;\; C^\nu_{\sqsubset\lambda\mu}C^\sigma_{\rho\sqsupset} = 0 \tag{2.5}$$

reducing to 9 the number of independent structure constants. One can rewrite them using the Ellis-MacCallum decomposition where intervenes a pseudo symmetric tensor $n^{\lambda\mu}$ and a vector a_μ such that:

$$C^\nu_{\lambda\mu} = \epsilon_{\sigma\lambda\mu}n^{\nu\sigma} + 2\delta^\nu_{\sqsubset\mu}a_{\lambda\sqsupset} \tag{2.6}$$

where the δ are the Kronecker symbols and the ϵ, the Levi-Civita symbols verifying in Minkwoskian coordinates:

$$\epsilon_{\sigma\lambda\mu} = -\epsilon^{\sigma\lambda\mu}, \;\; \epsilon_{123} = 1, \tag{2.7}$$

the square brackets indicating the anti-symmetrization operation on indexes appearing inside. One[2] deduces that:

$$\begin{aligned} a_\mu &= \frac{1}{2}C^\nu_{\mu\nu} \\ n^{\lambda\mu} &= \frac{1}{2}C^{(\lambda}_{\sigma\tau}\epsilon^{\mu)\sigma\tau}, \end{aligned} \tag{2.8}$$

the brackets indicating the symmetrization operation on indexes appearing inside.

The above decomposition verifies the antisymmetry property, and the Jacobi identities provide the equation:

$$n^{\lambda\mu}a_\mu = 0 \tag{2.9}$$

that we have to solve in order to find all the possible structures for a $3-$ dimensional Lie algebra and that are not mutually equivalent under any change of base ξ_λ. The matrix $n^{\lambda\mu}$ is real and symmetric, so we can not apply the *JJ.Sylvester theorem* which states that the rank l and the absolute value $|s|$ of its signature are invariant under the action of a change of basis. We need consequently to search various possible combinations of rank and signature of the matrix $n^{\lambda\mu}$. **The Bianchi models split in two classes**:

- The **Bianchi class** A such that:

$$a_\lambda = 0. \tag{2.10}$$

One chooses a basis in which the tensor $n^{\lambda\mu}$ is diagonal and whose eigenvalues $n^{(i)}$ are diagonal elements of $n^{\lambda\mu}$ and are equal to 0, 1 or -1. We then have six ways to combine the rank and the signature of the matrix $n^{\lambda\mu}$ corresponding to six models: $I,\ II,\ VI_0,\ VII_0,\ VIII$ and IX.

- The **Bianchi class** B such that:

$$a_\lambda \neq 0. \tag{2.11}$$

In this case, a_λ is an eigenvector of $n^{\lambda\mu}$ relatively to the null eigenvalue. One chooses a basis in witch the tensor $n^{\lambda\mu}$ is diagonal with the eigenvalues $n^{(i)}$ and such that the vectors a_λ be oriented along the third axis. One deduces that:

$$n^{(3)} = 0 \tag{2.12}$$

since $a \neq 0$ and that the rank of the matrix is less than or equal to 2.

If furthermore one uses the scale transformation

$$\xi_i = k_i \xi_i' \tag{2.13}$$

where k_i is a constant, one shows that the quantity

$$h^{-1} = n^{(1)} n^{(2)} a^{-2} \tag{2.14}$$

is invariant. The four models obtained will be denoted by : $IV,\ V,\ VI_h$ and VII_h.

- The **dimension of the Lie algebras** obtained.

Every class of equivalence of the structure constants $C^\nu_{\lambda\mu}$ of a Lie algebra constitutes a sub manifold of the 3 indexes tensors space and consequently of 27 dimensions. The structure constants being antisymmetric $(27 - 18 = 9)$ and obeying the three Jacobi identities $(9 - 3 = 6)$, each Bianchi model is associated with a $6-$dimensional sub manifold at most. For the class A models, this corresponds to the 6 components of the matrix $n^{\lambda\mu}$, and for the class B models, to the three components of the vector a_μ and to the three components of $n^{\lambda\mu}$ in the orthogonal plan to the third axis. One deduces that:

- for the Bianchi type $VI_h,\ VII_h,\ VIII$ and IX models, there is no restriction and there exists $6-$ dimensional sets of structure constants at most.
- for the Bianchi types $VI_h,\ VII_h$, if h is fixed, one has a constraint and consequently their sets of structure constants are of 5 dimensions.
- for the Bianchi types $II,\ V$, a vector is given (a_μ for the type V and the first row of $n^{\lambda\mu}$ for the type II). Their sets of structure constants are of 3 dimensions.

– for the Bianchi type I, the structure constants are all null and the dimension is zero.

The above classification can be summarized by the table:

Class	type	$n^{(1)}$	$n^{(2)}$	$n^{(3)}$	a	dimension
A	I	0	0	0	0	0
A	II	1	0	0	0	3
A	VI_0	1	-1	0	0	5
A	VII_0	1	1	0	0	5
A	$VIII$	1	1	-1	0	6
A	IX	1	1	1	0	6
B	V	0	0	0	1	3
B	IV	1	0	0	1	5
B	$III = VI_{-1}$	1	-1	0	1	5
B	$VI_h\ (h < 0)$	1	-1	0	$\sqrt{-h}$	(6) 5 if h fixed
B	$VII_h\ (h > 0)$	1	1	0	$\sqrt{h}$	(6) 5 if h fixed.

(2.15)

2.3. Metrics of Bianchi spatially homogeneous models

A **congruence** is a set of contours completely fulfilling at last one domain locally delimited of a given manifold. To write a metric, we have to choose a temporal congruence and a spatial basis.

2.3.1 Temporal congruence

Let us consider a set of spatial hyper-surfaces invariant under an action of the elements of a group of isometries $G_{r\geq 3}$. Let S be one of the surfaces and P a point belonging to S. One traces a normal temporal geodesic to S and passing through P. n^α is the unit vector tangent to this geodesic on which one measures a proper distance s. One obtains then another point Q and builds thereby a surface S' to which this point belongs. Let P' be any other point of S, since the group of isometries is transitive, there exists a transformation $\phi \in G_r$ such that

$$\phi(P) = P'. \tag{2.16}$$

Again $Q' \in S'$ is deduced from P' by carrying the same distance s along the temporal geodesic perpendicular to S and through P'. This generates thereby a space, tangent to the spatial hyper-surfaces invariant under G_r.

Let $\xi_{(m)}$, $m = 1 \ldots r$, be the killing vectors generating in all points of the space-time, tangent space to the invariant spatial hyper-surfaces. The vectors $\xi_{(m)}$ obey the killing equations

$$\xi_{(m)\alpha;\beta} + \xi_{(m)\beta;\alpha} = 0 \tag{2.17}$$

and n^α is subject to the geodesics equation

$$n^\alpha_{;\beta} n^\beta = 0. \tag{2.18}$$

One deduces that

$$n^{\alpha}\xi_{(m)\alpha} = 0 \tag{2.19}$$

and that the *temporal geodesic with the tangent vector n^{α} is orthogonal to any homogeneous surface cut by the geodesic*, because

$$n^{\alpha} \perp \xi_{(m)\alpha},\ m = 1 \ldots r. \tag{2.20}$$

Consequently, *normals to the* hyper-surfaces of homogeneity *constitute the vectors field tangent to a time-like geodesic congruence, orthogonal to spatial hyper-surfaces*. One chooses then direction of n^{α} to define the temporal variable t.

The homogeneous spatial hyper-surfaces are then the surfaces $S(t)$ where t is constant. Those surfaces are parametrized by the distance measured along the temporal geodesics, so:

$$n_{\alpha} = -\frac{\partial t}{\partial x^{\alpha}} = (-1, 0, 0, 0)\,. \tag{2.21}$$

The above choice defines a synchronized reference frame with

$$g_{00} = -1,\ \ g_{0m} = 0,\ m = 1, 2, 3. \tag{2.22}$$

$x^0 = t$ is the proper time of each point of the space and the metric in the synchronized reference frame reads:

$$ds^2 = -dt^2 + g_{mn}dx^m dx^n,\ m, n = 1, 2, 3. \tag{2.23}$$

The spatial and temporal variables are not mixed and due to the spatial homogeneity, the vectors field n^{α} is invariant under the action of the elements of G_r. This invariance of the group implies the vanishing of the Lie derivative relatively to any infinitesimal generator of isometries. It follows that n^{α} commutes with any killing vector:

$$\left[\xi_{(\mu)}, n\right] = 0. \tag{2.24}$$

2.3.2 Spatial basis

Let G_r be a group of infinitesimal transformations and $\left(\xi_{(\mu)}\right)$ a basis of killing vectors. One defines the orbit at a point P of the manifold M as being a sub manifold constituted of the points of M resulting from the action of all the elements of the group on P. We will search the set of vectors $\chi_{(m)}, m = 1, 2, 3$ that under tightens the tangent space to the orbit or equivalently:

$$\left[\chi_{(n)}, \xi_{(m)}\right] = 0,\ (m, n) = 1 \ldots r. \tag{2.25}$$

The above equality indicates that they constitute an invariant basis whose structure constants D^l_{mn} are introduced by the means of commutation operators:

$$\left[\chi_{(n)}, \chi_{(m)}\right] = D^l_{mn}\chi_{(l)}. \tag{2.26}$$

In order to build an invariant basis, let us consider r independent vectors $\chi_{(n)}$ at a point P_0 with the initial conditions

$$\chi_{(n)0} = \xi_{(n)}\left(P_0\right), \tag{2.27}$$

r standing for the number of parameters of the group of isometries, and let us translate them using the Lie derivative in view to define r vector fields on M on which the group G_r acts. If C^l_{mn} stands for the structure constants of the killing vectors, one finds that

$$\begin{array}{l} D^l_{mn} = -C^l_{mn}, \ \forall P \in M \\ \left[\chi_{(m)}, \chi_{(n)}\right] = -C^l_{mn}\chi_{(l)}. \end{array} \tag{2.28}$$

One deduces that the Lie algebra formed of invariant vectors of the vector fields tangent to the orbit, is equivalent to the Lie algebra of killing vectors of the group G_r. One can then show that the scalar product of any of two invariant vector fields is constant on each orbit:

$$\left(\chi^\alpha_{(m)}\chi^\beta_{(n)}\right)_{;\gamma}\xi^\gamma = 0, \tag{2.29}$$

for any given killing vector ξ_γ.

Consequently, the invariant basis $\left(\chi_{(m)}\right)$, built at a point of each homogeneous surface becomes a vector's field on the space-time, by translating the invariant vectors by means of the Lie derivative with respect to the field vectors $n_\alpha = (-1, 0, 0, 0)$, orthogonal to the hyper-surfaces $S\left(t\right)$:

$$\left[\chi_{(\mu)}, n\right] = 0 \Longleftrightarrow \frac{\partial}{\partial t}\left(\chi^a_{(\mu)}\right) = 0. \tag{2.30}$$

It follows that the invariant vectors are independent of the time and the scalar products

$$g_{ab}\chi^a_{(m)}\chi^b_{(n)},$$

simply denoted by g_{mn}, are constant on each surface of transitivity and depend only on time. One can henceforth explicitly write the formulation of the cosmological homogeneous Bianchi models metric. It suffices to choose the $\chi^{(m)}_a$ such that:

$$\chi^{(m)}_a\chi^a_{(n)} = \delta^m_n. \tag{2.31}$$

The spatial-temporal metric then writes:

$$ds^2 = -dt^2 + g_{mn}\left(t\right)\chi^{(m)}_a\chi^{(n)}_b dx^a dx^b. \tag{2.32}$$

A differential ***one-form*** is a linear operator acting on vector fields. Thus if ω in a $1-$form and $\overrightarrow{U}$ a vector, $\omega\left(\overrightarrow{U}\right)$ is a function such that $\omega\left(\overrightarrow{U}\right)(P)$ gives a real, P being a point. One introduces then the differential $1-$forms $\left(\omega^{(m)}\right)$ such that:

$$\omega^{(m)}_a\chi^a_{(n)} = \delta^m_n. \tag{2.33}$$

The 1− forms constitute the ***dual basis*** of the $\left(\chi_{(m)}\right)$. Then the inverse matrix

$$\left\| \chi_a^{(m)} \right\| \tag{2.34}$$

where the upper index (m) stands for a line index, can be interpreted as providing the co-variant components of the 1− forms $\omega^{(m)}$. The 1− forms of the basis verify the **Cartan equations** and one writes

$$\omega^{(m)} = \chi_a^{(m)} dx^a. \tag{2.35}$$

The final shape of the metric can then be written:

$$ds^2 = -dt^2 + g_{mn}(t)\,\omega^m \omega^n. \tag{2.36}$$

2.3.3 Invariants vectors and metrics of Bianchi models

The vectors being the group invariants, and then commuting with the killing vectors, one has in term of components:

$$\xi_{(m)}^a \chi_{(n),a}^b - \chi_{(n)}^a \xi_{(m),a}^b = 0. \tag{2.37}$$

Since the determinant of $\left\| \chi_{(m)}^a \right\|$ does not vanish, the lower index (m) standing for the column index, one can define three co-variant vectors, $\chi^{(m)}$, with components $\chi_a^{(m)}$ such that

$$\chi_{(m)}^a \chi_b^{(m)} = \delta_b^a. \tag{2.38}$$

Moreover, we know that

$$\xi_{(m)}^a \xi_b^{(m)} = \delta_b^a \tag{2.39}$$

and reporting the relation (39) in (37) , one finds:

$$\xi_{(n),c}^b - \chi_{(n)}^a \xi_{(m),a}^b \xi_c^{(m)} = 0. \tag{2.40}$$

The equation we will use for calculation of invariant vectors is then written:

$$\xi_{(n),b}^a - \xi_{(m),c}^a \xi_b^{(m)} \chi_{(n)}^c = 0 \tag{2.41}$$

with the following initial conditions given at a point of spatial coordinates (0, 0, 0) :

$$\chi_{(m)}^a(0) = \xi_{(m)}^a(0).$$

Moreover, the killing vectors $\xi_{(m)}$ of the group G_3 of spatial homogeneity corresponding to the various Bianchi types and having C_{mn}^l as structure constants verify:

$$\xi_{(m)}^a \xi_{(n),a}^b - \xi_{(n)}^a \xi_{(m),a}^b = C_{mn}^l \xi_{(l)}^b, \tag{2.42}$$

the Lie product of two killing vectors being a killing vector.

2.4. Application: Bianchi type II model

The method needed to follow, for obtaining invariant bases of Bianchi cosmological models is the next:

a) one supposes that the structure constants of the model are known;

b) one solves eq (42) in order to obtain the killing vectors $\xi^a_{(m)}$;

c) one solves eq (41) in order to obtain the invariant basis vectors $\chi^a_{(m)}$;

d) one writes explicitly the metric using $(36)-(37)$.

Notice that, using (15) , one can easily compute the structure constants C^l_{mn} appearing in (42) to obtain:

$$
\begin{array}{ll}
\textbf{Class A} & \textbf{Structure constants} \\
I & C^{\lambda}_{\mu\nu} = 0 \\
II & C^1_{23} = -C^1_{32} = 1 \\
VI_0 & C^1_{23} = -C^1_{32} = 1,\ C^2_{13} = -C^2_{31} = 1 \\
VII_0 & C^1_{23} = -C^1_{32} = 1,\ C^2_{13} = -C^2_{31} = -1 \\
VIII & C^1_{23} = -C^1_{32} = 1,\ C^2_{31} = -C^2_{13} = 1,\ C^3_{12} = -C^3_{21} = -1 \\
IX & C^1_{23} = -C^1_{32} = 1,\ C^2_{31} = -C^2_{13} = 1,\ C^3_{12} = -C^3_{21} = 1 \\
\textbf{Class B} & \textbf{Structure constants} \\
V & C^1_{13} = -C^1_{31} = -1,\ C^2_{23} = -C^2_{32} = -1 \\
IV & C^1_{13} = -C^1_{31} = -1,\ C^1_{23} = -C^1_{32} = 1,\ C^2_{23} = -C^2_{32} = -1 \\
VI_h & C^1_{23} = -C^1_{32} = 1,\ C^2_{13} = -C^2_{31} = 1,\ C^1_{13} = -C^1_{31} = -\sqrt{-h}, \\
 & \quad C^2_{23} = -C^2_{32} = -\sqrt{-h} \\
VII_h & C^1_{23} = -C^1_{32} = 1,\ C^2_{13} = -C^2_{31} = -1,\ C^1_{13} = -C^1_{31} = -\sqrt{h}, \\
 & \quad C^2_{23} = -C^2_{32} = -\sqrt{h}.
\end{array}
\tag{2.43}
$$

Thus, for the Bianchi type II model, the only non vanishing structure constants are

$$C^1_{23} = -C^1_{32} = 1.$$

The equation (42) gives:

$$
\begin{aligned}
\xi^a_{(1)}\xi^b_{(3),a} - \xi^a_{(3)}\xi^b_{(1),a} &= 0 \\
\xi^a_{(1)}\xi^b_{(2),a} - \xi^a_{(2)}\xi^b_{(1),a} &= 0 \\
\xi^a_{(2)}\xi^b_{(3),a} - \xi^a_{(3)}\xi^b_{(2),a} &= 0
\end{aligned}
\tag{2.44}
$$

whose particular solution is:

$$\left\|\xi^{a}_{(m)}\right\| = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & x^{3} \\ 0 & 1 & 0 \end{pmatrix}, \quad \left\|\xi^{a}_{(m)}\right\|^{-1} = \left\|\xi^{(m)a}\right\| = \begin{pmatrix} -x^{3} & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.45}$$

where the lower index (m) stands for a column index and the upper index (m) stands for a line one.

The equation (41) then writes:

$$\begin{aligned} \chi^{1}_{(n),b} &= 0 \\ \chi^{3}_{(n),b} &= 0 \\ \chi^{2}_{(n),b} &= \xi^{(3)}\chi^{3}_{(n)} \end{aligned} \tag{2.46}$$

where $\xi^{(3)}_{b} = 0$ for every b except for $b = 1$, and in this case $\xi^{(3)}_{1} = 1$.

The equation (46) implies that

$$\begin{aligned} \chi^{1}_{(n)} &= constant, \ \forall n \\ \chi^{3}_{(n)} &= constant, \ \forall n. \end{aligned} \tag{2.47}$$

From $(46) - (47)$, one gets

$$\begin{aligned} \chi^{2}_{(n),1} &= \chi^{3}_{(n)} \\ \chi^{2}_{(n),2} &= 0 \\ \chi^{2}_{(n),3} &= 0 \end{aligned} \tag{2.48}$$

where one deduces that

$$\chi^{2}_{(n)} = \chi^{3}_{(n)}x^{1} + const, \ \forall n. \tag{2.49}$$

Using the relation (49), one builds three invariant basis vectors:

$$\left\|\chi^{a}_{(n)}\right\| = \begin{pmatrix} 0 & 0 & 1 \\ 1 & x^{1} & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \left\|\chi^{a}_{(n)}\right\|^{-1} = \left\|\chi^{(n)}_{a}\right\| = \begin{pmatrix} 0 & 1 & -x^{1} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.50}$$

The relation (35) allows to write:

$$\begin{aligned} \omega^{1} &= dx^{2} - x^{1}dx^{3} \\ \omega^{2} &= dx^{3} \\ \omega^{3} &= dx^{1} \end{aligned} \tag{2.51}$$

giving then the diagonal Bianchi type II metric

$$ds^{2} = -dt^{2} + g_{11}(t)\left(dx^{2} - x^{1}dx^{3}\right)^{2} + g_{22}(t)\left(dx^{3}\right)^{2} + g_{33}(t)\left(dx^{1}\right)^{2}. \tag{2.52}$$

Using the same method as the one displayed above, one obtains the following table of the *one-forms* defining each type of Bianchi models

Class A	ω^1	ω^2	ω^3
I	dx^1	dx^2	dx^3
II	$dx^2 - x^1 dx^3$	dx^3	dx^1
VI_0	$\cosh x^1 dx^2 + \sin x^1 dx^3$	$\sinh x^1 dx^2 + \cosh x^1 dx^3$	$-dx^1$
VII_0	$\cos x^1 dx^2 + \sin x^1 dx^3$	$-\sin x^1 dx^2 + \cos x^1 dx^3$	$-dx^1$
$VIII$	$dx^1 + \left(\left(x^1\right)^2 - 1\right) dx^2 +$	$2dx^1 dx^2 + \left(1 - 2x^1 x^2\right) dx^3$	$-dx^1 - \left(1 + \left(x^1\right)^2\right) dx^2 +$
	$\left(x^1 + x^2 - \left(x^1\right)^2 x^2\right) dx^3$		$\left(x^2 - x^1 + \left(x^1\right)^2 x^2\right) dx^3$
IX	$-\sin x^3 dx^1 + \sin x^1 \cos x^3 dx^2$	$\cos x^3 dx^1 + \sin x^1 \sin x^3 dx^2$	$\cos x^1 dx^2 + dx^3$
Class B	ω^1	ω^2	ω^3
V	$e^{-x^1} dx^2$	$e^{-x^1} dx^3$	$-dx^1$
IV	$e^{-x^1} dx^2 + x^1 e^{-x^1} dx^3$	$e^{x^1} dx^3$	$-dx^1$
VI_h	$e^{-ax^1} \cosh x^1 dx^2 +$	$e^{-ax^1} \sinh x^1 dx^2 +$	$-dx^1$
	$e^{-ax^1} \sinh x^1 dx^3$	$e^{-ax^1} \cosh x^1 dx^3$	
VII_h	$e^{-ax^1} \cos x^1 dx^2 +$	$-e^{-ax^1} \sin x^1 dx^2 +$	$-dx^1$.
	$e^{-ax^1} \sin x^1 dx^3$	$e^{-ax^1} \cos x^1 dx^3$	

(2.53)

3. Field equations for a perfect-fluid scalar-tensor theory in a reformulated Lagrangian formalism

In this section, we show first of all how to obtain rapidly the non vanishing components of the curvature tensor using the Cartan method. Secondly we partially reformulate the mathematical theory of Lagrangian formalism, and after we give field equations of generalized scalar-tensor theory in Lagrangian formalism, with application to the *class A* Bianchi models.

3.1. Calculation of the curvature of a manifold

3.1.1 Differentiation of the basis $1-$ forms

We are going to establish the Cartan structure equations which allow to obtain the curvature of a manifold without computing the null components of the curvature tensor. To this end, let us introduce the concept of differentiation of the basis differential $1-$ forms.

Let $\left\{\overrightarrow{e_i}\right\}$, be a basis vectors for a Riemannian space and $\{\tilde{\omega_i}\}$ its dual basis. One has:

$$\begin{aligned} \overrightarrow{e_i} &= a_i^s \frac{\partial}{\partial x^s} \\ \tilde{\omega_i} &= b_s^i d\tilde{x}_s \end{aligned} \tag{3.1}$$

a_s and b_s representing functions of the time t, and by self-duality:

$$< \tilde{\omega_i}, \overrightarrow{e_j} >= b_s^i a_j^t \delta_t^s = \delta_j^i \tag{3.2}$$

which implies

$$b^i_s a^s_j = \delta^i_j. \tag{3.3}$$

We recall the definition of the exterior product of two 1− forms as follows:

$$\begin{aligned} \tilde{\mu} \wedge \tilde{\nu} &= \tilde{\mu} \otimes \tilde{\nu} - \tilde{\mu} \otimes \tilde{\nu} \\ \tilde{\mu} \wedge \tilde{\nu} &= -\tilde{\nu} \wedge \tilde{\mu} \\ \tilde{\mu} \wedge \tilde{\mu} &= 0 \end{aligned} \tag{3.4}$$

where $\otimes$ stands for the tonsorial product. The exterior differential of a 1− form is then written

$$\tilde{d}\tilde{\omega}^i = \tilde{d}b^i_s \wedge \tilde{d}x^s = b^i_{s,t}\tilde{d}x^t \wedge \tilde{d}x^s \tag{3.5}$$

since $\tilde{d}\left(\tilde{d}x^s\right) = 0$. Using (56) one obtains

$$\tilde{d}\tilde{\omega}^i = b^i_{s,t}a^t_j a^s_k \tilde{\omega}^j \wedge \tilde{\omega}^k \tag{3.6}$$

and

$$\left(b^i_s a^s_k\right)_{,t} = \left(\delta^i_k\right)_{,t} = 0 \Rightarrow b^i_{s,t}a^s_k = -b^i_s a^s_{k,t}. \tag{3.7}$$

Consequently, (59) yields:

$$\tilde{d}\tilde{\omega}^i = -b^i_s a^s_{k,t} a^t_j \tilde{\omega}^j \wedge \tilde{\omega}^k. \tag{3.8}$$

But only the antisymmetric part of the 1− form $\tilde{\omega}^j \wedge \tilde{\omega}^k$ is essential, since the whole expression is antisymmetric on the j and k indexes, so:

$$\tilde{d}\tilde{\omega}^i = -\frac{1}{2}b^i_s \left(a^t_j a^s_{k,t} - a^t_k a^s_{j,t}\right)\tilde{\omega}^j \wedge \tilde{\omega}^k.$$

Moreover one has:

$$\left[\overrightarrow{e_j}, \overrightarrow{e_k}\right] = \left(a^t_j a^s_{k,t} - a^t_k a^s_{j,t}\right) b^i_s \overrightarrow{e_i} = C^i_{jk}\overrightarrow{e_i},$$

so:

$$\tilde{d}\tilde{\omega}^i = -\frac{1}{2}C^i_{jk}\tilde{\omega}^j \wedge \tilde{\omega}^k. \tag{3.9}$$

The above equation gives the ***exterior differential of the basis*** 1−***forms*** in terms of the exterior product of those 1− forms.

3.1.2 Cartan structure equations

One defines the set of the 1− forms of the affine connection $\tilde{\omega}^i_j$ by:

$$\tilde{\omega}^i_j = \Gamma^i_{jk}\tilde{\omega}^k \tag{3.10}$$

where $\nabla_i \overrightarrow{e_j} = \Gamma^i_{jk} \overrightarrow{e_k}$, ∇ being the affine connection of components Γ and $\nabla_i = \nabla_{\overrightarrow{e_i}}$, for any basis vector $\overrightarrow{e_i}$ defined in (54). We only consider the single case of affine symmetric connections, meaning that:

$$\nabla_{\overrightarrow{u}} \overrightarrow{v} - \nabla_{\overrightarrow{v}} \overrightarrow{u} = \left[\overrightarrow{u}, \overrightarrow{v}\right] \tag{3.11}$$

for any vector fields $\overrightarrow{u}$ and $\overrightarrow{v}$. The above symmetry condition allows us to deduct using the basis vectors:

$$\Gamma^i_{kj} - \Gamma^i_{jk} = C^i_{jk}. \tag{3.12}$$

So the relation (62) becomes

$$\tilde{d}\tilde{\omega}^i = -\Gamma^i_{kj}\tilde{\omega}^j \wedge \tilde{\omega}^k$$

for which one extracts the ***first Cartan structure equation***:

$$\tilde{d}\tilde{\omega}^i = -\tilde{\omega}^i_k \wedge \tilde{\omega}^k. \tag{3.13}$$

In the view of obtaining the second Cartan structure equation, one has to compute the exterior differential of the affine connection 1− forms

$$\tilde{d}\tilde{\omega}^i_j = \Gamma^i_{js,t}\tilde{\omega}^t \wedge \tilde{\omega}^s - \frac{1}{2}\Gamma^i_{jl}C^l_{ts}\tilde{\omega}^t \wedge \tilde{\omega}^s. \tag{3.14}$$

Moreover

$$\tilde{\omega_l}^i \wedge \tilde{\omega}^l_j = \Gamma^i_{lt}\Gamma^l_{js}\tilde{\omega}^t \wedge \tilde{\omega}^s. \tag{3.15}$$

Only the antisymmetric part of the components $\tilde{\omega}^t \wedge \tilde{\omega}^s$ is useful in relations (67) − (68). Summing, one gets:

$$\tilde{d}\tilde{\omega}^i_j + \tilde{\omega_s}^i \wedge \tilde{\omega}^s_j = \frac{1}{2}\left(\Gamma^i_{js,t} - \Gamma^i_{jt,s} - \Gamma^i_{jl}C^l_{ts} + \Gamma^i_{lt}\Gamma^l_{js} - \Gamma^i_{ls}\Gamma^l_{jt}\right)\tilde{\omega}^t \wedge \tilde{\omega}^s. \tag{3.16}$$

Now the curvature operator is defined by

$$\begin{aligned} R\left(\overrightarrow{e_s}, \overrightarrow{e_t}\right)\overrightarrow{e_j} &= \nabla_s\left(\nabla_t \overrightarrow{e_j}\right) - \nabla_t\left(\nabla_s \overrightarrow{e_j}\right) - \nabla_{\left[\overrightarrow{e_s}, \overrightarrow{e_t}\right]}\overrightarrow{e_j} \\ &= \nabla_s\left(\nabla_t \overrightarrow{e_j}\right) - \nabla_t\left(\nabla_s \overrightarrow{e_j}\right) - C^l_{st}\nabla_l \overrightarrow{e_j} \\ &= R^i_{jst}\overrightarrow{e_i} \end{aligned} \tag{3.17}$$

since $\left[\overrightarrow{e_s}, \overrightarrow{e_t}\right] = C^l_{st}\overrightarrow{e_l}$ and R^i_{jst} represents the components of the Riemann tensor. So:

$$R^i_{jst} = -\Gamma^i_{js,t} + \Gamma^i_{jt,s} + \Gamma^i_{jl}C^l_{ts} - \Gamma^i_{lt}\Gamma^l_{js} + \Gamma^i_{ls}\Gamma^l_{jt}. \tag{3.18}$$

One obtains the ***second Cartan structure equation***

$$\tilde{d}\tilde{\omega}^i_j + \tilde{\omega_s}^i \wedge \tilde{\omega}^s_j = \frac{1}{2}R^i_{jst}\tilde{\omega}^s \wedge \tilde{\omega}^t. \tag{3.19}$$

In what follows, the two Cartan structure equations (66) and (72) will be very helpful for the computation of the curvature tensor.

3.1.3 The Cartan method

Let M be a manifold and g its Riemannian metric. (M, g) induces an associated symmetric covariant derivation ∇. The ***Riemann compatibility condition*** written below guarantees the compatibility between ∇ and g :

$$\nabla_{\vec{w}}\left(g\left(\vec{u}, \vec{v}\right)\right) = g\left(\nabla_{\vec{w}}\vec{u}, \vec{v}\right) + g\left(\vec{u}, \nabla_{\vec{w}}\vec{v}\right) \tag{3.20}$$

where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vector fields and g the metric tensor of components g_{ij}. The condition (73) together with the first Cartan structure equation (66) allows to compute the connection forms and symbols using the metric and basis differential forms:

$$\tilde{d}g_{ij} = \tilde{\omega}_{ij} + \tilde{\omega}_{ji} \tag{3.21}$$

where $\tilde{\omega}_{ij} = g_{is}\tilde{\omega}^s_j = \Gamma_{ijk}\tilde{\omega}^k$. Thus, if one chooses a basis in which g_{ij} is a constant, one gets:

$$\tilde{d}g_{ij} = 0, \ \ \tilde{\omega}_{ij} = -\tilde{\omega}_{ji}. \tag{3.22}$$

The **Cartan method** consist in:

a) choosing a tetra of basis vectors $\left\{\vec{e_i}\right\}$ and a tetra of the corresponding dual $1-$forms basis $\left\{\tilde{\omega}^i\right\}$ such that $g_{ij} = \vec{e_i}.\vec{e_j} = const$ in order to use (75) ;

b) solving the first Cartan structure equation using (75) and obtaining then the six affine connection $1-$forms $\tilde{\omega}^i_j$;

c) using those $2-$ forms in view to compute the six curvature $2-$ forms $\tilde{\theta}^i_j = \tilde{d}\tilde{\omega}^i_j + \tilde{\omega}^i_l \wedge \tilde{\omega}^l_j$ and the second Cartan structure equation in view to obtain Riemann curvature tensor components in the chosen basis.

3.1.4 Application of the Cartan method in the Bianchi type IX homogeneous model

We now display the first steps of calculation in the Cartan basis (in witch the most structure constants are equal to 0 or ± 1) of the invariant forms $\left\{\tilde{\omega}^i\right\}$ under the group $SO(3)$ and, that generate homogeneous Bianchi type IX space-time, of the Riemann curvature tensor components applying the Cartan method. For this end, let us write the metric in the following form

$$ds^2 = -dt^2 + e^{2\alpha}\left(\tilde{\omega}^1\right)^2 + e^{2\beta}\left(\tilde{\omega}^2\right)^2 + e^{2\gamma}\left(\tilde{\omega}^3\right)^2 \tag{3.23}$$

where α, β and γ are functions depending only on t and the ω^i, the $1-$ forms defining the homogeneous Bianchi type IX model:

$$\begin{gathered}\tilde{\omega}^0 = \tilde{d}t \\ \tilde{\omega}^1 = -\sin(z)\,\tilde{d}x + \sin(x)\cos(z)\,\tilde{d}y \\ \tilde{\omega}^2 = \cos(z)\,\tilde{d}x + \sin(x)\sin(z)\,\tilde{d}y \\ \tilde{\omega}^3 = \cos(x)\,\tilde{d}y + \tilde{d}z.\end{gathered} \tag{3.24}$$

One gets then

$$\begin{array}{c}\tilde{d}\tilde{\omega}^0 = 0\\ \tilde{d}\tilde{\omega}^1 = -\cos(z)\,\tilde{d}z\wedge\tilde{d}x + \cos(x)\cos(z)\,\tilde{d}x\wedge\tilde{d}y - \sin(x)\sin(z)\,\tilde{d}z\wedge\tilde{d}y\\ \tilde{d}\tilde{\omega}^2 = -\sin(z)\,\tilde{d}z\wedge\tilde{d}x + \cos(x)\sin(z)\,\tilde{d}x\wedge\tilde{d}y - \sin(x)\cos(z)\,\tilde{d}z\wedge\tilde{d}y\\ \tilde{d}\tilde{\omega}^3 = -\sin(x)\,\tilde{d}x\wedge\tilde{d}y.\end{array} \tag{3.25}$$

Consequently

$$\begin{array}{c}\tilde{d}\tilde{\omega}^1 = \tilde{\omega}^2\wedge\tilde{\omega}^3\\ \tilde{d}\tilde{\omega}^2 = \tilde{\omega}^3\wedge\tilde{\omega}^1\\ \tilde{d}\tilde{\omega}^3 = \tilde{\omega}^1\wedge\tilde{\omega}^2.\end{array} \tag{3.26}$$

One chooses a new basis of $1-$ forms such that the metric functions g_{ij} be constants and a new τ time coordinate:

$$\begin{array}{c}\tilde{\nu}^0 = \tilde{d}t = e^{\alpha+\beta+\gamma}\tilde{d}\tau\\ \tilde{\nu}^i = e^{\alpha_i}\tilde{\omega}^i\end{array} \tag{3.27}$$

with $i = 1, 2, 3$ and $\alpha_i = \alpha, \beta, \gamma$. The metric then writes

$$ds^2 = -\left(\tilde{\nu}^0\right)^2 + \left(\tilde{\nu}^1\right)^2 + \left(\tilde{\nu}^2\right)^2 + \left(\tilde{\nu}^3\right)^2 \tag{3.28}$$

and one computes that

$$\begin{array}{c}\tilde{d}\tilde{\nu}^1 = \alpha' e^{-\alpha-\beta-\gamma}\tilde{\nu}^0\wedge\tilde{\nu}^1 + e^{\alpha-\beta-\gamma}\tilde{\nu}^2\wedge\tilde{\nu}^3\\ \tilde{d}\tilde{\nu}^2 = \beta' e^{-\alpha-\beta-\gamma}\tilde{\nu}^0\wedge\tilde{\nu}^2 + e^{\beta-\alpha-\gamma}\tilde{\nu}^3\wedge\tilde{\nu}^1\\ \tilde{d}\tilde{\nu}^3 = \gamma' e^{-\alpha-\beta-\gamma}\tilde{\nu}^0\wedge\tilde{\nu}^3 + e^{\gamma-\alpha-\beta}\tilde{\nu}^1\wedge\tilde{\nu}^2.\end{array} \tag{3.29}$$

Now by antisymmetry one has:

$$\begin{array}{c}\tilde{\nu}^\eta_\eta = 0\\ \tilde{\nu}^0_\eta = \tilde{\nu}^\eta_0\\ \tilde{\nu}^n_m = \tilde{\nu}^m_n\end{array} \tag{3.30}$$

without summation on $\eta = 0, 1, 2, 3$.

Using relations (83) with the first Cartan structure equation, we are allow to write

$$\begin{array}{c}-\tilde{d}\tilde{\nu}^0 = \tilde{\nu}^0_1\wedge\tilde{\nu}^1 + \tilde{\nu}^0_2\wedge\tilde{\nu}^2 + \tilde{\nu}^0_3\wedge\tilde{\nu}^3\\ -\tilde{d}\tilde{\nu}^1 = \tilde{\nu}^1_0\wedge\tilde{\nu}^1 + \tilde{\nu}^1_2\wedge\tilde{\nu}^2 + \tilde{\nu}^1_3\wedge\tilde{\nu}^3 = -\left(\alpha' e^{-\alpha-\beta-\gamma}\tilde{\nu}^0\wedge\tilde{\nu}^1 + e^{\alpha-\beta-\gamma}\tilde{\nu}^2\wedge\tilde{\nu}^3\right)\\ -\tilde{d}\tilde{\nu}^2 = \tilde{\nu}^2_0\wedge\tilde{\nu}^0 + \tilde{\nu}^2_1\wedge\tilde{\nu}^1 + \tilde{\nu}^2_3\wedge\tilde{\nu}^3 = -\left(\beta' e^{-\alpha-\beta-\gamma}\tilde{\nu}^0\wedge\tilde{\nu}^2 + e^{\beta-\alpha-\gamma}\tilde{\nu}^3\wedge\tilde{\nu}^1\right)\\ -\tilde{d}\tilde{\nu}^3 = \tilde{\nu}^3_0\wedge\tilde{\nu}^0 + \tilde{\nu}^3_1\wedge\tilde{\nu}^1 + \tilde{\nu}^3_2\wedge\tilde{\nu}^2 = -\left(\gamma' e^{-\alpha-\beta-\gamma}\tilde{\nu}^0\wedge\tilde{\nu}^3 + e^{\gamma-\alpha-\beta}\tilde{\nu}^1\wedge\tilde{\nu}^2\right).\end{array} \tag{3.31}$$

Examining the system (84) , one finds that:

$$\begin{aligned}
\tilde{\nu}_0^1 &= \alpha' e^{-\alpha-\beta-\gamma}\tilde{\nu}^1 \\
\tilde{\nu}_0^2 &= \beta' e^{-\alpha-\beta-\gamma}\tilde{\nu}^2 \\
\tilde{\nu}_0^3 &= \gamma' e^{-\alpha-\beta-\gamma}\tilde{\nu}^3 \\
\tilde{\nu}_2^1 &= \frac{1}{2}e^{-\alpha-\beta-\gamma}\left(e^{2\alpha}+e^{2\beta}-e^{2\gamma}\right)\tilde{\nu}^3 \\
\tilde{\nu}_1^3 &= \frac{1}{2}e^{-\alpha-\beta-\gamma}\left(e^{2\alpha}-e^{2\beta}+e^{2\gamma}\right)\tilde{\nu}^2 \\
\tilde{\nu}_3^2 &= \frac{1}{2}e^{-\alpha-\beta-\gamma}\left(-e^{2\alpha}+e^{2\beta}+e^{2\gamma}\right)\tilde{\nu}^1 .
\end{aligned} \tag{3.32}$$

We have now to use the second Cartan structure equation to compute the exterior differentials and their exterior product $\tilde{\nu}_j^i$ from which we extract the curvature 2− forms

$$\tilde{\theta}_v^u = \tilde{d}\tilde{\nu}_v^u + \tilde{\nu}_s^u \wedge \tilde{\nu}_v^s = \frac{1}{2}R_{vst}^u \tilde{\nu}^s \wedge \tilde{\nu}^t, \;\; u, v, s, t = 0, 1, 2, 3. \tag{3.33}$$

By identification in (86) , one obtains the Riemann tensor components.

3.2. Reformulation of the Lagrangian formalism

We assume in all what follows the 4− dimensional Riemann space (M, g) invariance under any isomorphism, where g stands for a metric tensor which can be either external or dynamical. We also assume that there exists other fields collectively referred to as *matter fields* denoted by φ. Of course, g occupies a special status as the mediator of gravitational interactions with all the other fields appearing in the theory. The dynamic of the theory is governed by a total Lagrangian L, assumed to be the sum of two terms, a "purely gravitational" part L_g depending only on the metric tensor g and its first and second order partial derivatives but not on the matter fields or their derivatives, and a "matter field part" L_m depending on the matter fields and their first and second order partial derivatives as well as on the metric tensor g and its first and second order partial derivatives.

$$L\left(g, R, \varphi, \nabla\varphi\right) = L_g + L_m = L_g\left(g, \partial g, \partial^2 g\right) + L_m\left(g, \partial g, \partial^2 g, \varphi, \partial\varphi\right). \tag{3.34}$$

A standard additional hypothesis is that the purely gravitational part, L_g depends on the metric tensor g and its first and second order partial derivatives only through combinations constructed from g itself and the Riemann curvature R :

$$L_g = L_g\left(g, R\right). \tag{3.35}$$

Similarly, the matter part L_m is allowed to depend on the metric tensor g and its first and second derivatives only through combinations constructed from g itself, the Riemann curvature and the co-variant derivatives of the matter fields:

$$L_m = L_m\left(g, R, \varphi, \nabla\varphi\right). \tag{3.36}$$

We now define the total energy-momentum tensor T to be the sum:

$$T = T_g + T_m \tag{3.37}$$

where T_g and T_m are given by the variational derivatives of L_g and L_m with respect to the metric tensor

$$\begin{aligned} T_g^{\mu\nu} &= -2\frac{\delta L_g}{\delta g_{\mu\nu}} \\ T_m^{\mu\nu} &= -2\frac{\delta L_m}{\delta g_{\mu\nu}} \\ T^{\mu\nu} &= T_g^{\mu\nu} + T_m^{\mu\nu} \end{aligned} \tag{3.38}$$

in the condition that the respective fields involved satisfy their *equations of motion* and that T satisfies the conservation law:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{3.39}$$

The relation (92) is obtained from the *required general coordinate invariance of the fields action* S_K defined by

$$S_K[g, \varphi] = \int_K d^n x \sqrt{|\det g|} L(g, R, \varphi, \nabla\varphi) \tag{3.40}$$

where K is a compact subset of M.

This means that T is the rank 2 tensor field on space-time M depending on the fields of the theory which satisfies:

$$\delta_g \int_K d^n x \sqrt{|\det g|} L(g, R, \varphi, \nabla\varphi) = -\frac{1}{2}\int_K d^n x \sqrt{|\det g|} T^{\mu\nu} \delta g_{\mu\nu} \tag{3.41}$$

for every compact subset K of M and every variation $\delta g_{\mu\nu}$ of the metric tensor with support contained in K, and where the integrand on the $r.h.s$ of equation (94) is understood to contain no derivatives of $\delta g_{\mu\nu}$.

If we use the fact that, for any given variation $\delta g_{\mu\nu}$ of the metric tensor, the induced variations of the inverse metric tensor, the metric determinant, the Christoffel symbols and the curvature are given by:

$$\begin{aligned} \delta g^{\mu\nu} &= -g^{\mu\alpha} g^{\nu\lambda} \delta g_{\alpha\lambda} \\ \delta\sqrt{|\det g|} &= \frac{1}{2}\sqrt{|\det g|} g^{\mu\nu} \delta g_{\mu\nu} \\ \delta\Gamma^\alpha_{\mu\lambda} &= \frac{1}{2} g^{\alpha\nu}\left(\nabla_\mu \delta g_{\lambda\nu} + \nabla_\lambda \delta g_{\mu\nu} - \nabla_\nu \delta g_{\mu\lambda}\right) \\ \delta R^\alpha_{\lambda\mu\nu} &= \nabla_\mu \delta\Gamma^\alpha_{\lambda\nu} - \nabla_\nu \delta\Gamma^\alpha_{\lambda\mu}, \end{aligned} \tag{3.42}$$

we obtain, after discarding all surface terms coming from the partial integrations, because they vanish due to our support assumption, the explicit formula:

$$T^{\mu\nu} = -2\frac{\partial L}{\partial g_{\mu\nu}} - g^{\mu\nu}L + \nabla_\lambda \left(g^{\alpha\mu}\frac{\partial L}{\partial \Gamma^\alpha_{\lambda\nu}} + g^{\alpha\nu}\frac{\partial L}{\partial \Gamma^\alpha_{\lambda\mu}} - g^{\alpha\lambda}\frac{\partial L}{\partial \Gamma^\alpha_{\mu\nu}} \right)$$
$$+\nabla_\rho\nabla_\sigma \left(g^{\alpha\mu}\frac{\partial L}{\partial R^\alpha_{\rho\nu\sigma}} + g^{\alpha\mu}\frac{\partial L}{\partial R^\alpha_{\nu\rho\sigma}} + g^{\alpha\nu}\frac{\partial L}{\partial R^\alpha_{\rho\mu\sigma}} + g^{\alpha\nu}\frac{\partial L}{\partial R^\alpha_{\mu\rho\sigma}} - g^{\alpha\rho}\frac{\partial L}{\partial R^\alpha_{\mu\nu\sigma}} \right.\ .$$
$$\left. -g^{\alpha\rho}\frac{\partial L}{\partial R^\alpha_{\nu\mu\sigma}} \right) \tag{3.43}$$

One must notice that the simplest class of field Lagrangian is of course formed by those that depend only on the metric tensor g itself but not on its derivatives. In this case, the variational derivative in equation (96) reduces to an ordinary partial derivative, plus contribution from the metric determinant:

$$T^{\mu\nu} = -2\frac{\partial L}{\partial g_{\mu\nu}} - g^{\mu\nu}L. \tag{3.44}$$

One must also notice that, in the classical theory, we have

$$T^{\mu\nu} = -2\frac{\delta L_m}{\delta g_{\mu\nu}} \tag{3.45}$$

instead of (91) .

3.3. Perfect-fluid scalar- tensor theory field equations in a total Lagrangian formalism

3.3.1 General form of field equations

The action of a generalized perfect-fluid scalar-tensor theory can be written in the form:

$$S = \int \left[G^{-1}R - \frac{1}{2}\frac{2\omega+3}{\phi^2}\phi_{,\mu}\phi^{,\mu} - U + 16\pi L_m \right] \sqrt{|\det g|} \tag{3.46}$$

where $G^{-1}R - \frac{1}{2}\frac{2\omega+3}{\phi^2}\phi_{,\mu}\phi^{,\mu} - U + 16\pi L_m = L\left(g, R, \varphi, \nabla\varphi\right)$ and L_m represents this time a Lagrangian for a perfect fluid whose state equation is:

$$p = (\delta - 1)\,\rho,$$

in which p and ρ are respectively the pressure and the fluid density. G, ω and U are three functions of the scalar field ϕ whose signification is given below:

a) G is the gravitation function. In the case where G is a constant, the ***scalar field is said to be minimally coupled***.

b) ω is the ***Bran's-Dike coupling function*** which represents the coupling of the scalar field with the metric. When ω is a constant, we obtain the ***Jordan-Bran-Dike theory***.

c) U is the potential and represents the self coupling of the scalar field. If $U \neq 0$, the ***scalar field is said to be massive.***

The action (99) is not the most general at all, but represents most of the scalar-tensor theories in the literature. In this way:

– The General Theory of Relativity with a cosmological constant and a perfect fluid, often considered as the cosmological model likely to describe our actual Universe, is such that G represents the gravitational constant, ω does not figure and $U = 2\wedge$, $\wedge$ being the cosmological constant.

– The Bran s-Dike theory is recovered for $G = \phi^{-1}$ and ω constant. This theory has initially been imagined to obtain a relativistic gravitation theory compatible with the ideas of Mach and such that the gravitational function varies as the inverse of the scalar field.

– The low energy string theory without its antisymmetric tensor is defined by $G = e^{\phi}$ and $3 + 2\omega = \phi e^{-\phi}$.

Now varying the action (99) with respect to the metric functions, using (91) and (96) , one obtains the field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = G\left[\frac{1}{2}\frac{2\omega+3}{\phi^2}\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\frac{2\omega+3}{2\phi^2}\phi_{,\lambda}\phi^{,\lambda}g_{\mu\nu} + \left(G^{-1}\right)_{,\mu;\nu} - g_{\mu\nu}\Box\left(G^{-1}\right) - \frac{1}{2}Ug_{\mu\nu} + \frac{8\pi}{c^4}T_{\mu\nu}\right]. \tag{3.47}$$

The scalar curvature is equal to:

$$R = G\left[\frac{1}{2}\frac{2\omega+3}{\phi^2}\phi_{,\lambda}\phi^{,\lambda} + 3\Box\left(G^{-1}\right) + 2U - \frac{8\pi}{c^4}T\right]. \tag{3.48}$$

It is fundamental to note that in equations (100) :

$$T^{\mu\nu} = -2\frac{\delta L_m}{\delta g_{\mu\nu}}$$

where L_m is the Lagrangian for a perfect fluid.

One deduces using (101) an alternative form of field equations (100) :

$$R_{\mu\nu} = G\left[\frac{1}{2}\frac{2\omega+3}{\phi^2}\phi_{,\mu}\phi_{,\nu} + \left(G^{-1}\right)_{,\mu;\nu} + \frac{1}{2}g_{\mu\nu}\Box\left(G^{-1}\right) + \frac{1}{2}Ug_{\mu\nu} + \frac{8\pi}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)\right]. \tag{3.49}$$

Varying the action with respect to the scalar field, one easily obtains the well known *Klein-Gordon equation*:

$$GG_{\phi}^{-1}\left(\frac{1}{2}\frac{2\omega+3}{\phi^{2}}\phi_{,\lambda}\phi^{,\lambda}+3\Box\left(G^{-1}\right)+2U-\frac{8\pi}{c^{4}}T\right)+\left(-\frac{\omega_{\phi}}{\phi^{2}}-\frac{3+2\omega}{\phi^{2}}\right)\phi_{,\lambda}\phi^{,\lambda}-U_{\phi}+\frac{2\omega+3}{\phi^{2}}\Box\phi=0. \tag{3.50}$$

3.3.2 Field equations for *class A* Bianchi models

The various class A Bianchi model metrics can be written in the general form

$$ds^{2}=e^{2\alpha+2\beta+2\gamma}d\tau^{2}+e^{2\alpha}\left(\omega^{1}\right)^{2}+e^{2\beta}\left(\omega^{2}\right)^{2}+e^{2\gamma}\left(\omega^{3}\right)^{2} \tag{3.51}$$

where $dt=e^{\alpha+\beta+\gamma}d\tau$.

Introducing (104) in (102) , it comes

$$\begin{aligned}
&\alpha''G^{-1}+\alpha'\left(G^{-1}\right)'+\left(1/2G^{-1}\right)''-e^{6\Omega}\left[1/2U+4\pi\left(2-\delta\right)\rho_{0}V^{-3\delta}\right]=G^{-1}C_{L_{ag_1}}\\
&\beta''G^{-1}+\beta'\left(G^{-1}\right)'+\left(1/2G^{-1}\right)''-e^{6\Omega}\left[1/2U+4\pi\left(2-\delta\right)\rho_{0}V^{-3\delta}\right]=G^{-1}C_{L_{ag_2}}\\
&\gamma''G^{-1}+\gamma'\left(G^{-1}\right)'+\left(1/2G^{-1}\right)''-e^{6\Omega}\left[1/2U+4\pi\left(2-\delta\right)\rho_{0}V^{-3\delta}\right]=G^{-1}C_{L_{ag_3}}\\
&\alpha'\beta'+\alpha'\gamma'+\beta'\gamma'+3\left(\alpha'+\beta'+\gamma'\right)G\left(G^{-1}\right)'-1/2GV^{2}U\\
&\quad-\frac{1}{4}G\left(3+2\omega\right)\frac{\phi'^{2}}{\phi^{2}}-8\pi\rho_{0}GV^{2-\delta}=C_{L_{ag_4}},
\end{aligned} \tag{3.52}$$

where $\Omega=\alpha+\beta+\gamma$ stands for the isotropic part of the metric, $V=e^{3\Omega}$ represents the 3− volume of the Universe and the fluid density has been computed using the conservation equation (92) and writes

$$\rho=V^{-3\delta}. \tag{3.53}$$

Notice that following Misner in [20] , one can set:

$$\begin{aligned}
\alpha&=\Omega+\beta_{+}\\
\beta&=\Omega+\beta_{-}\\
\gamma&=\Omega-\beta_{+}-\beta_{-},
\end{aligned}$$

where $\beta_{\pm}$describe the anisotropic part of the metric..

Also notice that the $C_{L_{ag_i}}$ represent the curvature potentials of various *class A* Bianchi models and are given in the table below:

$$
\begin{array}{ll}
I & C_{L_{ag_i}} = 0 \\
II & -C_{L_{ag_1}} = C_{L_{ag_2}} = C_{L_{ag_3}} = 2C_{L_{ag_4}} = \frac{1}{2}e^{4\alpha} \\
VI_0,\ VII_0 & -C_{L_{ag_1}} = C_{L_{ag_2}} = \frac{1}{2}\left(e^{4\alpha} - e^{4\beta}\right),\ C_{L_{ag_3}} = 2C_{L_{ag_4}} = \frac{1}{2}\left(e^{2\alpha} \pm e^{2\beta}\right)^2 \\
VIII,\ IX & C_{L_{ag_1}=\frac{1}{2}\left[\left(e^{2\beta}\pm e^{2\gamma}\right)^2-e^{4\alpha}\right]} \\
& C_{L_{ag_2}=\frac{1}{2}\left[\left(e^{2\alpha}\pm e^{2\gamma}\right)^2-e^{4\beta}\right]},\ C_{L_{ag_3}=\frac{1}{2}\left[\left(e^{2\alpha}-e^{2\beta}\right)^2-e^{4\gamma}\right]}, \\
& C_{L_{ag_4}=\frac{1}{4}\left[e^{4\alpha}+e^{4\beta}+e^{4\gamma}-2\left(e^{2(\alpha+\beta)}\pm e^{2(\alpha+\gamma)}\pm e^{2(\beta+\gamma)}\right)\right]}.
\end{array}
\tag{3.54}
$$

3.3.3 Application: exact solutions in the empty Bianchi type I model for *Hyper-extended scalar-tensor theory(HST)*

The line element in a Bianchi type I model is given following (53) and (76) by:

$$ds^2 = -dt^2 + e^{2\alpha}\left(dx^1\right)^2 + e^{2\beta}\left(dx^2\right)^2 + e^{2\gamma}\left(dx^3\right)^2 \tag{3.55}$$

where t is the proper time and e^{α}, e^{β}, e^{γ} are the metric functions depending on t. The Lagrangian of the HST is written

$$L = G(\phi)^{-1} R - \frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi^{,\mu} \tag{3.56}$$

where G and ω are depending on the scalar field, specifying then the theory.

Setting

$$dt = e^{\alpha+\beta+\gamma}d\tau$$

in (108) , and varying the action with respect to the space-time metric and scalar field we obtain the field equations and the Klein-Gordon equation that can be written following (105) , (106) :

$$\alpha'' + \alpha' G\left(G^{-1}\right)' + \frac{1}{2}G\left(G^{-1}\right)'' = 0 \tag{3.57}$$

$$\beta'' + \beta' G\left(G^{-1}\right)' + \frac{1}{2}G\left(G^{-1}\right)'' = 0 \tag{3.58}$$

$$\gamma'' + \gamma' G\left(G^{-1}\right)' + \frac{1}{2}G\left(G^{-1}\right)'' = 0 \tag{3.59}$$

$$\alpha'\beta' + \alpha'\gamma' + \beta'\gamma' + G\left(G^{-1}\right)'\left(\alpha' + \beta' + \gamma'\right) - \omega\frac{G}{2}\frac{\phi'^2}{\phi^2} = 0 \tag{3.60}$$

$$\phi'^2\left[-\frac{\omega_\phi}{\phi} + \frac{\omega}{\phi^2} - G\left(G^{-1}\right)_\phi\frac{\omega_\phi}{\phi}\right] - 2\omega\frac{\phi''}{\phi} - 3G\left(G^{-1}\right)_\phi\left(G^{-1}\right)'' = 0 \tag{3.61}$$

a prime standing for a derivative with respect to τ. Since the functions α, β and γ play equivalent roles in the field equations, we will only consider the metric function e^{α}.

We define new variables a, b, c and f in order to transform the second order field equations into a first order system:

$$\begin{aligned} a &= \alpha' G^{-1} \\ b &= \beta' G^{-1} \\ c &= \gamma' G^{-1} \\ f &= \frac{1}{2}\left(G^{-1}\right)'. \end{aligned} \tag{3.62}$$

Then, after integration we obtain the following relations linking the spatial components of the field equations

$$\begin{aligned} a + f &= a_0 \\ b + f &= b_0 \\ c + f &= c_0 \end{aligned} \tag{3.63}$$

where a_0, b_0 and c_0 are integration constants.

Integrating the Klein-Gordon equation, one gets:

$$\frac{3}{4}\left(G^{-1}\right)'^{2} + \frac{1}{2}G^{-1}\omega\phi^{-1}\phi'^{2} = -\Pi \tag{3.64}$$

Π being an integration constant. The relation (117) yields

$$\left[\frac{3}{4}\left(G^{-1}\right)^{2}_{\phi} + \frac{G^{-1}\omega}{2\phi}\right]\phi'^{2} = -\Pi. \tag{3.65}$$

The following relation between the integration constants is deduced from the constraint equation (113) :

$$a_0 b_0 + a_0 c_0 + b_0 c_0 = -\Pi. \tag{3.66}$$

Since the quantity between the square brackets in the left hand-side of (118) is proportional and has the same sign as the energy density of the scalar field in the *Einstein frame*, we impose for physical reasons that the energy density is positive:

$$\frac{3}{4}\left(G^{-1}\right)^{2}_{\phi} + \frac{G^{-1}\omega}{2\phi} > 0. \tag{3.67}$$

If we choose $G^{-1} = \phi$, we recover the usual relation for a positive energy density for generalized scalar-tensor theory(GST)

$$3 + 2\omega > 0. \tag{3.68}$$

The sign of ϕ' is constant and depends on the sign of the square root of the energy density: if we take it positive or negative , the scalar field will increase or decrease , hence the scalar field being a monotone function of time, it will be considered as a time variable.

Using the first relations in (115) and (116) , we deduce the exact solution for $\alpha(\tau)$:

$$\alpha - \alpha_0 = \int \frac{a_0}{G^{-1}} d\tau - \frac{1}{2} \ln \left(G^{-1}\right) \tag{3.69}$$

α_0 being an integration constant. If we write

$$d\tau = \phi'^{-1} d\phi$$

and express ϕ' using (118) , we obtain $\alpha(\phi)$:

$$\alpha - \alpha_0 = \int \frac{a_0}{G^{-1}} \sqrt{-\frac{1}{\Pi}\left[\frac{3}{4}\left(G^{-1}\right)_\phi^2 + \frac{G^{-1}\omega}{2\phi}\right]} d\phi - \frac{1}{2} \ln \left(G^{-1}\right) \tag{3.70}$$

and analogous relations for β and γ with couples of constants (β_0, b_0) and (γ_0, c_0) instead of (α_0, a_0) .

Notice that there are two interesting asymptotic values for the couple (a, f) . The first one is $(a, f) \to (0, a_0)$. It means that $G^{-1} \to 2(a_0\tau + a_1)$. Then we deduce from (122) that the metric function tends toward a constant. The point $(0, a_0)$ consequently, stands for the static solution for e^α. The second one is $(a, f) \to (a_0, 0)$. Then G^{-1} tends toward a constant and we get from (122) $\alpha \to \alpha_1\tau + \alpha_2$, α_1, α_2 constants. The functions β and γ will behave in the same way in respectively $(b_0, 0)$ and $(c_0, 0)$. In the t time, this solution for the metric functions corresponds to power laws of t.

4. Conclusion

In this work, we have studied the homogeneous but anisotropic cosmological Bianchi models in perfect-fluid scalar-tensor theories. The introduction was devoted to a historical and physical justification of the scalar-tensor theories and anisotropic cosmological models. The second section details the mathematical tools, that are necessary to better understand the anisotropic cosmologies classification. In the third section we have given in detail all the steps leading to the calculation of the Riemannian curvature, the metric tensor, the form of the field equations in a proposed reformulated and classical Lagrangian formalism and we have investigated exact solutions for a given particular Lagrangian in the empty Bianchi type I model. The present studies are important in the sense that they can allow us to determine the properties of scalar-tensor theories so that these anisotropic homogeneous models asymptotically behave as our current Universe or bring some responses to some of its problems namely the cosmological constant and coincidence problems. The study can also bring a light to the process of *isotropization* of anisotropic models which can explains the primordial state of our current Universe.

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