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Connections in affine frame bundles[1]

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Abstract

The (affine) development of a smooth curve in a smooth manifold M with respect to an arbitrarily given affine connection in the bundle $A(M)$ of affine frames over M is well known (cf. S.Kobayashi and K.Nomizu, Foundations of Differential Geometry, Vol.1). In this paper, we get the generalized affine development of a smooth curve in M with respect to an arbitrarily given generalized affine connection in $A(M)$, and then investigate relationships among the covariant derivatives with respect to an arbitrarily given linear connection ω in the bundle $L(M)$ ($\subset A(M)$) of linear frames, the affine connection and a generalized affine connection in $A(M)$ which are related to the linear connection ω.

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0. Introduction

Let $L(M)$ and $A(M)$ be the linear frame and the affine frame bundles over an n-dimensional C^∞ manifold M respectively. Let $\tilde{\gamma} : L(M) \hookrightarrow A(M)$ be the (principal fiber bundle) homomorphism of $L(M)$ into $A(M)$ with the group homomorphism $\gamma : GL(n; R) \hookrightarrow A(n; R)$, ω (resp. φ) an arbitrarily given linear connection (resp. an arbitrarily given tensorial 1-form of type $(GL(n; R), R^n)$ ([2, p. 75])) which is defined on $L(M)$, and θ the canonical 1-form on $L(M)$. Let $\tilde{\omega}$ (resp. $\bar{\omega}$) be the affine (resp. the generalized affine) connection such that $\tilde{\gamma}^\star\tilde{\omega} =: \omega + \theta$ (resp. $\tilde{\gamma}^\star\bar{\omega} =: \omega + \varphi$) on $L(M)$.

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Let $\tau = x_t (0 \leq t \leq 1)$ be a C^∞ curve in M, and $\tilde{\tau}_0^t$ (resp. $\bar{\tau}_0^t$) the affine (resp. the generalized affine) parallel displacement of the affine tangent space $A_{x_t}(M)$ into $A_{x_0}(M)$ with respect to $\tilde{\omega}$ (resp. $\bar{\omega}$) in $A(M)$. Then, the affine development $\tilde{C}_t = \tilde{\tau}_0^t(x_t)(0 \leq t \leq 1)$ of the curve $\tau = x_t (0 \leq t \leq 1)$ in M into $A_{x_0}(M)$ is well known ([2, Proposition 4.1, p. 131]).

First of all in this paper, we get the generalized affine development $\bar{C}_t = \bar{\tau}_0^t(x_t)$ of the curve $\tau = x_t$ $(0 \leq t \leq 1)$ in M into $A_{x_0}(M)$ as follows.

Theorem 2.3. Let $\bar{\omega}$ be an arbitrarily given generalized affine connection in $A(M)$, and let $\tau = x_t,\ 0 \leq t \leq 1$, be a smooth curve in M. Let $\bar{\tau}_0^t$ be the parallel displacement of $A_{x_t}(M)$ into $A_{x_0}(M)$ along τ with respect to the generalized affine connection $(form)$ $\bar{\omega}$. Then the generalized affine development $\bar{C}_t = \bar{\tau}_0^t(x_t)$ $(0 \leq t \leq 1)$ of the curve $\tau = x_t$ $(0 \leq t \leq 1)$ in M into $A_{x_0}(M)$ is given as follows:

$$\bar{C}_t = \bar{\tau}_0^t(x_t) = \bar{\tau}_0^t(\dot{x}_t) - \tau_0^t(\dot{x}_t)\ (0 \leq t \leq 1),$$

where $\dot{x}_t := dx_t/dt$ and τ_0^t is the linear parallel displacement along τ from x_t to x_0 with respect to the linear connection ω in $L(M)$ which is corresponding to $\bar{\omega}$ $(\tilde{\gamma}^\star(\bar{\omega}) = \omega + \varphi\ on\ L(M))$ in $A(M)$.

Let Y be a smooth cross section of M into the tangent bundle TM (or the affine tangent bundle $A(M) \times_{A(n;R)} A^n$) over M. Let $\nabla_{\dot{x}_t} Y,\ \tilde{\nabla}_{\dot{x}_t} Y$ and $\bar{\nabla}_{\dot{x}_t} Y$ be the covariant derivatives of Y in the direction of the curve $\tau = x_t (0 \leq t \leq 1)$ with respect to $\omega,\ \tilde{\omega}$ and $\bar{\omega}$ respectively. Then we obtain the following results.

Theorem 2.2. Let ω be an arbitrarily given linear connection in $L(M)$ and θ the canonical 1-form on $L(M)$. Let $\tilde{\omega}$ be the affine connection in $A(M)$ such that $\tilde{\gamma}^\star(\tilde{\omega}) = \omega + \theta$ on $L(M)$, and $\tau = x_t$ $(0 \leq t \leq 1)$ a C^∞ curve in M. Let Y be a cross section of M into TM (or $A(M) \times_{A(n;R)} A^n$). Let $\nabla_{\dot{x}_t}$ $(resp.\ \tilde{\nabla}_{\dot{x}_t})$ be the covariant differentiation along τ with respect to ω $(resp.\ \tilde{\omega})$. Then

$$(\tilde{\nabla}_{\dot{x}_t} Y_t)_{t=0} = (\nabla_{\dot{x}_t} Y_t + \frac{d\tilde{C}_t}{dt})_{t=0},$$

where $\tilde{C}_t$ is the affine development of $\tau = x_t$ $(0 \leq t \leq 1)$ into $A_{x_0}(M)$.

Theorem 2.6. Let ω be an arbitrarily given linear connection in $L(M)$ and φ an arbitrarily given tensorial 1-form on $L(M)$ of type $(GL(n;\ R),\ R^n)$. Let $\bar{\omega}$ be the generalized affine connection in $A(M)$ such that $\tilde{\gamma}^\star(\bar{\omega}) = \omega + \varphi$ on $L(M)$, and $\tau = x_t$ $(0 \leq t \leq 1)$ a C^∞ curve in M. Let Y be a cross section of M into TM (or $A(M) \times_{A(n;R)} A^n$). Let $\nabla_{\dot{x}_t}$ $(resp.\ \bar{\nabla}_{\dot{x}_t})$ be the covariant differentiation along τ with respect to ω $(resp.\ \bar{\omega})$. Then

$$(\bar{\nabla}_{\dot{x}_t} Y_t)_{t=0} = (\nabla_{\dot{x}_t} Y_t + \frac{d\bar{C}_t}{dt})_{t=0},$$

where $\bar{C}_t$ is the generalized affine development of $\tau = x_t$ $(0 \leq t \leq 1)$ into $A_{x_0}(M)$.

1. Preliminaries

In general, when we regard R^n as an affine space, we denote it by A^n. The group $A(n; R)(= GL(n; R) \times R^n)$ of all affine transformations of A^n is represented by the group of all matrices of the form

$$\tilde{a} = \begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix}, \tag{1.1}$$

where $a = (a^i_j)_{i,j} \in GL(n; R)$ and $\xi = (\xi^i),\ \xi \in R^n$, is a column vector. The element $\widetilde{a}$ in (1.1) maps a point η of A^n into $a\eta + \xi$. We have the following exact sequence (cf. [2, p.125]):

$$0 \hookrightarrow R^n \overset{\alpha}{\hookrightarrow} A(n; R) \overset{\beta}{\longrightarrow} GL(n; R) \longrightarrow 1. \tag{1.2}$$

The tangent space $T_x(M)$ of an n-dimensional smooth manifold M at x $(\in M)$, regarded as an affine space, is denoted by $A_x(M)$ and is called the *affine tangent space*. An *affine frame* of the manifold M at x $(\in M)$ consists of a point $p \in A_x(M)$ and a linear frame $(X_1, \dots, X_n)$ at x; it is denoted by $\tilde{u} := (X_1, ..., X_n; p)$. We denote by $A(M)$ the set of all affine frames of M and define the projection $\tilde{\pi} : A(M) \to M$ by setting $\tilde{\pi}(\tilde{u}) = x$ for every affine frame $\tilde{u}$ at x. Then, $A(M)(M, A(n; R), \tilde{\pi})$ is a principal fiber bundle over M with group $A(n; R)$. We call $A(M)(M, A(n; R), \tilde{\pi})$ the *bundle of affine frames* over M (cf. [1, 2]).

Let $L(M)$ be the bundle of linear frames over M. Corresponding to the natural group homomorphisms $\beta : A(n; R) \to GL(n; R)$ and $\gamma : GL(n; R) \hookrightarrow A(n; R)$, we have principal fiber bundle homomorphisms $\tilde{\beta} : A(M) \to L(M)$ and $\tilde{\gamma} : L(M) \hookrightarrow A(M)$. Namely, $\tilde{\beta} : A(M) \to L(M)$ maps $(X_1, \dots, X_n; p)$ into $(X_1, \dots, X_n)$, and $\tilde{\gamma} : L(M) \hookrightarrow A(M)$ maps $(X_1, \dots, X_n)$ into $(X_1, \dots, X_n; 0_x)$, where $0_x \in A_x(M)$ is the point corresponding to the origin of $T_x(M)$. In particular, $L(M)$ can be considered as a subbundle of $A(M)$.

A *generalized affine connection* of M is a connection in the principal fiber bundle $A(M)$ of affine frames over M. We denote by R^n the Lie algebra of the vector group R^n. Corresponding to the exact sequence (1.2) of groups, we have the following exact sequence of the Lie algebras (cf. [2, p.127]):

$$0 \hookrightarrow R^n \hookrightarrow \mathfrak{a}(n; R) \longrightarrow \mathfrak{gl}(n; R) \longrightarrow 0. \tag{1.3}$$

Therefore,

$$\mathfrak{a}(n; R) = \mathfrak{gl}(n; R) + R^n \quad \text{(semidirect sum) (cf. [2, p.127]).} \tag{1.4}$$

Let $\bar{\omega}$ be the connection form of a generalized affine connection of M. Then $\tilde{\gamma}^\star\bar{\omega}$ is an $\mathfrak{a}(n; R)$-valued 1-form on $L(M)$, where $\tilde{\gamma}^\star\bar{\omega}$ is the pull back of $\bar{\omega}$ by $\tilde{\gamma}$. Let

$$\tilde{\gamma}^\star\bar{\omega} = \omega + \varphi, \text{ (cf. [2, Proposition 3.1, p.127]),} \tag{1.5}$$

be the decomposition corresponding to $\mathfrak{a}(n;\, R) = \mathfrak{gl}(n;\, R) + R^n$, so that ω is a $\mathfrak{gl}(n;\, R)$-valued 1-form on $L(M)$ and φ is an R^n-valued 1-form on $L(M)$. Here φ is a tensorial 1-form on $L(M)$ of type $(GL(n;\, R),\, R^n)$ ([2, §5 of Chapter II]), and hence corresponds to a tensor field of type (1,1) on M.

A generalized affine connection (form) $\bar{\omega}$ is called an *affine connection* (*form*) if, in (1.5), the R^n-valued 1-form φ is the canonical 1-form θ on $L(M)$, i.e.,

$$\theta(X) = u^{-1}(\pi_\star(X)) \text{ for } X \in T_u(P), \quad (u \in L(M), P = (L(M)). \tag{1.6}$$

From now on, we denote by $\tilde{\omega}$ and $\bar{\omega}$ affine connections (forms) and generalized affine connections (forms) in the principal fiber bundle $A(M)$ of all affine frames over M respectively.

For later use, we introduce the following lemmas.

Lemma 1.1 ([2, Proposition 3.1, p.127]). Let $\bar{\omega}$ be a generalized affine connection $(form)$ on $A(M)$ and let

$$\tilde{\gamma}^\star \bar{\omega} = \omega + \varphi,$$

where ω is $\mathfrak{gl}(n;\, R)$-valued and φ is R^n-valued. Then

(1) The correspondence between the set of all generalized affine connection forms on $A(M)$ and the set of all pairs consisting of a connection form on $L(M)$ and a tensorial 1-form on $L(M)$ of type $(GL(n;\, R),\, R^n)$ given by $\bar{\omega} \to (\omega, \varphi)$ is 1 : 1.

(2) The homomorphism $\tilde{\beta} : A(M) \to L(M)$ maps horizontal subspaces in $A(M)$ into horizontal subspaces in $L(M)$.

The following lemma is an immediate consequence of Lemma 1.1.

Lemma 1.2 ([2, Theorem 3.3, p.129]). The principal fiber bundle homomorphism $\tilde{\beta}$: $A(M) \to L(M)$ maps every affine connection on $A(M)$ into a linear connection on $L(M)$. Moreover, the map, which is defined by (1) of $Lemma 1.1$, between the set of all affine connections in $A(M)$ and the set of all linear connections in $L(M)$ is a one-to-one correspondence.

2. Connections in the bundle of affine frames

2.1. The bundle of affine frames over a C^∞ manifold

Let M be an n-dimensional C^∞ manifold. For each $x \in M$, an *affine frame* at x consists of a point $p \in A_x(M)$ and a linear frame $u = (X_1, \ldots, X_n)$ at x; it will be denoted by $(u;\, p) = (X_1, \ldots, X_n;\, p)$. We denote by $A(M)$ the set of all affine frames of M. An affine frame $\tilde{u} = (u;\, p) \in A_x(M)$ is considered as a map of A^n onto the affine tangent space $A_x(M)$;

$$\tilde{u} = (u;\, p) : A^n \ni \eta \longmapsto u(\eta) + p \in A_x(M).$$

We define an action of $A(n; R)$ on $A(M)$ by

$$\tilde{u}\tilde{a} := \tilde{u} \circ \tilde{a} \quad (\tilde{u} \in A(M), \tilde{a} \in A(n; R)),$$

where $\tilde{u} \circ \tilde{a}$ is the composite of the affine transformations $\tilde{a} : A^n \to A^n$ and $\tilde{u} : A^n \to A_x(M)$. Hence, we get

$$\begin{aligned} \tilde{u}\tilde{a}(= (u; p)(a; \xi)) = (ua; u\xi + p) \in A_x(M) \subset A(M), \\ (u \in L(M),\ p \in A_x(M),\ a \in GL(n; R),\ \xi \in R^n). \end{aligned} \tag{2.1}$$

We define the projection $\tilde{\pi} : A(M) \to M$ by setting $\tilde{\pi}(\tilde{u}) = x$ for every affine frame $\tilde{u}$ at x. Let $(U; x^1, \ldots, x^n)$ and $(V; y^1, \ldots, y^n)$, $U \cap V \neq \emptyset$, be local coordinate neighborhoods of M. Since, for $\tilde{u} = (u; p) \in \tilde{\pi}^{-1}(U \cap V)$,

$$\begin{aligned} \tilde{u} = (u; p) &= (\partial/\partial x^1 \cdots \partial/\partial x^n)[a(u)\ \xi(p)] \\ &= (\partial/\partial y^1 \cdots \partial/\partial y^n)[b(u)\ \eta(p)], \end{aligned} \tag{2.2}$$

$(a(u), b(u) \in GL(n; R);\ \xi(p), \eta(p) \in R^n)$, we get the transition function ϕ_{UV} on $U \cap V$ which is defined by

$$\phi_{UV} = [(\partial x^i/\partial y^j)]_{ij} \equiv \begin{pmatrix} (\partial x^i/\partial y^j)_{ij} & 0 \\ 0 & 1 \end{pmatrix} \in A(n; R). \tag{2.3}$$

Then, $A(M)(M, A(n; R), \tilde{\pi})$ becomes a principal fiber bundle over M with group $A(n; R)$. Such a principal fiber bundle $A(M)$ is called the *bundle of affine frames* over M.

Here, we define a fiber bundle associated with a principal fiber bundle as follows ([1, 2, 3]). Let $P(M, G, \pi)$ be a principal fiber bundle and F a manifold on which G acts on the left. On the product manifold $P \times F$, we let G act on the right as follows: an element $a \in G$ maps $(u, \xi) \in P \times F$ into $(ua, a^{-1}\xi) \in P \times F$. The quotient space of $P \times F$ by this group action is denoted by $E = P \times_G F$. We call E or more precisely $E(M, F, G, P)$ the *fiber bundle* over the base manifold M, with fiber F and structure group G, which is associated with the principal fiber bundle $P(M, G, \pi)$.

Now, we define a map ν between the affine tangent bundle and the tangent (vector) bundle over the base manifold M by

$$\nu : (A(M) \times_{A(n;R)} A^n) \ni [\tilde{u}, \eta] \longmapsto u\eta + p \in (L(M) \times_{GL(n;R)} R^n) = TM, \tag{2.4}$$

$(\tilde{u} = (u; p) \in A(M),\ \eta \in A^n)$. This map ν is well defined. In fact, if

$$[\tilde{u}, \eta] = [\tilde{v}, \zeta] \quad \text{for } \tilde{u} = (u; p) \text{ and } \tilde{v} = (v; q)\ (\tilde{\pi}(\tilde{u}) = \tilde{\pi}(\tilde{v})),$$

then there exists $\tilde{a} = (a; \xi)$ $(\tilde{a}^{-1} = (a^{-1}; -a^{-1}\xi)) \in A(n; R)$ such that

$$[\tilde{u}, \eta]\, \tilde{a} = [\tilde{v}, \zeta]. \tag{2.5}$$

From (2.1) and (2.5), we have

$$\tilde{u}\tilde{a} = (ua; u\xi + p) = (v; q) = \tilde{v}, \quad \tilde{a}^{-1}\eta = a^{-1}\eta - a^{-1}\xi = \zeta. \tag{2.6}$$

We get from (2.4) and (2.6)

$$\begin{aligned} &\nu([\tilde{u}, \eta]) = u\eta + p, \\ &\nu([\tilde{v}, \zeta]) = \nu([(ua; u\xi + p); a^{-1}(\eta - \xi)]) = u\eta + p. \end{aligned} \tag{2.7}$$

So, the map ν is well defined.

The map ν is bijective. In fact, ν is evidently surjective. In order to show the fact that ν is injective, we assume that

$$\nu([\tilde{u}; \eta]) = \nu([\tilde{v}; \zeta]) \quad (\tilde{u} = (u; p), \tilde{v} = (v; q) \in A(M);\ \eta, \zeta \in A^n). \tag{2.8}$$

Then, since $\tilde{\pi}(\tilde{u}) = \tilde{\pi}(\tilde{v})$, there uniquely exists $a \in GL(n; R)$ such that $v = ua$. From this fact, (2.4) and (2.8), we get

$$ua = v, \quad u\eta + p = v\zeta + q = ua\zeta + q. \tag{2.9}$$

From (2.9), we have

$$\zeta = a^{-1}\eta + a^{-1}u^{-1}(p - q). \tag{2.10}$$

Putting $\tilde{a} = (a; u^{-1}(q - p))$ $(\tilde{a}^{-1} = (a^{-1}; -a^{-1}u^{-1}(q - p))) \in A(n; R)$, then we have from (2.1), (2.9) and (2.10),

$$(\tilde{u}; \eta)\tilde{a} = (\tilde{u}\tilde{a}; \tilde{a}^{-1}\eta) = ((ua; q); a^{-1}\eta + a^{-1}u^{-1}(p - q)) = (\tilde{v}; \zeta).$$

So ν is injective. Hence the map ν is bijective.

Eventually, the total space (i.e., the bundle space) of the affine tangent bundle over M is naturally homeomorphic with that of the tangent (vector) bundle over M; the distinction between the two is that the affine tangent bundle is associated with $A(M)$ whereas the tangent (vector) bundle is associated with $L(M)$. Thus,

$$\{Y \mid Y : M \longrightarrow TM = (L(M) \times_{GL(n;R)} R^n) \text{ is a } C^\infty \text{ cross section}\}$$

and

$$\{Y \mid Y : M \longrightarrow A(M) \times_{A(n;R)} A^n \text{ (the affine tangent bundle) is a } C^\infty \text{ cross section}\}$$

are naturally 1:1 correspondent.

2.2. Affine connections

Let $\tau = x_t,\ 0 \le t \le 1$, be a smooth curve in M. The *affine parallel displacement* along τ is an affine transformation of the affine tangent space $A_{x_0}(M)$ at x_0 onto the affine tangent space $A_{x_1}(M)$ at x_1 which is defined by the given affine connection in $A(M)$. Let $\tilde{\tau}_s^t$ be the affine parallel displacement along the curve τ from x_t to x_0. A cross section of M into the affine tangent bundle (associated with $A(M)$) is called a *point field*. Let p be a point field defined along τ so that p_{x_t} is an element of $A_{x_t}(M)$ for each t. Then $\tilde{\tau}_0^t(p_{x_t})$ describes a curve in $A_{x_0}(M)$. We identify the curve $\tau = x_t$ with the trivial point field along τ, that is, the point field corresponding to the zero vector field along τ. Then the *affine development* (cf. [1, 2, 4]) of the curve τ in M into the affine tangent space $A_{x_0}(M)$ is the curve $\tilde{\tau}_0^t(x_t)$ in $A_{x_0}(M)$, where $\tilde{\tau}_0^t$ is the affine parallel displacement $A_{x_t}(M) \to A_{x_0}(M)$ along τ (in the reversed direction) from x_t to x_0. The following lemma is well known.

Lemma 2.1 ([2, Proposition 4.1, p.131]). Given a curve $\tau = x_t,\ 0 \le t \le 1$, in M, set $Z_t := \tau_0^t(\dot{x}_t)$, where τ_0^t is the parallel displacement with respect to an arbitrarily given linear connection $(form)$ ω $(\tilde{\gamma}^\star\tilde{\omega} = \omega + \theta)$ along τ from x_t to x_0 and $\dot{x}_t = dx_t/dt$. Let $\tilde{C}_t,\ 0 \le t \le 1$, be the curve in $A_{x_0}(M)$ starting from the origin (that is $\tilde{C}_0 = x_0$) such that $d\tilde{C}_t/dt = Z_t$ for every t. Then $\tilde{C}_t$ is the affine development of τ into $A_{x_0}(M)$.

Proof. Let u_0 be any point in $L(M)$ such that $\pi(u_0) = x_0$, and u_t the horizontal lift of x_t in $L(M)$ with respect to the linear connection ω. Let $\tilde{u}_t$ be the horizontal lift of x_t in $A(M)$ with respect to the affine connection (form) $\tilde{\omega}$ such that $\tilde{u}_0 = u_0$. Since, by virtue of Lemma 1.1, the homomorphism $\tilde{\beta} : A(M) \to L(M) = A(M)/R^n$ maps $\tilde{u}_t$ into u_t, there is a curve $\tilde{a}_t$ in $R^n \subset A(n; R)$ such that $\tilde{u}_t = u_t\tilde{a}_t$ and $\tilde{a}_0$ is the identity. Then, since $\tilde{a}_t \in R^n \subset A(n; R)$, we can put

$$\tilde{a}_t = \begin{pmatrix} I_n & \tilde{\xi}(t) \\ 0 & 1 \end{pmatrix}, \quad \tilde{a}_t^{-1} = \begin{pmatrix} I_n & -\tilde{\xi}(t) \\ 0 & 1 \end{pmatrix} \tag{2.11}$$

for each t.

Here we shall find a necessary and sufficient condition for $\tilde{a}_t$ in order that $\tilde{u}_t$ be horizontal with respect to the affine connection (form) $\tilde{\omega}$. From Leibniz's formula, we get

$$\dot{\tilde{u}}_t = \dot{u}_t\tilde{a}_t + u_t\dot{\tilde{a}}_t. \tag{2.12}$$

We obtain by virtue of (2.12) and Lemma 1.1

$$\begin{aligned} \tilde{\omega}(\dot{\tilde{u}}_t) &= Ad(\tilde{a}_t^{-1})(\tilde{\omega}(\dot{u}_t)) + \tilde{a}_t^{-1}\dot{\tilde{a}}_t \\ &= Ad(\tilde{a}_t^{-1})(\omega(\dot{u}_t) + \theta(\dot{u}_t)) + \tilde{a}_t^{-1}\dot{\tilde{a}}_t \\ &= Ad(\tilde{a}_t^{-1})(\theta(\dot{u}_t)) + \tilde{a}_t^{-1}\dot{\tilde{a}}_t, \end{aligned} \tag{2.13}$$

since the curve u_t in $L(M)$ is a horizontal curve with respect to the linear connection (form) ω in $L(M)$ by the above assumption. Thus, by virtue of (2.11) and (2.13), we get the fact that $\tilde{u}_t$ is horizontal with respect to the affine connection (form) $\tilde{\omega}$ if and only if

$$\theta(\dot{u_t}) = -\dot{\tilde{a}}_t\ \tilde{a}_t^{-1} = \tilde{a}_t\ (d\tilde{a}_t{}^{-1}/dt) = d\tilde{a}_t{}^{-1}/dt. \tag{2.14}$$

Now, in order to obtain the affine development, we assume that the curve $\tilde{u}_t = u_t\tilde{a}_t$ in $A(M)$ is horizontal with respect to the affine connection $\tilde{\omega}$. Then from (2.11) and (2.14), we obtain

$$Z_t := \tau_0^t(\dot{x}_t) = (u_0 \circ u_t{}^{-1})(\dot{x}_t) = u_0(\theta(\dot{u}_t)) = -u_0(d\tilde{\xi}(t)/dt). \tag{2.15}$$

Since $\tilde{a}_t^{-1} \in A(n;\, R)$ and $u_t{}^{-1}(x_t) \in A^n$, we have from (2.11)

$$\begin{aligned}\tilde{C}_t &= \tilde{\tau}_0^t(x_t) = \tilde{u}_0(\tilde{u}_t^{-1}(x_t))\\ &= u_0(\tilde{a}_t^{-1}(u_t{}^{-1}(x_t))) = u_0(u_t{}^{-1}(x_t) - \tilde{\xi}(t)) = -u_0(\tilde{\xi}(t)).\end{aligned} \tag{2.16}$$

By the help of (2.15) and (2.16), we obtain $d\tilde{C}_t/dt = \tau_0^t(\dot{x}_t) = Z_t$. ■

We investigate relationships between covariant differentiations with respect to a linear connection ω and the affine connection $\tilde{\omega}$ $(\gamma^\star(\tilde{\omega}) = \omega + \theta)$ which is defined by ω.

Let ω be an arbitrarily given linear connection in the bundle $L(M)$ of linear frames over an n-dimensional C^∞ manifold M. Let $\tilde{\omega}$ be the affine connection in $A(M)$ such that

$$\tilde{\gamma}^\star(\tilde{\omega}) = \omega + \theta \ \text{ on } L(M). \tag{2.17}$$

Let Y be a C^∞ cross section of M into TM (or $A(M) \times_{A(n;R)} A^n$) and $\tau = x_t$ $(0 \le t \le 1)$ a (piecewise) C^∞ curve in M. The covariant derivative $(\nabla_{\dot{x}_t} Y_t)_{t=0}$, $(Y_t := Y_{x_t})$, of Y along τ with respect to the linear connection (form) ω is defined by

$$(\nabla_{\dot{x}_t} Y_t)_{t=0} = \lim_{t\to 0} \frac{\tau_0^t(Y_t) - Y_0}{t}, \tag{2.18}$$

where τ_0^t is the linear parallel displacement with respect to the linear connection (form) ω along the curve τ from x_t to x_0. Similarly the covariant derivative $(\tilde{\nabla}_{\dot{x}_t} Y_t)_{t=0}$ of Y along τ with respect to the affine connection (form) $\tilde{\omega}$ is defined by

$$(\tilde{\nabla}_{\dot{x}_t} Y_t)_{t=0} = \lim_{t\to 0} \frac{\tilde{\tau}_0^t(Y_t) - Y_0}{t}, \tag{2.19}$$

where $\tilde{\tau}_0^t$ is the affine parallel displacement with respect to the affine connection (form) $\tilde{\omega}$ along the curve τ from x_t to x_0. Then, using the notations as in the course of the proof of Lemma 2.1, we get from (2.11)

$$\tilde{\tau}_0^t(Y_t) = \tilde{u}_0(\tilde{u}_t^{-1}(Y_t) = \tilde{u}_0(\tilde{a}_t^{-1}u_t{}^{-1})(Y_t) = u_0(u_t{}^{-1}(Y_t) - \tilde{\xi}(t)), \tag{2.20}$$

since $\tilde{a}_t^{-1} \in A(n;\, R)$ and $u_t{}^{-1}(Y_t) \in A^n$. From (2.19) and (2.20), we obtain

$$(\tilde{\nabla}_{\dot{x}_t} Y_t)_{t=0} = \lim_{t\to 0} \frac{u_0 u_t{}^{-1}(Y_t) - Y_0 - u_0(\tilde{\xi}(t))}{t}. \tag{2.21}$$

Since $\tilde{a}_0 \in A(n;\, R)$ is the identity, from (2.11) we have $\tilde{\xi}(0) = 0(\in R^n)$. From this fact, (2.18) and (2.21), we get

$$(\tilde{\nabla}_{\dot{x}_t} Y_t)_{t=0} = \left(\nabla_{\dot{x}_t} Y_t - u_0 \left(\frac{d\tilde{\xi}(t)}{dt} \right) \right)_{t=0} . \tag{2.22}$$

By the help of (2.16) and (2.22), we obtain

$$(\tilde{\nabla}_{\dot{x}_t} Y_t)_{t=0} = \left(\nabla_{\dot{x}_t} Y_t + \frac{d\tilde{C}_t}{dt} \right)_{t=0} . \tag{2.23}$$

Therefore we get

Theorem 2.2. Let ω be an arbitrarily given linear connection in $L(M)$ and θ the canonical 1-form on $L(M)$. Let $\tilde{\omega}$ be the affine connection in $A(M)$ such that $\tilde{\gamma}^{\star}(\tilde{\omega}) = \omega + \theta$ on $L(M)$, and $\tau = x_t$ $(0 \leq t \leq 1)$ a C^{∞} curve in M. Let Y be a cross section of M into TM (or $A(M) \times_{A(n;R)} A^n$). Let $\nabla_{\dot{x}_t}$ $(resp.\ \tilde{\nabla}_{\dot{x}_t})$ be the covariant differentiation along τ with respect to ω $(resp.\ \tilde{\omega})$. Then

$$(\tilde{\nabla}_{\dot{x}_t} Y_t)_{t=0} = \left(\nabla_{\dot{x}_t} Y_t + \frac{d\tilde{C}_t}{dt} \right)_{t=0} ,$$

where $\tilde{C}_t$ is the affine development of $\tau = x_t$ $(0 \leq t \leq 1)$ into $A_{x_0}(M)$.

2.3. Generalized affine connections

As in the subsection 2.2, let $\tau = x_t,\ 0 \leq t \leq 1$, be a smooth curve in M. The *generalized affine parallel displacement* along τ is a generalized affine transformation of the affine tangent space $A_{x_0}(M)$ at x_0 onto the affine tangent space $A_{x_1}(M)$ at x_1 which is defined by a given generalized affine connection in $A(M)$. Let $\bar{\tau}_s^t$ be the generalized affine parallel displacement along the curve τ from x_t to x_s. In particular, $\bar{\tau}_0^t$ is the generalized affine parallel displacement $A_{x_t}(M) \rightarrow A_{x_0}(M)$ along τ (in the reversed direction) from x_t to x_0. The *generalized affine development* of the curve τ in M into the affine tangent space $A_{x_0}(M)$ is the curve $\bar{\tau}_0^t(x_t)$ in $A_{x_0}(M)$. Now we obtain the following theorem.

Theorem 2.3. Let $\bar{\omega}$ be an arbitrarily given generalized affine connection in $A(M)$, and let $\tau = x_t,\ 0 \leq t \leq 1$, be a smooth curve in M. Let $\bar{\tau}_0^t$ be the parallel displacement of $A_{x_t}(M)$ into $A_{x_0}(M)$ along τ with respect to the generalized affine connection $(form)$ $\bar{\omega}$. Then the generalized affine development $\bar{C}_t = \bar{\tau}_0^t(x_t)$ $(0 \leq t \leq 1)$ of the curve $\tau = x_t$ $(0 \leq t \leq 1)$ in M into $A_{x_0}(M)$ is given as follows:

$$\bar{C}_t = \bar{\tau}_0^t(x_t) = \bar{\tau}_0^t(\dot{x}_t) - \tau_0^t(\dot{x}_t)\ (0 \leq t \leq 1),$$

where $\dot{x}_t := dx_t/dt$ and τ_0^t is the linear parallel displacement along τ from x_t to x_0 with respect to the linear connection ω in $L(M)$ which is corresponding to $\bar{\omega}$ $(\tilde{\gamma}^{\star}(\bar{\omega}) = \omega + \varphi)$ in $A(M)$.

Proof. For the generalized affine connection $\bar{\omega}$ in $A(M)$, $\tilde{\gamma}^{\star}(\bar{\omega}) = \omega + \varphi$, where ω (resp. φ) is the linear connection (resp. R^n-valued 1-form) on $L(M)$ (cf. Lemma 1.1). Let u_0 be a point in $L(M)$ such that $\pi(u_0) = x_0$, and u_t the horizontal lift of x_t in $L(M)$ with respect to the linear connection ω. Let $\bar{u}_t$ be the horizontal lift of x_t in $A(M)$ with respect to the generalized affine connection (form) $\bar{\omega}$ such that $\bar{u}_0 = u_0$. Since, by virtue of Lemma 1.1, the homomorphism $\tilde{\beta} : A(M) \to L(M) = A(M)/R^n$ maps $\bar{u}_t$ into u_t, there exists a curve $\bar{a}_t$ in $R^n \subset A(n; R)$ such that $\bar{u}_t = u_t\bar{a}_t$ and $\bar{a}_0$ is the identity. Then, since $\bar{a}_t \in R^n \subset A(n; R)$, we can put

$$\bar{a}_t = \begin{pmatrix} I_n & \bar{\xi}(t) \\ 0 & 1 \end{pmatrix}, \quad \bar{a}_t^{-1} = \begin{pmatrix} I_n & -\bar{\xi}(t) \\ 0 & 1 \end{pmatrix} \tag{2.24}$$

for each t.

Here we shall find a necessary and sufficient condition for $\bar{a}_t$ in order that $\bar{u}_t$ be horizontal with respect to the generalized affine connection (form) $\bar{\omega}$. From Leibniz's formula, we get

$$\dot{\bar{u}}_t = \dot{u}_t\bar{a}_t + u_t\dot{\bar{a}}_t. \tag{2.25}$$

Since the curve u_t in $L(M)$ is a horizontal lift with respect to ω, we obtain by virtue of (2.25) and Lemma 1.1

$$\begin{aligned} \bar{\omega}(\dot{\bar{u}}_t) &= Ad(\bar{a}_t^{-1})(\bar{\omega}(\dot{u}_t)) + \bar{a}_t^{-1}\dot{\bar{a}}_t \\ &= Ad(\bar{a}_t^{-1})(\omega(\dot{u}_t) + \varphi(\dot{u}_t)) + \bar{a}_t^{-1}\dot{\bar{a}}_t \\ &= Ad(\bar{a}_t^{-1})(\varphi(\dot{u}_t)) + \bar{a}_t^{-1}\dot{\bar{a}}_t. \end{aligned} \tag{2.26}$$

Thus, by virtue of (2.24) and (2.26), we get the fact that $\bar{u}_t$ is horizontal with respect to the generalized affine connection (form) $\bar{\omega}$ if and only if

$$\varphi(\dot{u}_t) = -\dot{\bar{a}}_t\, \bar{a}_t^{-1} = \bar{a}_t\, (d\bar{a}_t^{-1}/dt) = d\bar{a}_t^{-1}/dt. \tag{2.27}$$

Now, in order to obtain the generalized affine development, we assume that the curve $\bar{u}_t = u_t\bar{a}_t$ in $A(M)$ is horizontal with respect to the generalized affine connection $\bar{\omega}$. Then we get

$$\bar{\tau}_0^t(\dot{x}_t) = (\bar{u}_0 \circ \bar{u}_t^{-1})(\dot{x}_t) = (u_0 \circ \bar{a}_t^{-1} \circ u_t^{-1})(\dot{x}_t). \tag{2.28}$$

Since $u_t^{-1}(\dot{x}_t) \in A^n$ and $\bar{a}_t^{-1} \in A(n; R)$, we get from (2.24)

$$\bar{a}_t^{-1}(u_t^{-1}(\dot{x}_t)) = u_t^{-1}(\dot{x}_t) - \bar{\xi}(t). \tag{2.29}$$

By virtue of (2.28) and (2.29), we obtain

$$\bar{\tau}_0^t(\dot{x}_t) = \tau_0^t(\dot{x}_t) - u_0(\bar{\xi}(t)). \tag{2.30}$$

On the other hand, we have

$$\bar{C}_t = \bar{\tau}_0^t(x_t) = \bar{u}_0(\bar{u}_t^{-1}(x_t)) = u_0(\bar{a}_t^{-1}(u_t^{-1}(x_t))) = u_0(\bar{a}_t^{-1}(0))), \tag{2.31}$$

since ${u_t}^{-1} \in L(M) \subset A(M)$ and $x_t = 0_{x_t} \in T_{x_t}(M)$. We get from (2.24) and (2.31)

$$\bar{C}_t = -\bar{u}_0(\bar{\xi}(t)) = -u_0(\bar{\xi}(t)). \tag{2.32}$$

Therefore, by virtue of (2.30) and (2.32), the generalized affine development $\bar{C}_t$ of a curve $\tau = x_t$ $(0 \leqq t \leqq 1)$ in M into $A_{x_0}(M)$ is given as follows:

$$\bar{C}_t = \bar{\tau}_0^t(\dot{x}_t) - \tau_0^t(\dot{x}_t), \tag{2.33}$$

where $\bar{\tau}_0^t$ and τ_0^t are parallel displacements along τ from x_t to x_0 with respect to the generalized affine and the linear connections respectively. ■

From the course of the proof of Theorem 2.3, we get the following corollary.

Corollary 2.4. Let $\bar{\omega}$ be an arbitrarily given generalized affine connection in $A(M)$ such that $\tilde{\gamma}^{\star}\bar{\omega} = \omega + \varphi$. Let $\tau = x_t$ $(0 \le t \le 1)$ be a smooth curve in M, and u_t a horizontal lift of $\tau = x_t$ in $L(M)$ with respect to ω. Let $\bar{u}_t$ be a smooth curve in $A(M)$ such that $\tilde{\pi}(\bar{u}_t) = x_t$ and $\bar{u}_0 = u_0$. Then, $\bar{u}_t$ is the horizontal lift of $\tau = x_t$ in $A(M)$ with respect to $\bar{\omega}$ if and only if, for each t,

$$\bar{u}_t = u_t\bar{a}_t \ (\bar{a}_t \in R^n \subset A(n; R)) \quad \text{and} \quad \varphi(\dot{u}_t) = d\bar{a}_t^{-1}/dt.$$

Proof. This is clear from (2.27) and Lemma 1.1. ■

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.5. Let $\bar{\omega}$ be an arbitrarily given generalized affine connection in $A(M)$ such that $\tilde{\gamma}^{\star}\bar{\omega} = \omega + \varphi$. Let $\bar{C}_t$ be the generalized affine development of a curve $\tau = x_t$ $(0 \le t \le 1)$ in M into $A_{x_0}(M)$. Then (i) if $\dot{x}_t$ is parallel along $\tau = x_t$ with respect to the generalized affine connection $\bar{\omega}$ in $A(M)$, then $\bar{C}_t = \dot{x}_t|_{t=0} - \tau_0^t(\dot{x}_t)$, (ii) if $\dot{x}_t$ is parallel along $\tau = x_t$ with respect to the linear connection ω in $L(M)$, then $\bar{C}_t = \bar{\tau}_0^t(\dot{x}_t) - \dot{x}_t|_{t=0}$.

Finally, we investigate relationships among covariant derivatives with respect to connections in $L(M)(\subset A(M))$ and $A(M)$.

Let ω be an arbitrarily given linear connection in $L(M)(\subset A(M))$, φ an arbitrarily given tensorial 1-form on $L(M)$ of type $(GL(n; R), R^n)$ (cf. [2, §5 of Chapter II). Let $\tilde{\omega}$ and $\bar{\omega}$ be the affine and the generalized affine connections in $A(M)$ respectively such that

$$\tilde{\gamma}^{\star}(\tilde{\omega}) = \omega + \theta \quad \text{and} \quad \tilde{\gamma}^{\star}(\bar{\omega}) = \omega + \varphi \quad \text{on } L(M). \tag{2.34}$$

The covariant derivative $(\bar{\nabla}_{\dot{x}_t} Y_t)_{t=0}$, $(Y_t := Y_{x_t})$, of a cross section Y along a curve $\tau = x_t$ $(0 \le t \le 1)$ with respect to the generalized affine connection (form) $\bar{\omega}$ is defined by

$$(\bar{\nabla}_{\dot{x}_t} Y_t)_{t=0} = \lim_{t \to 0} \frac{\bar{\tau}_0^t(Y_t) - Y_0}{t}, \tag{2.35}$$

where $\bar{\tau}_0^t$ is the generalized affine parallel displacement with respect to the generalized affine connection (form) $\bar{\omega}$ along the curve τ from x_t to x_0. Using the notations as in the course of the proof of Theorem 2.3, we get from (2.24)

$$\bar{\tau}_0^t(Y_t) = \bar{u}_0(\bar{u}_t^{-1}(Y_t) = \bar{u}_0(\bar{a}_t^{-1}{u_t}^{-1})(Y_t) = u_0({u_t}^{-1}(Y_t) - \bar{\xi}(t)), \tag{2.36}$$

since $\bar{a}_t^{-1} \in A(n;\, R)$ and ${u_t}^{-1}(Y_t) \in A^n$. From (2.35) and (2.36), we obtain

$$(\bar{\nabla}_{\dot{x}_t} Y_t)_{t=0} = \lim_{t\to 0} \frac{u_0{u_t}^{-1}(Y_t) - Y_0 - u_0(\bar{\xi}(t))}{t}. \tag{2.37}$$

Since $\bar{a}_0 \in A(n;\, R)$ is the identity, from (2.24) we have $\bar{\xi}(0) = 0 (\in R^n)$. From this fact, (2.35) and (2.37), we get

$$(\bar{\nabla}_{\dot{x}_t} Y_t)_{t=0} = \left(\nabla_{\dot{x}_t} Y_t - u_0\left(\frac{d\bar{\xi}(t)}{dt}\right)\right)_{t=0}. \tag{2.38}$$

By virtue of (2.32) and (2.38), we get

$$(\bar{\nabla}_{\dot{x}_t} Y_t)_{t=0} = \left(\nabla_{\dot{x}_t} Y_t + \frac{d\bar{C}_t}{dt}\right)_{t=0}. \tag{2.39}$$

Therefore we obtain

Theorem 2.6. Let ω be an arbitrarily given linear connection in $L(M)$ and φ an arbitrarily given tensorial 1-form on $L(M)$ of type $(GL(n;\, R),\, R^n)$. Let $\bar{\omega}$ be the generalized affine connection in $A(M)$ such that $\tilde{\gamma}^\star(\bar{\omega}) = \omega + \varphi$ on $L(M)$, and $\tau = x_t$ $(0 \le t \le 1)$ a C^∞ curve in M. Let Y be a cross section of M into TM (or $A(M) \times_{A(n;R)} A^n$). Let $\nabla_{\dot{x}_t}$ $(resp.\ \bar{\nabla}_{\dot{x}_t})$ be the covariant differentiation along τ with respect to ω $(resp.\ \bar{\omega})$. Then

$$(\bar{\nabla}_{\dot{x}_t} Y_t)_{t=0} = \left(\nabla_{\dot{x}_t} Y_t + \frac{d\bar{C}_t}{dt}\right)_{t=0},$$

where $\bar{C}_t$ is the generalized affine development of $\tau = x_t$ $(0 \le t \le 1)$ into $A_{x_0}(M)$.

By the help of Theorems 2.2 and 2.6, we get

Corollary 2.7. Let ω be an arbitrarily given linear connection in $L(M)$ and θ the canonical 1-form on $L(M)$. Let φ be an arbitrarily given tensorial 1-form on $L(M)$ of type $(GL(n;\, R),\, R^n)$. Let $\tilde{\omega}$ $(resp.\ \bar{\omega})$ be the affine $(resp.\ the\ generalized\ affine)$ connection in $A(M)$ such that $\tilde{\gamma}^\star(\tilde{\omega}) = \omega + \theta$ $(resp.\ \tilde{\gamma}^\star(\bar{\omega}) = \omega + \varphi)$ on $L(M)$, and $\tau = x_t$ $(0 \le t \le 1)$ a C^∞ curve in M. Let Y be a cross section of M into TM (or $A(M) \times_{A(n;R)} A^n$). Let $\tilde{\nabla}_{\dot{x}_t}$ $(resp.\ \bar{\nabla}_{\dot{x}_t})$ be the covariant differentiation along τ with respect to $\tilde{\omega}$ $(resp.\ \bar{\omega})$. Then

$$(\bar{\nabla}_{\dot{x}_t} Y_t)_{t=0} = \left(\tilde{\nabla}_{\dot{x}_t} Y_t + \frac{d(\bar{C}_t - \tilde{C}_t)}{dt}\right)_{t=0},$$

where $\tilde{C}_t$ $(resp.\ \bar{C}_t)$ is the affine $(resp.\ the\ generalized\ affine)$ development of $\tau = x_t$ $(0 \le t \le 1)$ into $A_{x_0}(M)$.

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