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Strongly Prime and Strongly Semiprime ideals in Posets

J. Catherine Grace John and B. Elavarasan[1]

Department of Mathematics,
Karunya University, Coimbatore-641114, Tamilnadu, India.
E-mail: catherine@karunya.edu; belavarasan@gmail.com

Abstract

In this paper, we study the notions of strongly prime and strongly semi prime ideals of posets, and explore the various properties of strongly primeness and strongly semi primeness in posets.

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1. Preliminaries

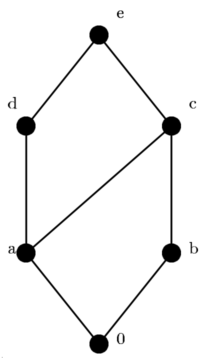
Throughout this paper $(P, \leq)$ denotes a poset with smallest element 0. For basic terminology and notation for posets, we refer [1] and [2]. For $M \subseteq P$, let $L(M) = \{x \in P : x \leq m \text{ for all } m \in M\}$ denote the lower cone of M in P and dually, let $U(M) = \{x \in P : m \leq x \text{ for all } m \in M\}$ be the upper cone of M in P. Let $A, B \subseteq P$, we write $L(A, B)$ instead of $L(A \cup B)$ and dually for the upper cones. If $M = \{x_1, x_2, \ldots, x_n\}$is finite, then we use the notation $L(x_1, x_2, ldots, x_n)$ instead of $L(\{x_1, x_2, \ldots, x_n\})$(and dually). It is clear that for any subset A of P, we have $A \subseteq L(U(A))$ and $A \subseteq U(L(A))$. If $A \subseteq B$, then $L(B) \subseteq L(A)$ and $U(B) \subseteq U(A)$. Moreover, $LUL(A) = L(A)$ and $ULU(A) = U(A)$. Following [1], a subset I of P is called ideal if for any $a, b \in I$, we have $L(U(a, b)) \subseteq I$. Following [3], a non empty subset I of P is called a semi-ideal if $b \in I$ and $a \leq b$, then $a \in I$. It is clear that every ideal is semi-ideal, but converse need not be true in general. A proper ideal I of P is called prime if $L(a, b) \subseteq I$ implies that either $a \in I$ or $b \in I$[1]. Let I be an ideal of P. Then the extension of I by $x \in P$ is meant the set $\langle x, I\rangle = \{a \in P : L(a, x) \subseteq I\}$.

An ideal I of P is called semi prime if $L(a, b) \subseteq I$ and $L(a, c) \subseteq I$ together imply $L(a, U(b, c)) \subseteq I$[1]. For $a \in P$, the subset $\{x \in P : x \leq a\}$ is an ideal of P generated

[1]Corresponding author.

by a , denoted by $(a]$. For any subset A of P, we denote $A^* = A \backslash \{0\}$. An ideal I of P is called strongly prime if $L(A^*, B^*) \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for any different proper ideals A, B of P. An ideal I of P is called strongly semi-prime if $L(A^*, B^*) \subseteq I$ and $L(A^*, C^*) \subseteq I$ together imply $L(A^*, U(B^*, C^*)) \subseteq I$ for any different ideals A, B and C of P. It is clear the intersection of any nonempty family of strongly prime ideals of P is a strongly semi prime ideal and every strongly prime ideal of P is prime ideal of P. The following example shows that prime ideal of P not necessarily to be strongly prime in general.

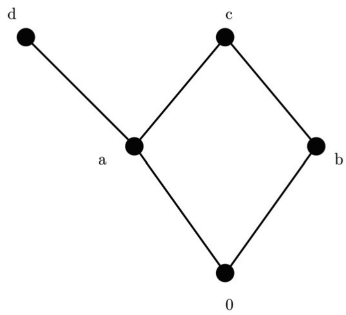
Example 1.1. Consider $P = \{0, a, b, c, d, e, \}$ and define a relation $\leq$ on P as follows.



Then $(P, \leq)$ is a poset and $I = \{0, a, d\}$ is a prime ideal of P, but not strongly prime, since for ideals $A = \{0, b\}$ and $B = \{0, a, b, c\}$ of P, we have $L(A^*, B^*) \subseteq I$, but neither A nor B contained in I.

Let I be a semi-ideal of P. Then I satisfies (*) condition if whenever $L(A, B) \subseteq I$ implies $A \subseteq \langle B, I \rangle$ for any subsets A and B of P. In Example 1.1, let $A = \{0, a, b, c\}$, $B = \{0, b\}$ and $I = \{0, a, d\}$. Then $L(A, B) \subseteq I$, but $A \nsubseteq \langle B, I \rangle = \{0, a, d\}$. So there exists an ideal which not satisfies (*) condition. It is also clear that every strongly semi prime ideal with (*) condition of P is semi prime, but converse not true in general.

Example 1.2. Let $P = \{0, a, b, c, d\}$ and define a relation $\leq$ on P as follows.



Then $(P, \leq)$ is a poset and $I = \{0\}$ is a semi prime ideal of P, but not strongly semi prime since for ideals $A = \{0, a\}$; $B = \{0, b\}$; $C = \{0, a, b, c\}$ of P, we have $L(A^*, B^*) \subseteq I$ and $L(A^*, C^*) \subseteq I$, but $L(A^*, U(B^*, C^*)) \nsubseteq I$.

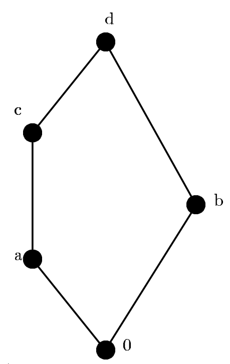
2. Main Results

Lemma 2.1. Let I be an ideal of a poset P. If I is strongly prime, then I is strongly semi prime.

Proof. Let A, B and C be ideals of P such that $L(A^*, B^*) \subseteq I$ and $L(A^*, C^*) \subseteq I$. If $A \subseteq I$, then $L(A^*, U(B^*, C^*)) \subseteq L(A^*) \subseteq I$. If $A \nsubseteq I$, then $B \subseteq I$ and $C \subseteq I$ which imply $L(U(b, c)) \subseteq I$ for all $b \in B^*$; $c \in C^*$ which implies $L(U(B^*, C^*)) \subseteq I$. So $L(A^*, U(B^*, C^*)) \subseteq L(U(B^*, C^*)) \subseteq I$. ■

The following example shows that the converse of Lemma 2.1 does not true in general. That is strongly semi prime ideal of P is not necessarily to be a strongly prime ideal of P in general.

Example 2.2. Let $P = \{0, a, b, c, d\}$ and define a relation $\leq$ on P as follows.



Then $(P, \leq)$ is a poset and $I = \{0\}$ is a strongly semi prime ideal of P, but not strongly prime since for ideals $A = \{0, a, c\}$; $B = \{0, b\}$ of P, $L(A^*, B^*) \subseteq I$, but $A \nsubseteq I$ and $B \nsubseteq I$.

Following [4], a proper ideal I of P is called irreducible if for any ideals J and K of P, $I = J \cap K$ implies $J = I$ or $K = I$.

Theorem 2.3. Every strongly prime ideal of P is an irreducible ideal of P.

Proof. Let I be a strongly prime ideal of P, and J, K be ideals of P such that $I = J \cap K$. Suppose there exists $x \in J \setminus I$ and $y \in K \setminus I$. Then $L((x]^*, (y]^*) \subseteq L(x, y) \subseteq J \cap K \subseteq I$. Since I is a strongly prime, we have either $x \in (x] \subseteq I$ or $y \in (y] \subseteq I$, a contradiction. ■

Theorem 2.4. Let I be an ideal of P. Then the following statements are holds for any ideals A, B, C of P.

(i) If I satisfies (*) condition, then $L(A^*, B^*) \subseteq \langle C^*, I \rangle$ if and only if $L(C^*, A^*, B^*) \subseteq I$.

(ii) If I is strongly semi prime with (*) condition, then $L(C^*, U(A^*, B^*)) \subseteq I$ if and only if $L(U(A^*, B^*)) \subseteq \langle C^*, I\rangle$.

(iii) $\langle A^*, I\rangle = P$ if and only if $A \subseteq I$.

Proof.

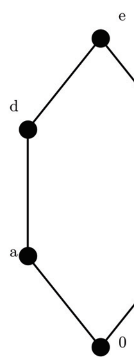
(i) Suppose that $L(A^*, B^*) \subseteq< C^*, I >$ and let $z \in L(C^*, A^*, B^*)$. Then $z \in< C^*, I >$ and $z \leq c$ for all $c \in C^*$ which imply $z \in L(z, C^*) \subseteq I$. Suppose that $L(C^*, A^*, B^*) \subseteq I$ and $z \in L(A^*, B^*)$. Then $z \in \langle C^*, I\rangle$.

(ii) Suppose $L(C^*, U(A^*, B^*)) \subseteq I$ and let $z \in L(U(A^*, B^*))$. Then $L(C^*, z) \subseteq L(C^*, U(A^*, B^*)) \subseteq I$ which implies $z \in \langle C^*, I\rangle$. If $L(U(A^*, B^*)) \subseteq \langle C^*, I\rangle$, then $A^* \subseteq \langle C^*, I\rangle$ and $B^* \subseteq \langle C^*, I\rangle$ which imply $L(C^*, U(A^*, B^*)) \subseteq I$.

(iii) If $x \in \langle A^*, I\rangle$ for all $x \in P$, then $a \in \bigcap_{x\in P} \langle x, I\rangle$ for $a \in A$ which implies $a \in L(a) \subseteq I$. ■

For an ideal I and a nonempty subset A of P, we define $\langle A, I\rangle = \{z \in P : L(a, z) \subseteq I$ for all $a \in A\}$. It is observe that $L(z, A^*) \subseteq I$ for $z \in \langle A^*, I\rangle$, and $I \subseteq \langle A^*, I\rangle = \bigcap_{x\in A^*} \langle x, I\rangle$ is always a semi ideal of P. But not necessary to be an ideal of P shows in the following example.

Example 2.5. Consider $P = \{0, a, b, c, d, e\}$ and define a relation $\leq$ on P as follows.



Then $(P, \leq)$ is a poset and $I = \{0, a\}$ is an ideal of P. For $A = \{a, d\}$, $\langle A, I\rangle$ is not an ideal of P.

Lemma 2.6. Let I and A be ideals of P. If I is strongly semi prime with (*) condition, then $\langle A^*, I\rangle$ is an ideal of P.

Proof. Let $a, b \in \langle A^*, I\rangle$. Then $L((a]^*, A^*) \subseteq L(a, A^*) \subseteq I$ and $L((b]^*, A^*) \subseteq L(b, A^*) \subseteq I$. Since I is strongly semi-prime, we have $L(A^*, U((a]^*, (b]^*)) \subseteq I$. By

Lemma 2.4 (ii), we have $L(U(a,b)) \subseteq L(U((a]^*,(b]^*)) \subseteq \langle A^*, I\rangle$, so $\langle A^*, I\rangle$ is an ideal of P. ■

Theorem 2.7. Let I be an ideal of P with (*) condition. Then I is strongly semi prime of P if and only if $\langle A^*, I\rangle$ is a strongly semi prime ideal of P for any ideal A of P.

Proof. Let I be a strongly semi prime and A be an ideal of P. Suppose that B, C and D are ideals of P such that $L(C^*, B^*) \subseteq \langle A^*, I\rangle$ and $L(C^*, D^*) \subseteq \langle A^*, I\rangle$. Then by Lemma 2.4(i), $L(A^*, C^*) \subseteq \langle B^*, I\rangle$ and $L(A^*, C^*) \subseteq \langle D^*, I\rangle$.

Let $z \in L(A^*, C^*, U(B^*, D^*))$. Then $z \in L(A^*, C^*)$ and $z \in L(U(B^*, D^*))$ which imply $L(B^*, (z]^*) \subseteq L(B^*, z) \subseteq I$ and $L(D^*, (z]^*) \subseteq L(D^*, z) \subseteq I$, so $B^*, D^* \subseteq \langle L(z)^*, I\rangle$. By Lemma 2.6, $\langle L(z)^*, I\rangle$ is an ideal of P which implies $z \in L(U(B^*, D^*)) \subseteq \langle (z]^*, I\rangle$, so $z \in I$. Thus $L(A^*, C^*, U(B^*, D^*)) \subseteq I$ and hence $L(C^*, U(B^*, D^*)) \subseteq \langle A^*, I\rangle$.

Conversely, let $\langle A^*, I\rangle$ is a strongly semi prime ideal for any ideal A of P. Suppose B, C and D are ideals of P such that $L(B^*, C^*) \subseteq I$ and $L(B^*, D^*) \subseteq I$. Then $L(B^*, C^*) \subseteq \langle B^*, I\rangle$ and $L(B^*, D^*) \subseteq \langle B^*, I\rangle$. Since $\langle B^*, I\rangle$ is strongly semi prime, we have $L(B^*, U(C^*, D^*)) \subseteq < B^*, I >$. Let $t \in L(B^*, U(C^*, D^*))$. Then $L(B^*, t) \subseteq I$ which implies $t \in I$. Hence $L(B^*, U(C^*, D^*)) \subseteq I$. ■

Theorem 2.8. Let I be a maximal strongly prime ideal of P with (*) condition. Then I is strongly prime.

Proof. Let I be a maximal and strongly semi prime ideal of P. Suppose that A and B are ideals of P such that $L(A^*, B^*) \subseteq I$. Then, by Theorem 2.7, $\langle B^*, I\rangle$ is a strongly semi prime ideal of P and $A^* \subseteq \langle B^*, I\rangle$. If $I \subseteq \langle B^*, I\rangle$, then, by maximality of I, $\langle B^*, I\rangle = P$. By Lemma 2.4(iii), we have $B \subseteq I$. ■

Corollary 2.9. Let I be a maximal ideal of P with (*) condition. Then I is strongly semi prime if and only if I is strongly prime.

Theorem 2.10. Let I be a strongly prime ideal of P. Then $\langle A^*, I\rangle = I$ for all ideal A of P not contained in I.

Proof. Suppose I is strongly prime and A is an ideal of P such that $A \nsubseteq I$. Let $z \in \langle A^*, I\rangle$. Then $L((z]^*, A^*) \subseteq (L(z, A^*) \subseteq I$ which imply $z \in (z]^* \subseteq I$. ■

Corollary 2.11. Let I be an ideal of P with (*) condition. Then I is strongly prime if and only if $\langle A^*, I\rangle = I$ for all ideal A of P not contained in I.

For an ideal I of a poset P, consider the set $F_I = \{x \in P : \langle x, I\rangle = I\}$.

Lemma 2.12. Let I be an ideal of a finite poset P. Then $\langle A^*, I\rangle \cap F_I = \phi$ for all ideals A of P not contained in I.

Proof. It is trivial. ■

Lemma 2.13. Let I be a proper ideal of P. Then $I \cap F_I = \phi$.

Theorem 2.14. Let I be a proper ideal of P with (*) condition. Then I is strongly prime if and only if $I \cup F_I = P$.

Proof. Suppose I is a strongly prime ideal of P and let $x \in P$ and $x \notin F_I$. Then there exists $y \in < x, I >$ with $y \notin I$ and $L((x]^*, (y]^*) \subseteq L(x, y) \in I$, which implies $x \in (x] \subseteq I$.

Let $I \cup F_I = P$. If A and B are ideals of P with $L(A^*, B^*) \subseteq I$ and $A \nsubseteq I$. Then $B^* \subseteq \langle A^*, I \rangle$ and there exits $a \in A \backslash I$ with $\langle a, I \rangle = I$ which imply $B^* \subseteq \bigcap_{t \in A^*} \langle t, I \rangle \subseteq \langle a, I \rangle = I$. ■

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