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A note on degenerate Boole numbers and polynomials

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Abstract

In this paper, we introduce the degenerate Boole numbers and polynomials and give some identities related to these numbers and polynomials. Specially we relate degenerate Boole polynomials to Euler polynomials and Boole polynomials. We also study higher order degenerate Boole polynomials.

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1. Introduction

As is well known, the Euler polynomials are given by the generating function to be

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}, \quad \text{(see [1-3,5,6,8-10])}.$$

When $x=0$, $E_n=E_n(0)$ are called the Euler numbers. In [3] and [10], L. Carlitz considered the degenerate Euler polynomials as follows:

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\xi_n(x\mid\lambda)\frac{t^n}{n!}. \tag{1.1}$$

When $x=0$, $\xi_n(\lambda)=\xi_n(0\mid\lambda)$ are called the degenerate Euler numbers. Note that $\lim_{\lambda\to 0}\xi_n(x\mid\lambda)=E_n(x)$, $(n\geq 0)$.

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$.

Let f be a continuous function on $\mathbb{Z}_p$. Then the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by T. Kim to be

$$\begin{aligned} I_{-1}(f)=\int_{\mathbb{Z}_p}f(x)d\mu_{-1}(x)&=\lim_{N\to\infty}\sum_{x=0}^{p^N-1}f(x)\mu_{-1}(x+p^N\mathbb{Z}_p)\\ &=\lim_{N\to\infty}\sum_{x=0}^{p^N-1}f(x)(-1)^x, \quad \text{(see [1-3,5,6,8-10])}. \end{aligned} \tag{1.2}$$

In [4, 7], the Boole polynomials are given by the generating function to be

$$\int_{\mathbb{Z}_p}(1+t)^{x+sy}d\mu_{-1}(y)=\frac{2}{1+(1+t)^s}(1+t)^x=\sum_{n=0}^{\infty}2Bl_n(x|s)\frac{t^n}{n!}. \tag{1.3}$$

When $x=0$, $Bl_n(s)=Bl_n(0|s)$ are called the Boole numbers.

From (1.3), we have

$$\int_{\mathbb{Z}_p}(x+sy)_n d\mu_{-1}(y)=2Bl_n(x|s). \tag{1.4}$$

In the viewpoint of (1.1), we consider the degenerate Boole numbers and polynomials and we investigate some properties of these numbers and polynomials.

2. Degenerate Boole polynomials

Let us assume that $\lambda, t \in \mathbb{C}_p$ such that $|\lambda t|_p < p^{\frac{-1}{p-1}}$. From (1.2), we note that

$$\int_{\mathbb{Z}_p} f(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2f(0) \quad \text{(see [1-3,5,6,8-10])}. \tag{2.1}$$

Let us take $f(x) = (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{sx}$ in (2.1), then we get

$$\int_{\mathbb{Z}_p} (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{x+sy} d\mu_{-1}(y) = \frac{2}{1+(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^s}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^x. \tag{2.2}$$

In the viewpoint of (1.3), we define the degenerate Boole polynomials as follows:

$$\frac{1}{1+((1+\log(1+\lambda t)^{\frac{1}{\lambda}}))^s}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^x = \sum_{n=0}^{\infty} \mathfrak{Bl}_{n,\lambda}(x|s)\frac{t^n}{n!}. \tag{2.3}$$

When $x=0$, $\mathfrak{Bl}_{n,\lambda}(s) = \mathfrak{Bl}_{n,\lambda}(0|s)$ are called the degenerate Boole numbers.

It is well-kown fact that the generating function of the Stirlng number of the first kind is given by

$$(\log(1+t))^m = m!\sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!}, \quad \text{(see [6, 8])},$$

and the Stirling number of the second kind is defined by the generating function to be

$$(e^t-1)^n = \sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!}, \quad \text{(see [6, 8])}.$$

The following observation is useful for our further theory and is well-known. For the completeness of this paper, we record as lemma.

Lemma 2.1. For $n \geq 0$, we have

$$(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^x = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \lambda^{n-m}(x)_m S_1(n,m)\right)\frac{t^n}{n!}.$$

Proof. By binomial theorem, we get

$$\begin{aligned}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^x &= \sum_{m=0}^{\infty}(x)_m\frac{1}{m!}(\log(1+\lambda t)^{\frac{1}{\lambda}})^m\\ &= \sum_{m=0}^{\infty}\frac{1}{\lambda^m}(x)_m\frac{1}{m!}(\log(1+\lambda t))^m\\ &= \sum_{m=0}^{\infty}\lambda^{-m}(x)_m\sum_{n=m}^{\infty}S_1(n,m)\frac{\lambda^n t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}(x)_m S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned}$$

■

Now we explain degenerate Boole polynomials by Boole polynomials.

Theorem 2.2. For $n \geq 0$, we have

$$\mathfrak{Bl}_{n,\lambda}(x|s) = \sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)Bl_m(x|s).$$

Proof. We take fermionic p-adic integration on both sides (2.2)and apply the proof of Lemma 2.1, then we have the following equation: for $n \geq 0$

$$\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}(x+sy)_m d\mu_{-1}(y) = 2\mathfrak{Bl}_{n,\lambda}(x|s). \tag{2.4}$$

Now apply (1.4), we have the result. ■

We can relate degenerate Boole polynomials to Euler polynomials as follwos:

Corollary 2.3. For $n \geq 0$, we have

$$2\mathfrak{Bl}_{n,\lambda}(x|s) = \sum_{m=0}^{n}\sum_{l=0}^{m}\lambda^{n-m}S_1(n,m)S_1(m,l)s^l E_l\left(\frac{x}{s}\right).$$

Proof. We observe that

$$\int_{\mathbb{Z}_p}(x+sy)_m d\mu_{-1}(y) = \sum_{l=0}^{m}S_1(m,l)\int_{\mathbb{Z}_p}(x+sy)^l d\mu_{-1}(y),$$

and

$$\int_{\mathbb{Z}_p}e^{(x+sy)t}d\mu_{-1}(y) = \frac{2}{e^{st}+1}e^{xt} = \sum_{n=0}^{\infty}s^n E_n\left(\frac{x}{s}\right)\frac{t^n}{n!}.$$

Then apply (2.4) in the proof of Theorem 2.2. ∎

The following theorem shows the inversion of Theorem 2.2

Theorem 2.4. For $m \geq 0$, we have

$$Bl_m(x|s) = \sum_{n=0}^{m} \mathfrak{Bl}_{n,\lambda}(x|s)\lambda^{m-n} S_2(m,n).$$

Proof. By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.3), we get

$$\begin{aligned}\frac{1}{1+(1+t)^s}(1+t)^x &= \sum_{n=0}^{\infty} \mathfrak{Bl}_{n,\lambda}(x|s)\lambda^{-n}\frac{1}{n!}(e^{\lambda t}-1)^n \\ &= \sum_{n=0}^{\infty} \mathfrak{Bl}_{n,\lambda}(x|s)\lambda^{-n}\sum_{m=n}^{\infty} S_2(m,n)\lambda^m \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} \mathfrak{Bl}_{n,\lambda}(x|s)\lambda^{m-n} S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned}$$

Now apply (1.3), we have the result. ∎

We list some properties on degenerate Boole polynomials.

Theorem 2.5. For $n \geq 0$, we have

(i) $\mathfrak{Bl}_{n,\lambda}(s(x+1)|s) + \mathfrak{Bl}_{n,\lambda}(sx|s) = \sum_{m=0}^{n} (sx)_m \lambda^{n-m} S_1(n,m).$

(ii) $\mathfrak{Bl}_{n,\lambda}(s|s) + \mathfrak{Bl}_{n,\lambda}(s) = \delta_{0,n}.$

Proof.

(i) From (2.3), we note that

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(Bl_{n,\lambda}(s(x+1)|s) + Bl_{n,\lambda}(sx|s)\right)\frac{t^n}{n!} \\ &= \frac{1}{1+(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^s}(1+(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^s)(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{sx} \\ &= (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{sx} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(sx)_m\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!},\end{aligned}$$

now comparing the coefficients of the above equations, we have the result.

(ii) Let us take $x = 0$ in (i), $\mathfrak{Bl}_{n,\lambda}(s|s) + \mathfrak{Bl}_{n,\lambda}(s) = \delta_{0,n}, \quad (n \geq 0)$. ∎

For the distribution relation, we consider the following integral equation: $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have

$$\int_{\mathbb{Z}_p} f(x+d)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2\sum_{l=0}^{d-1}(-1)^l f(l). \tag{2.5}$$

Theorem 2.6. For $n \geq 0$, we have

$$\begin{aligned}\mathfrak{Bl}_{n,\lambda}(s) + \mathfrak{Bl}_{n,\lambda}(sd|s) &= \mathfrak{Bl}_{n,\lambda}(s) + \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(sd)_l S_1(k,l)\lambda^{k-l}\mathfrak{Bl}_{n-k,\lambda}(s)\\ &= \sum_{l=0}^{d-1}\sum_{m=0}^{n}(-1)^l(sl)_m\lambda^{n-m}S_1(n,m).\end{aligned}$$

Proof. We apply $f(x) = (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{sx}$ in (2.5), then we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{sx}d\mu_{-1}(x)\\ &= \frac{2}{1+(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{sd}}\sum_{l=0}^{d-1}(-1)^l(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{sl}.\end{aligned}$$

This is equivalent to

$$\begin{aligned}&(1+(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{sd})\int_{\mathbb{Z}_p}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{sx}d\mu_{-1}(x)\\ &= 2\sum_{l=0}^{d-1}(-1)^l(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{sl}.\end{aligned} \tag{2.6}$$

Thus, by (2.6), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(2\mathfrak{Bl}_{n,\lambda}(s) + \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(sd)_l S_1(k,l)\lambda^{k-l}2\mathfrak{Bl}_{n-k,\lambda}(s)\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(2\sum_{l=0}^{d-1}(-1)^l\sum_{m=0}^{n}(sl)_m\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

By comparing the coefficients in (2.7), we get

$$\begin{aligned}&\mathfrak{Bl}_{n,\lambda}(s) + \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(sd)_l S_1(k,l)\lambda^{k-l}\mathfrak{Bl}_{n-k,\lambda}(s)\\ &= \sum_{l=0}^{d-1}\sum_{m=0}^{n}(-1)^l(sl)_m\lambda^{n-m}S_1(n,m).\end{aligned}$$

∎

For $k \in \mathbb{N}$, let us define the degenerate Boole polynomials of order k as follows:

$$\left(\frac{1}{1+(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^s}\right)^k \left(1+(1+\log(1+\lambda t)^{\frac{1}{\lambda}})\right)^x = \sum_{n=0}^{\infty} \mathfrak{Bl}_{n,\lambda}^{(k)}(x|s)\frac{t^n}{n!}. \quad (2.8)$$

When $x = 0$, $\mathfrak{Bl}_{n,\lambda}^{(k)}(s) = \mathfrak{Bl}_{n,\lambda}^{(k)}(0|s)$ are called the degenerate Boole numbers of order k.

We apply Lemma 2.1 to multiple fermionic p-adic multiple integral on $\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, then we have the following.

Lemma 2.7. For $m \geq 0$, we have

$$\begin{aligned} 2^k \mathfrak{Bl}_{m,\lambda}^{(k)}(x|s) &= \sum_{n=0}^{m} \lambda^{m-n} n! S_1(m,n) \\ &\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{sx_1+\cdots+sx_k+x}{n} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k). \end{aligned}$$

Proof. We note that

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{sx_1+\cdots+sx_k+x} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\ &= \left(\frac{2}{1+(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^s}\right)^k \left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x \quad (2.9) \\ &= \sum_{n=0}^{\infty} 2^k \mathfrak{Bl}_{n,\lambda}^{(k)}(x|s)\frac{t^n}{n!} \end{aligned}$$

We observe that

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{sx_1+\cdots+sx_k+x} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{sx_1+\cdots+sx_k+x}{n} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\lambda^{-n}(\log(1+\lambda t))^n \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{sx_1+\cdots+sx_k+x}{n} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\lambda^{-n} \quad (2.10) \\ &\quad \times n! \sum_{m=n}^{\infty} S_1(m,n)\frac{\lambda^m t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \lambda^{m-n} n! S_1(m,n) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{sx_1+\cdots+sx_k+x}{n} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\right)\frac{t^m}{m!}. \end{aligned}$$

By comparing the coefficients of (2.9) and (2.10), we have the result. ■

The higher-order Boole polynomials are given by the generating function to be

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{sx_1+\cdots+sx_k+x} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) = \left(\frac{2}{1+(1+t)^s}\right)^k (1+t)^x = \sum_{n=0}^{\infty} 2^k Bl_n^{(k)}(x|s)\frac{t^n}{n!}.$$

Thus we have

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (sx_1+\cdots+sx_k+x)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) = 2^k Bl_n^{(k)}(x|s), \ (n \geq 0). \tag{2.11}$$

Now we can see degenerate Boole polynomials via Boole polynomials.

Theorem 2.8. For $m \geq 0$, we have

$$\mathfrak{Bl}_{m,\lambda}^{(k)}(x|s) = \sum_{n=0}^{m} \lambda^{m-n} S_1(m,n) Bl_n^{(k)}(x|s).$$

Proof. Apply (2.11) to Lemma 2.7. ■

It is easy to show that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (sx_1+\cdots+sx_k+x)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) = \sum_{l=0}^{n} S_1(n,l) E_l^{(k)}\left(\frac{x}{s}\right) s^l, \tag{2.12}$$

where $E_l^{(k)}(x)$ are the higher-order Euler polynomials which are given by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}.$$

Therefore, by (2.11), (2.12) and Theorem 2.8, we obtain the following corollary.

Corollary 2.9. For $m \geq 0$, we have

$$2^k \mathfrak{Bl}_{m,\lambda}^{(k)}(x|s) = \sum_{n=0}^{m} \sum_{l=0}^{n} \lambda^{m-n} S_1(m,n) S_1(n,l) E_l^{(k)}\left(\frac{x}{s}\right) s^l.$$

Now we study some properties of higher order degenerate Boole numbers.

Theorem 2.10.

For $m \geq 0$, we have

$$\text{(i)}\quad \mathfrak{Bl}_{n,\lambda}^{(k)}(s) = \sum_{l_1+\cdots+l_k=n} \binom{n}{l_1,\cdots,l_k} \mathfrak{Bl}_{l_1,\lambda}(s)\cdots\mathfrak{Bl}_{l_k,\lambda}(s).$$

$$\text{(ii)}\quad \mathfrak{Bl}_{m,\lambda}^{(k)}(s) = \sum_{n=0}^{m}(-1)^n\lambda^{m-n}(k+n-1)_n S_1(m,n).$$

Proof.

(i) From (2.8), we easily note that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{Bl}_{n,\lambda}^{(k)}(s)\frac{t^n}{n!} &= \left(\frac{1}{1+(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^s}\right)^k \\ &= \sum_{n=0}^{\infty}\sum_{l_1+\cdots+l_k=n}\binom{n}{l_1,\cdots,l_k}\mathfrak{Bl}_{l_1,\lambda}(s)\cdots\mathfrak{Bl}_{l_k,\lambda}(s)\frac{t^n}{n!}.\end{aligned}$$

(ii)

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{Bl}_{n,\lambda}^{(k)}(s)\frac{t^n}{n!} &= \left(\frac{1}{1+(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^s}\right)^k \\ &= \sum_{n=0}^{\infty}(-1)^n\lambda^{-n}\binom{k+n-1}{n}(\log(1+\lambda t))^n \\ &= \sum_{n=0}^{\infty}(-1)^n\lambda^{-n}(k+n-1)_n\sum_{m=n}^{\infty}S_1(m,n)\lambda^m\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}(-1)^n\lambda^{m-n}(k+n-1)_n S_1(m,n)\right)\frac{t^m}{m!}.\end{aligned}$$

■

The following theorem shows the inversion of Theorem 2.4.

Theorem 2.11. For $m \geq 0$, we have

$$Bl_m^{(k)}(x|s) = \sum_{n=0}^{m}\mathfrak{Bl}_{n,\lambda}^{(k)}(x|s)\lambda^{m-n}S_2(m,n).$$

Proof. By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.9), we get

$$\begin{aligned}\sum_{m=0}^{\infty} 2^k \mathfrak{Bl}_{m,\lambda}^{(k)}(s)\frac{t^m}{m!} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{sx_1+\cdots+sx_k+x}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\\ &= \sum_{n=0}^{\infty} 2^k \mathfrak{Bl}_{n,\lambda}^{(k)}(x|s)\frac{1}{n!}\left(\frac{1}{\lambda}(e^{\lambda t}-1)\right)^n\\ &= \sum_{n=0}^{\infty} 2^k \mathfrak{Bl}_{n,\lambda}^{(k)}(x|s)\sum_{m=n}^{\infty}\lambda^{m-n}S_2(m,n)\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} 2^k \mathfrak{Bl}_{n,\lambda}^{(k)}(x|s)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned}$$

■

Now, we consider the degenerate Boole polynomials of the second kind with order k as follows:

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{-sx_1-x_2-\cdots-sx_k+x}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\\ &=\left(\frac{2}{1+(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^s}\right)^k\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{sk+x} = \sum_{n=0}^{\infty}2^k\widehat{\mathfrak{Bl}}_{n,\lambda}^{(k)}(x|s).\end{aligned} \tag{2.13}$$

In the following, we can see the relation higher order degenerate Boole polynomials of the second kind to higher order degenerate Boole numbers or polynomials.

Theorem 2.12. For $n \geq 0$, we have

$$\begin{aligned}\widehat{\mathfrak{Bl}}_{n,\lambda}^{(k)}(x|s) &= \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\binom{sk+x}{l}l!\lambda^{m-l}S_1(m,l)\mathfrak{Bl}_{n-m,\lambda}^{(k)}(s)\\ &= \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\binom{x}{l}l!\lambda^{m-l}S_1(m,l)\mathfrak{Bl}_{n-m,\lambda}^{(k)}(k|s).\end{aligned}$$

Proof. From (2.13) the definition of higher order degenerate Boole polynomials of the second kind, we can interpretate following two ways:

$$\begin{aligned}\sum_{n=0}^{\infty}2^k\widehat{\mathfrak{Bl}}_{n,\lambda}^{(k)}(x|s)\frac{t^n}{n!} &= \left(\sum_{m=0}^{\infty}2^k\mathfrak{Bl}_{m,\lambda}^{(k)}(s)\frac{t^m}{m!}\right)\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{sk+x}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{sk+x}{l}l!\lambda^{k-l}S_1(k,l)2^k\mathfrak{Bl}_{n-k,\lambda}^{(k)}(s)\right)\frac{t^n}{n!}.\end{aligned}$$

■

Now we give the relation higher order degenerate Boole polynomials of the second kind to higher order Euler polynomials as follows:

Theorem 2.13. For $m \geq 0$, we have

$$2^k \widehat{\mathfrak{Bl}}_{m,\lambda}^{(k)}(x|s) = \sum_{n=0}^{m} \sum_{l=0}^{n} \lambda^{m-n} S_1(m,n) S_1(n,l) E_l^{(k)}(x+sk).$$

Proof. We observe that

$$\begin{aligned}
&\sum_{m=0}^{\infty} 2^k \mathfrak{Bl}_{m,\lambda}^{(k)}(x|s) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{-sx_1-sx_2-\cdots-sx_k+x} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{-sx_1-sx_2-\cdots-sx_k+x}{n} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\
&\quad \times \lambda^{-n}(\log(1+\lambda t))^n \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-sx_1-sx_2-\cdots-sx_k+x)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\
&\quad \times \lambda^{-n} \sum_{m=n}^{\infty} S_1(m,n) \frac{\lambda^m t^m}{m!} \\
&= \sum_{m=0}^{\infty} \Bigg(\sum_{n=0}^{m} \lambda^{m-n} S_1(m,n) \\
&\quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-sx_1-sx_2-\cdots-sx_k+x)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \Bigg) \frac{t^m}{m!}.
\end{aligned}$$

We observe that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-sx_1-sx_2-\cdots-sx_k+x)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\
&= \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-sx_1-sx_2-\cdots-sx_k+x)^l d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\
&= \sum_{l=0}^{n} (-s)^l S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(x_1+x_2+\cdots+x_k-\frac{x}{s} \right)^l d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\
&= \sum_{l=0}^{n} (-s)^l S_1(n,l) E_l^{(k)}\left(-\frac{x}{s}\right) = \sum_{l=0}^{n} S_1(n,l) E_l^{(k)}(x+sk).
\end{aligned}$$

■

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