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On Minimum Path Cover of a Tree

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Abstract

Let G be a simple graph of order n. The path cover number $\mu(G)$ is defined to be the minimum number of vertex disjoint paths required to cover the vertices of G. I count the minimum number of vertex disjoint paths $\mu(T)$ required to cover the vertices of a tree T that corresponds to a simple assembly graph Γ_w. A relationship between $\mu(T)$ and the assembly number of Γ_w, $\mathrm{An}(\Gamma_w)$, is found and an upper bound for $\mu(T)$ is also proved.

AMS subject classification:
Keywords: Trees, Hamiltonian path, minimum path cover.

1. Introduction

A graph $G = (V, E)$ is a structure consisting of a finite set V of vertices and a finite set E of edges where the two endpoints of an edge in E are vertices in V. The endpoints of every edge is either a pair of vertices or a single vertex. An edge joining a vertex to itself is called a *loop*. If e is an edge and v is an end point of e, then e is said to be incident to v. The number of edges incident to a vertex v is called the *degree* of v.

A *rigid vertex* is a vertex of degree 4 for which a cyclic order of edges is specified. $(v, (e_1, e_2, e_3, e_4)^{cyc})$, e_1 with e_3 and e_2 with e_4 are not neighbors with respect to vertex v.

A finite connected graph where all vertices are rigid vertices and have degree 1 or 4 is called an assembly graph. A vertex of degree 1 is called an *end point*. The number of 4-valent vertices in an assembly graph Γ_w is called the size of Γ_w and is denoted by $|\Gamma_w|$ [3, 6].

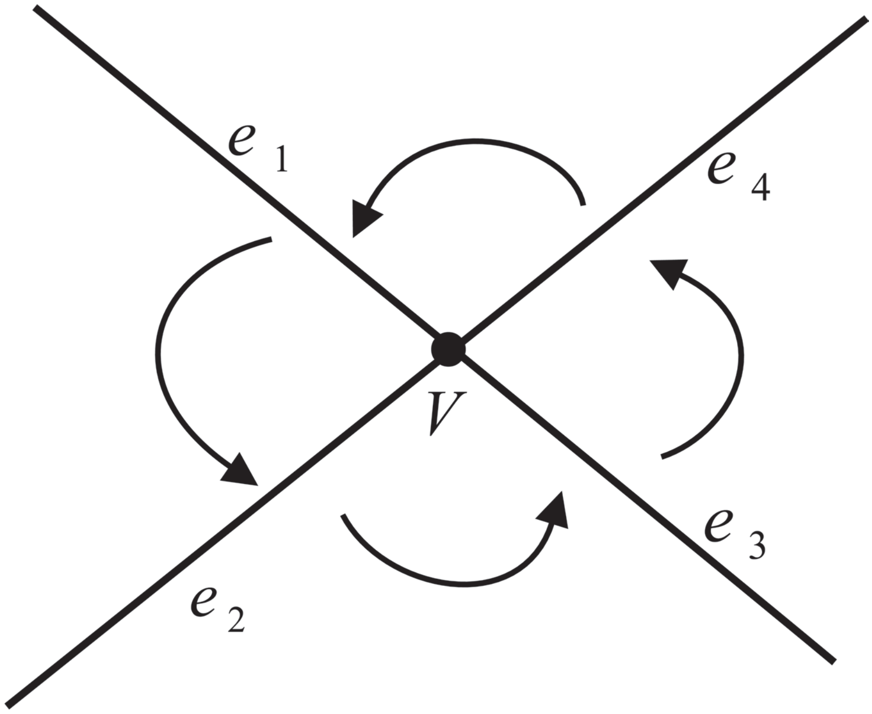

Figure 1: rigid vertex $(V, (e_1, e_2, e_3, e_4)^{cyc})$.

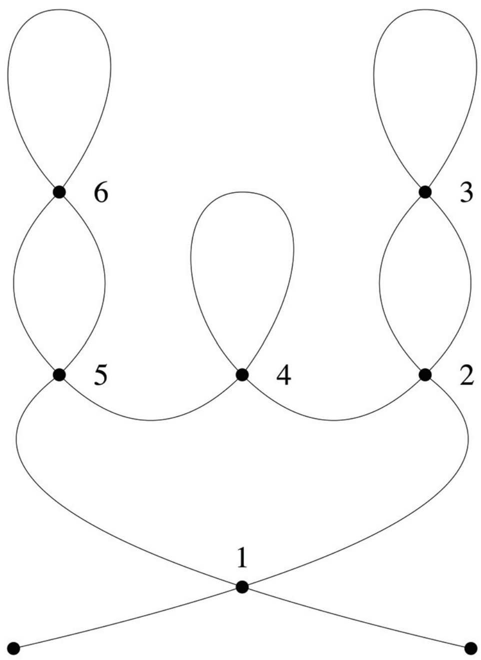

Figure 2: Assembly graph with 6 rigid vertices and 2 end-points.

2. Trees And Hereditary Graphs

Definition 2.1. A *polygonal path* is a path $\gamma = v_1 e_1 v_2 e_2 \cdots v_{m-1} e_{m-1} v_m e_m v_{m+1}$ in which every pair of consecutive edges e_i and e_{i+1} are neighbors with respect to their common incident vertex for each $i \in \{1, 2, \ldots, m-1\}$.

Note: $\gamma = v_1 e_1 v_2 e_2 \cdots v_{m-1} e_{m-1} v_m e_m v_{m+1} = (v_1, v_2, \cdots, v_{m-1}, v_m, v_{m+1})$.

Definition 2.2. A set of pairwise disjoint polygonal paths $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ in a simple assembly graph Γ is called *Hamiltonian* if their union contains all 4-valent vertices of Γ. A polygonal path γ is called *Hamiltonian* if the set $\{\gamma\}$ is Hamiltonian [1, 3].

Example 2.3. In Figure 3, an assembly graph is given with Hamiltonian set of polygonal path $\gamma = \{\gamma_1\}$.

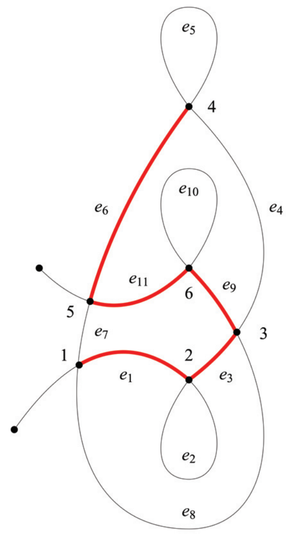

Figure 3: Hamiltonian set of polygonal path $\gamma = \{\gamma_1\}$.
$\gamma_1 = 4e_6 5e_{11} 6e_9 3e_3 2e_1 1 = (4, 5, 6, 3, 2, 1)$.

Definition 2.4. Let Γ be an assembly graph. The *assembly number* of Γ, denoted by $\text{An}(\Gamma)$, is $\text{An}(\Gamma) = \min\{\ k \mid$ there exists a Hamiltonian set of polygonal paths $\{\gamma_1, \ldots, \gamma_k\}$ in $\Gamma\}$.

The Hamiltonian set of polygonal paths that achieve the $\text{An}(\Gamma)$ is called a minimum Hamiltonian set of polygonal paths.

Example 2.5. Consider assembly graphs depicted at Figure 3. The assembly graph has a minimum Hamiltonian set of polygonal path $\{\gamma_1\}$ where $\gamma_1 = 4e_6 5e_{11} 6e_9 3e_3 2e_1 1 = (4, 5, 6, 3, 2, 1)$.

Given a simple assembly graph Γ with two endpoints, choose one of them to be initial (i) and the other to be terminal (t). We call Γ a *directed simple assembly graph* with direction from i to t. We consider the transverse path of a directed simple assembly graph as a path starting at the vertex i and terminating at the vertex t [2, 6].

Definition 2.6. The *composition* $\Gamma_1 \circ \Gamma_2$ of two directed simple assembly graphs Γ_1 and Γ_2 is the directed simple assembly graph obtained by identifying the terminal vertex of Γ_1 with the initial vertex of Γ_2.

Note that the initial vertex of $\Gamma_1 \circ \Gamma_2$ is the initial vertex of Γ_1 and terminal vertex of $\Gamma_1 \circ \Gamma_2$ is the terminal vertex of Γ_2. If the double-occurrence words u and v with disjoint domains correspond to the directed simple assembly graphs Γ_1 and Γ_2, respectively, then the concatenation uv corresponds to the composition $\Gamma_1 \circ \Gamma_2$. In general, $\Gamma_1 \circ \Gamma_2$ is not isomorphic to $\Gamma_2 \circ \Gamma_1$; for example, take Γ_1 represented by pp and Γ_2 by $rrsggs$.

Lemma 2.7. [1] For each pair of directed simple assembly graphs Γ_1 and Γ_2, one of the following equalities hold:

1. $\mathrm{An}(\Gamma_1 \circ \Gamma_2) = \mathrm{An}(\Gamma_1) + \mathrm{An}(\Gamma_2)$ or
2. $\mathrm{An}(\Gamma_1 \circ \Gamma_2) = \mathrm{An}(\Gamma_1) + \mathrm{An}(\Gamma_2) - 1$.

   That is $\mathrm{An}(\Gamma_1) + \mathrm{An}(\Gamma_2) - 1 \leq \mathrm{An}(\Gamma_1 \circ \Gamma_2) \leq \mathrm{An}(\Gamma_1) + \mathrm{An}(\Gamma_2)$.

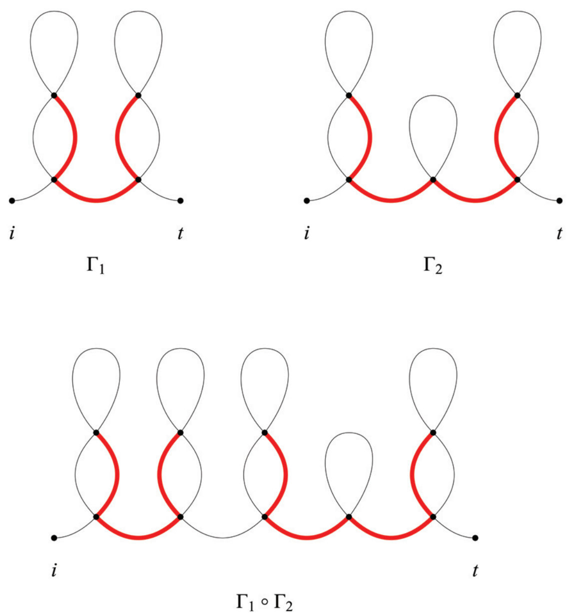

Figure 4: Composition of two simple assembly graphs.

3. Edge-unique simple assembly graphs and hereditary graphs

In this section, we define an edge-unique simple assembly graph Γ_w and a hereditary graph $D\Gamma_w$ that corresponds to an edge-unique simple assembly graph. We also introduce a tree representation of a hereditary graph $D\Gamma_w$ and count the minimum number of Hamiltonian set of paths for a hereditary graph $D\Gamma_w$ to find an assembly number of an edge-unique simple assembly graph Γ_w.

Definition 3.1. Let Γ_w be a simple assembly graph with two end points that corresponds to an assembly word $w = a_1a_2a_3 \cdots a_{2n}$ with $|\Gamma_w| = n$. We denote by $D\Gamma_w$ the graph with vertices $V(\Gamma_w)$ and a sequence of arcs $e_i = (a_i, a_{i+1})$ in the order of symbols in w such that $a_i \neq a_{i+1}$. The arc set is defined by $E(D\Gamma_w) = \{(a, b) \mid \text{distinct adjacent letters } a, b \in w\}$.

For instance, for the simple assembly graph Γ_w that corresponds to the double-occurrence word [4] $w = 122134435665$ depicted in Figure 5(a), the graph $D\Gamma_w$ is depicted in Figure 5(b).

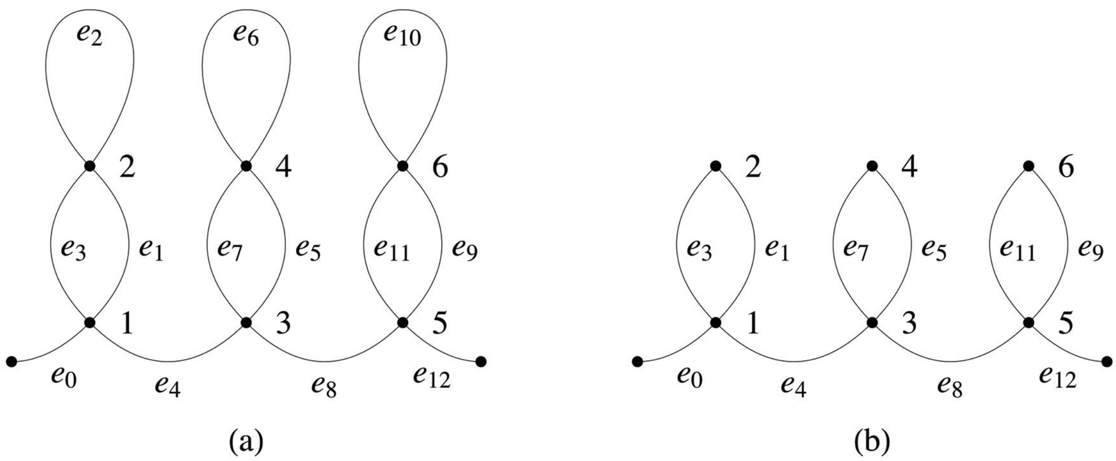

Figure 5: (a) Simple assembly graph Γ_w that corresponds to a double-occurrence word $w = 122134435665$ and (b), the hereditary graph $D\Gamma_w$ that corresponds to Γ_w.

Definition 3.2. We call two edges of a simple assembly graph *parallel* if their end vertices are the same.

Definition 3.3. A simple assembly graph Γ_w is called *edge-unique* if every pair of adjacent vertices are either a part of a unique cycle of size 2 or an edge.

Definition 3.4. We call a graph $D\Gamma_w$ *hereditary* if it corresponds to an edge-unique simple assembly graph Γ_w.

Remark 3.5. If a pair C_1 and C_2 of cycles in $D\Gamma_w$ intersect at two vertices x and y then the induced subgraph $(V(C_1) \cup V(C_2))$ contains two disjoint paths between x and y, and if $D\Gamma_w$ is hereditary graph, or Γ_w is edge-unique, then each pair C_1, C_2 of cycles in $D\Gamma_w$, or Γ_w are such that $(V(C_1) \cap V(C_2))$ is at most two vertices.

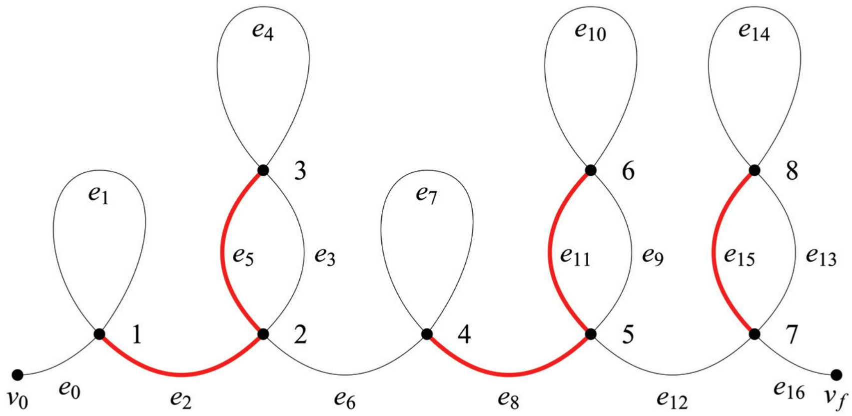

Figure 6: A simple assembly graph Γ_w with $w = 1123324456657887$ which corresponds to $D\Gamma_w$ depicted in Figure 7 (a).

Remark 3.6. The set of cycles $S = \{S_1, S_2, \dots, S_t\}$ of a hereditary graph $D\Gamma_w$ that corresponds to an edge-unique assembly graph Γ_w is the set of cycles of size 2. A hereditary graph $D\Gamma_w$ is acyclic if it does not contain a cycle. We denote by $D\Gamma_w \setminus S$ a subgraph of $D\Gamma_w$ obtained by removing only a single arc of each cycle $S_i \in S$. The subgraph $D\Gamma_w \setminus S$ is a tree, see Figure 7(b).

Example 3.7. Consider a double-occurrence word $w = 1123324456657887$. An edge-unique simple assembly graph Γ_w and the corresponding hereditary graph $D\Gamma_w$ are depicted in Figure 6 and Figure 7(a), respectively. Consider the Hamiltonian set of polygonal paths $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ for Γ_w with $\gamma_1 = 1e_2 2e_5 3$, $\gamma_2 = 4e_8 5e_{11} 6$, $\gamma_3 = 7e_{15} 8$, and the Hamiltonian set of paths $\beta = \{\beta_1, \beta_2, \beta_3\}$ for hereditary graph $D\Gamma_w$ from a tree $D\Gamma_w \setminus S$ with $\beta_1 = 1e_2 2e_3 3$, $\beta_2 = 4e_8 5e_9 6$, and $\beta_3 = 7e_{13} 8$. Since there is no Hamiltonian set of polygonal paths with cardinality 1 or 2 , γ is the minimum Hamiltonian set of polygonal paths for Γ_w, and β is the minimum Hamiltonian set of paths for $D\Gamma_w$ for there is no Hamiltonian set of paths with cardinality less than the cardinality of β. The cardinality of γ equals the cardinality of β.

If S_1 is a cycle for a hereditary graph $D\Gamma_w$ with arcs $e = (u, v)$ and $e' = (v, u)$, then we say e is parallel to e'. We denote by $\mathrm{Hn}(\mathrm{D\Gamma_w} \setminus \mathrm{S})$ the minimum cardinality of a Hamiltonian set of paths for $D\Gamma_w$.

Theorem 3.8. For an edge-unique simple assembly graph Γ_w, $\mathrm{An}(\Gamma_w) = \mathrm{Hn}(\mathrm{D\Gamma_w} \setminus \mathrm{S})$.

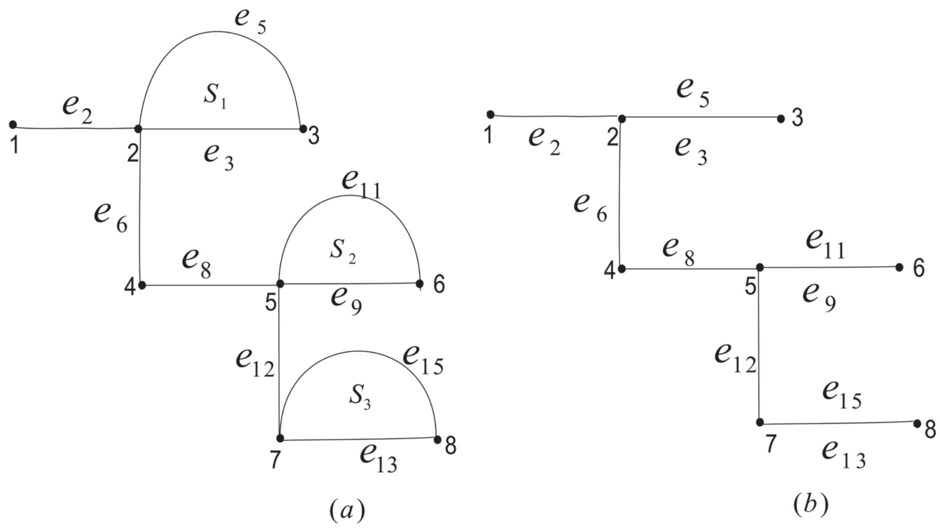

Figure 7: (a) Set of cycles $S = \{S_1, S_2, S_3\}$ for hereditary graph $D\Gamma_w$ corresponding to a simple assembly graph Γ_w depicted in Figure 6 and (b), a tree $D\Gamma_w \setminus S$.

Proof. Let Γ_w be an edge-unique simple assembly graph. Suppose $\mathrm{An}(\Gamma_w) = k$ obtained by a minimum Hamiltonian set of polygonal paths $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$, and $\beta = \{\beta_1, \beta_2, \ldots, \beta_s\}$ is a minimum Hamiltonian set of paths for $D\Gamma_w$. We show that $k = s$.

Let S be the set of cycles in $D\Gamma_w$. From the definition of hereditary graph $D\Gamma_w$, if two vertices are adjacent then they are either on a cycle of size 2 or it is an edge that does not belong to a cycle. For every pair of vertices $u, v \in V(\Gamma_w)$ either $e = (u, v)$ is a unique edge between u and v or $x = (u, v)$ and $e' = (v, u)$ are parallel edges. In the former case $e \in D\Gamma_w \setminus S$ and in the later case edge e or edge e' is in $D\Gamma_w \setminus S$. Let $\gamma_1 = a_1e_1a_2e_2 \ldots a_{t-1}e_{t-1}a_t$. Then a_i and a_{i+1} are vertices of two loops, or a_i is a vertex of a loop and a_{i+1} is a vertex of a 2 cycle, or a_i and a_{i+1} are parallel edges. In the first two cases edge $e_i = a_ia_{i+1}$ is a unique edge in Γ_w and it is also in $D\Gamma_w \setminus S$. In the latter case edges $e_i = a_ia_{i+1}$ and $e_i' = a_{i+1}a_i$ are edges of a 2 cycle in $D\Gamma_w$. This implies for each edge in a polygonal path γ there is a corresponding edge in $D\Gamma_w$. Hence every Hamiltonian set of polygonal paths in Γ_w is also Hamiltonian set of paths in $D\Gamma_w \setminus S$. This implies $k \leq s$. (*)

For each $1 \leq i \leq s$, set $\beta_i = a_1^ie_1^ia_2^ie_2^i \ldots a_{t_i}^ie_{t_i}^ia_{t_i+1}^i$ with $l(\beta_i) = t_i + 1$. Without loss of generality let $\beta = A \cup B$, $A \cap B = \emptyset$ where $A = \{\beta_1, \beta_2, \ldots, \beta_m\}$ and $B = \{\beta_{m+1}, \beta_{m+2}, \ldots, \beta_s\}$ such that $\beta_i \in A$ has two end-points and $\beta_j \in B$ for all j is just a vertex. Clearly the $\beta_j's$ are polygonal paths, see Figure 8 and Figure 9.

Consider $\beta_l \in A$. There is a set $S = \{S_1, S_2, \ldots, S_r\}$ of cycles for hereditary graph $D\Gamma_w$ with $S_p = (a_p, e_p, a_{p+1}, e_p')$ for each $1 \leq p \leq r$ such that either edge $x = a_pe_pa_{p+1}$, or edge $y = a_pe_p'a_{p+1}$ is in a Hamiltonian path β_l, see Figure 7. The edges x and y are parallel. This implies there is a corresponding polygonal path in

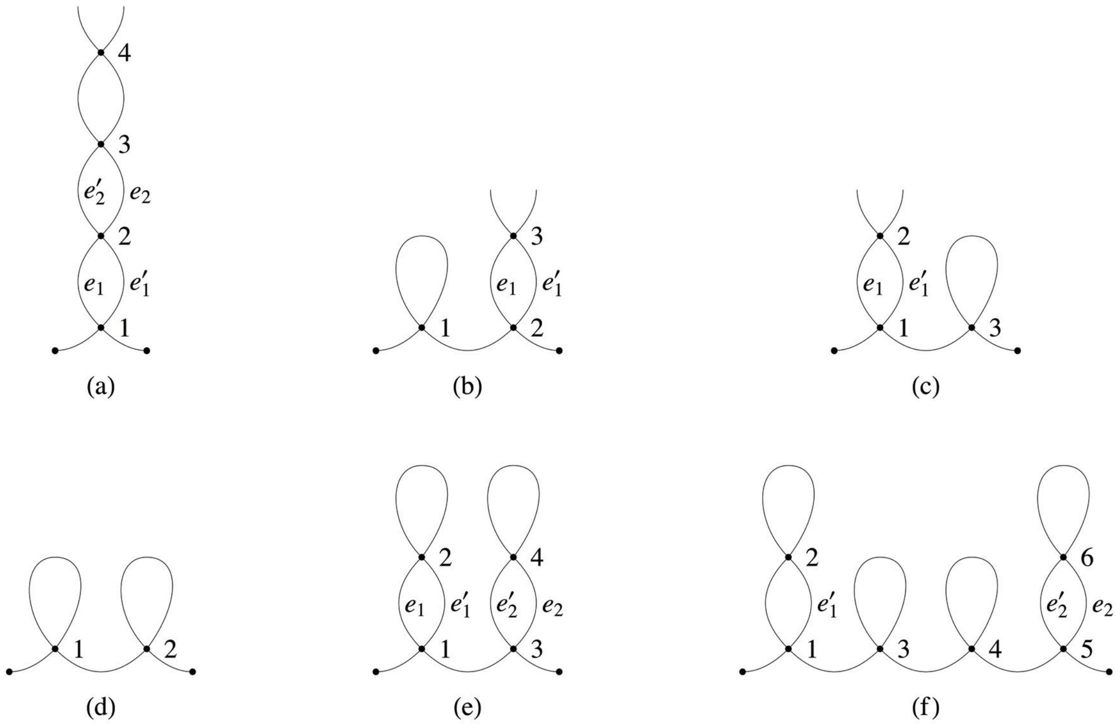

Figure 8: Subgraphs of edge-unique simple assembly graphs.

Γ_w such that if z and x are consecutive edges in β_l but not neighbors in Γ_w then the corresponding edges will be z and y which are neighboring edges in Γ_w. So for each pair of adjacent edges but non-neighboring in the Hamiltonian path β_l, we can choose a corresponding neighboring edge in S_p so that β_l is a polygonal path. Consequently A is a set of polygonal paths.This implies $s \leq k$. (**)

From (*) and (**) we have $k = s$. ■

Definition 3.9. Two 3-valent vertices v and u are called 2-*nhbd* if there is no a 3-valent vertex between v and u but at least one 2-valent vertex t exists so that $\beta = (v, t, u)$ is a path. If neither a 3-valent nor a 2-valent vertex exists between v and u but $\chi = (v, u)$ is a path then v and u are called 0-*nhbd* or *adjacent*.

Example 3.10. From Figure 6, vertices labeled 2 and 5 are 2-*nhbd* vertices because a vertex labelled 4 is a 2-valent vertex and $\beta = (2, 4, 5)$ is a path. Three-valent vertices labeled 2 and 7 are not 2-*nhbd* and vertices labelled 5 and 7 are adjacent but not 2-*nhbd*.

Notation: $T = D\Gamma_w \setminus S$ for the rest of the sections represents a tree defined above.

Definition 3.11. A *zero vertex* p between two vertices $u, v \in V(T)$ is a point on the edge $e = uv \in E(T)$ and $p \notin V(T)$.

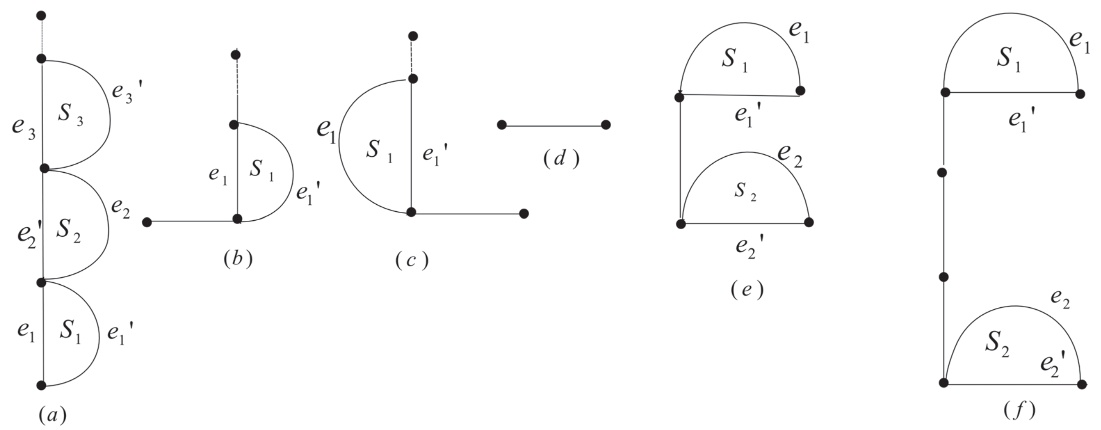

Figure 9: Hereditary graphs $D\Gamma_w$ correspond to Figures depicted in 8.

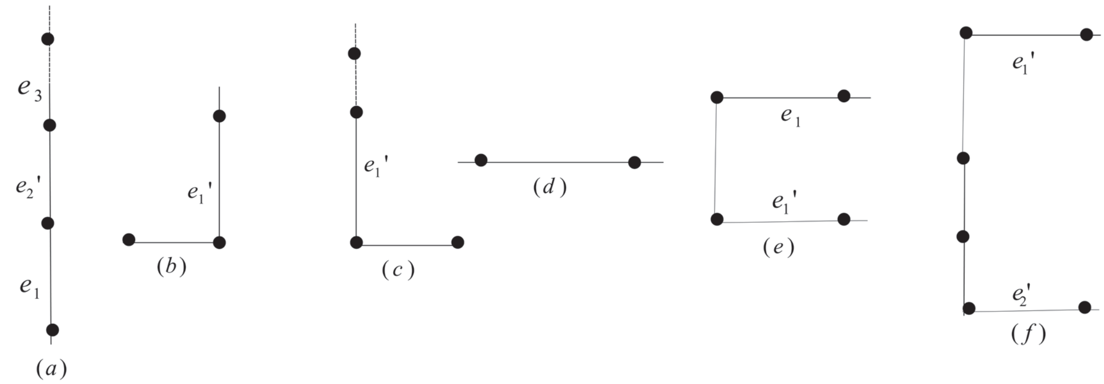

Figure 10: Subgraphs $D\Gamma_w \setminus S$ for hereditary graphs $D\Gamma_w$ depicted in Figure 9.

Definition 3.12. Composition of Trees Let T be a tree and v_1 and $v_2 \in V(T)$. Let p be a zero vertex on $e = v_1v_2 \in E(T)$ and T_1, T_2 are subgraphs of T such that $v_1 \in T_1$, $v_2 \in T_2$. We define the composition of T_1 and T_2 by $T_3 = T_1 \circ_p T_2$.

Note: $|T_3| = |T_1| + |T_2|$.

Definition 3.13. Let $S_n = \{w : w = a_1a_2\cdots a_{2n}, n \geq 1, u = a_ia_i$ is not a subword of w$\}$. We call S_n the set of *loop free* assembly words.

Theorem 3.14. The number of loop free assembly words $|S_n|$ is

$$|S_n| = (2n-1)!! - \sum_{k=1}^{n}(-1)^k \binom{n}{k} 2^{k-n}(2n-k)!.$$

Proof. The set of assembly words on n letters is $(2n-1)!!$. Let $A_i = \{\ w_i \mid w_i$ is a double-occurrence word formed with the two letters x_i are adjacent$\}$.

Thus, the desired number

$$|S_n| = (2n-1)!! - |A_1 \cup A_2 \cup \cdots \cup A_n| \tag{(*)}$$

We apply the Principle of Inclusion and Exclusion to find the cardinality of $|A_1 \cup A_2 \cup \cdots \cup A_n|$.

Suppose a word $w \in (A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})$. Then $w \in A_{i_j}$ for each $j \in \{1, 2, \ldots, k\}$ and thus the letters $x_{i_1}x_{i_1}; x_{i_2}x_{i_2}; \cdots; x_{i_k}x_{i_k}$ appearing as a subword. The words for which these k pairs of letters are adjacent are obtained in the following manner:

Form all the words having $(2n-k)$ letters taken from an alphabet obtained from alphabet Σ by suppressing one copy of each letter from $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$. Then in each word thus formed, repeat the letters $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ by adding the letters x_{ij} immediately after itself for $j \in \{1, 2, \ldots, k\}$. It follows that

$$\begin{aligned} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| &= \frac{(2n-k)!}{(2!)^{(n-k)}} \\ &= \frac{2^k(2n-k)!}{2^n} \end{aligned} \tag{(**)}$$

Since the indices $i_1, i_2, \ldots, i_k$ satisfy $1 \leq i_1 < \cdots < i_k \leq n$, they can be selected in $\binom{n}{k}$ ways and hence the total number of loop free words can be written using $(*)$ and $(**)$,

$$\begin{aligned} |S_n| &= (2n-1)!! - \binom{n}{1}\frac{2(2n-1)!}{2^n} + \binom{n}{2}\frac{2^2(2n-2)!}{2^n} - \cdots + \frac{(-1)^n 2^n n!}{2^n} \\ &= (2n-1)!! - \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k 2^k (2n-k)!}{2^n}. \end{aligned}$$

■

4. Composition of paths and path cover of a graph

Let $D\Gamma_w$ be a hereditary graph that corresponds to a simple assembly graph Γ_w. If v is any vertex of the tree $T = D\Gamma_w \setminus S$ then $1 \leq deg(v) \leq 3$. Call a pair of vertices $\{a, r\} \subseteq V(T)$ or $\{r, s\} \subseteq V(T)$ with $deg(a) = deg(r) = deg(s) = 1$ *adjacent* if there are at most two 2-*nhbd* vertices $p, v \in V(T)$ between a and r or between r and s such that either $\beta = (a, d_i, \ldots, d_{i+t}, p, d_{i+t+2}, \ldots, d_{i+t+m}, r)$ or $\chi = (r, d_{i+t+m}, \ldots, d_{i+t+2}, p, d_j, \ldots d_{j+m}, v, d_{j+m+2},$
$\ldots, d_{j+m+f}, s)$ is a path in T for some $d_i, \ldots, d_{i+t+m}, \ldots, d_{i+m+f} \in V(T), \{f, i, j, m, t\} \subseteq \mathbb{N}$ and $deg(d_i) = \cdots = deg(d_{i+t+m}) = \cdots = deg(j+m+f) = 2$.

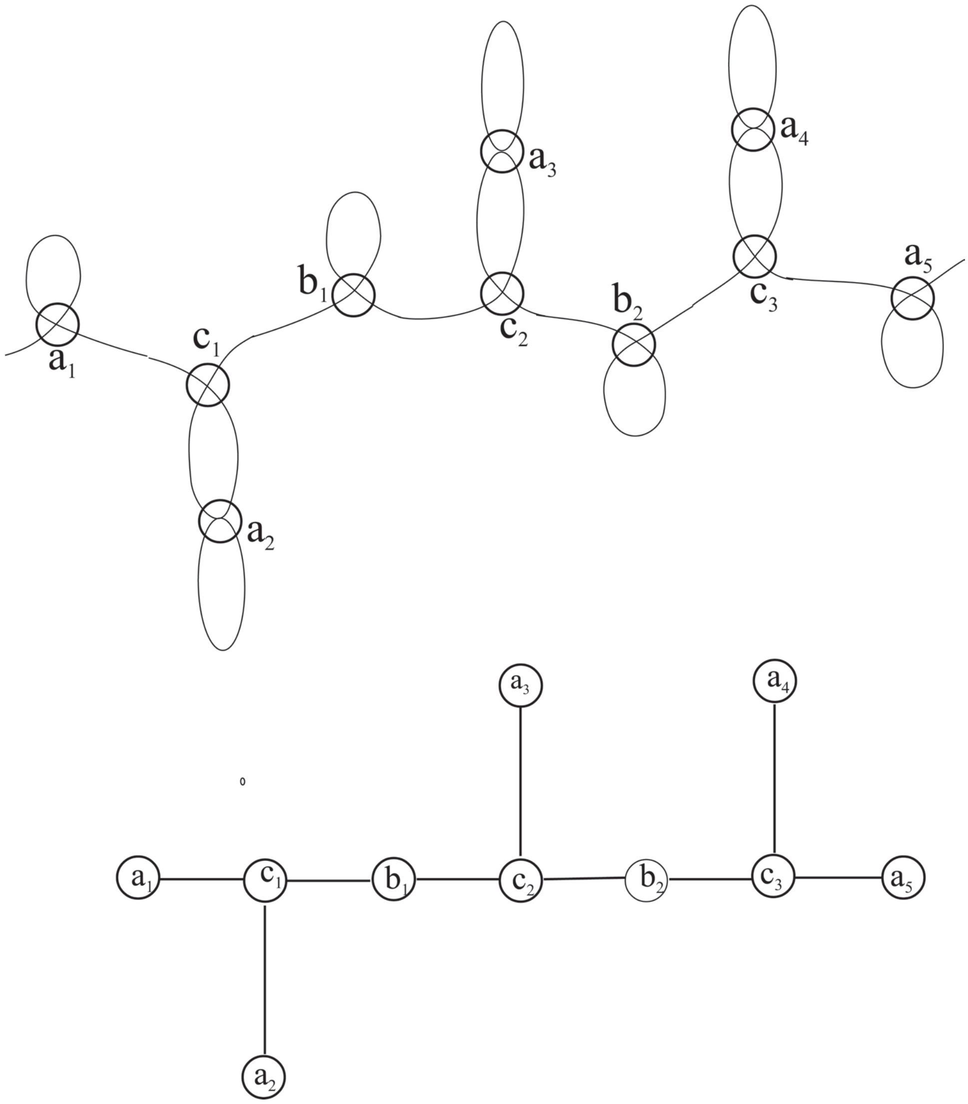

Figure 11: Loop saturated graph(top) and its tree representation (bottom).

Example 4.1. See Figure 12.

Let G be a graph. A path ϕ in G is an ordered sequence $\phi = (a_1, a_2 \ldots, a_n)$ of distinct points, where if $n \geq 2$, a_i is adjacent to a_{i+1} for $1 \leq i \leq n-1$. A path $\phi = (a_1, a_2, \ldots, a_n)$ is the same path as $\beta = (a_n, a_{n-1}, \ldots, a_1)$. If ϕ and β are paths, by $\phi \circ \beta$ we shall mean that one end-point, a of ϕ, is adjacent to one end-point, b of β, and that $\phi \circ \beta$ is formed by joining a to b. More specifically we may write $\phi a \circ b\beta$ to specify which end-point of ϕ is joined to which end-point of β. Also $(a_1, a_2, \ldots, a_n) \circ (b_1, b_2, \ldots, b_n)$ = $(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$ where a_n must be adjacent to b_1 [5, 8, 9].

A *path-cover* of G is a collection Φ of vertex-disjoint induced paths such that every

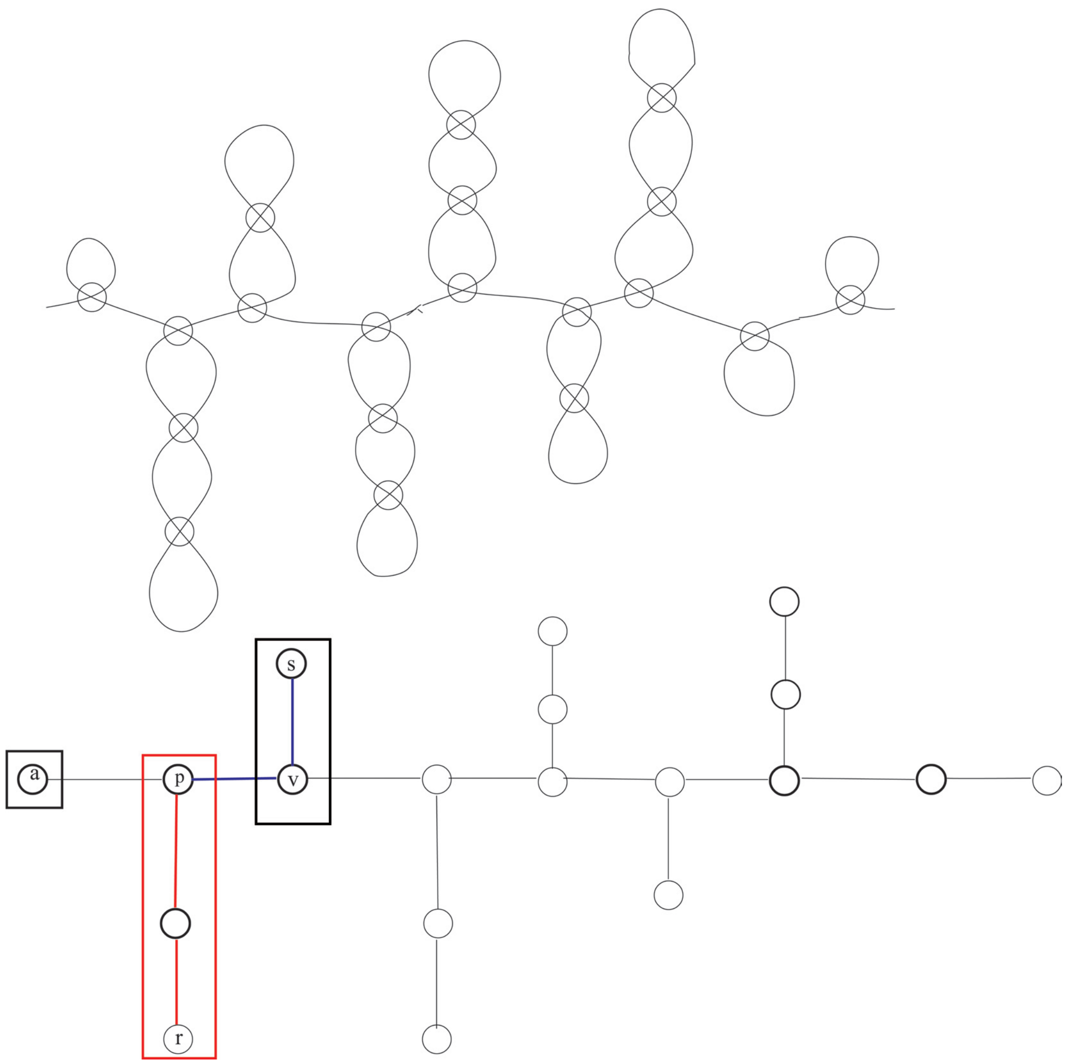

Figure 12: Simple assembly graph (Top) and its tree representation (bottom).

vertex of G lies on some path in Φ. The *path-covering number*, denoted by $\mu(G)$, of G, is defined by:

$$\mu(G) = \min\{|\Phi| : \Phi \text{ is a } path - cover \text{ of } G\}$$

Example 4.2. Consider the tree T, Figure 13 (top), and set $A = \{\phi_1, \phi_2, \phi_3, \phi_4\}$ where $\phi_1 = (a_1, c_1, a_2)$, $\phi_2 = (b_2)$, $\phi_3 = (a_3, c_2, b_2, c_3, a_4)$, $\phi_4 = (a_5)$. Set A is a collection of vertex disjoint paths that contains every vertex of T; moreover, it is a minimum path-cover of T as there is no a set of at most three paths that cover all vertices of T. Hence $\mu(T) = 4$.

A minimum path-cover need not be unique as $B = \{\phi_1, \phi_2, \phi_3, \phi_4\}$ where $\phi_1 = (a_1, c_1, b_1, c_2, a_3)$, $\phi_2 = (a_2)$, $\phi_3 = (b_2)$, $\phi_4 = (a_4, c_3, a_5)$ is also a minimum *path-cover* of T, Figure 13 ($bottom$).

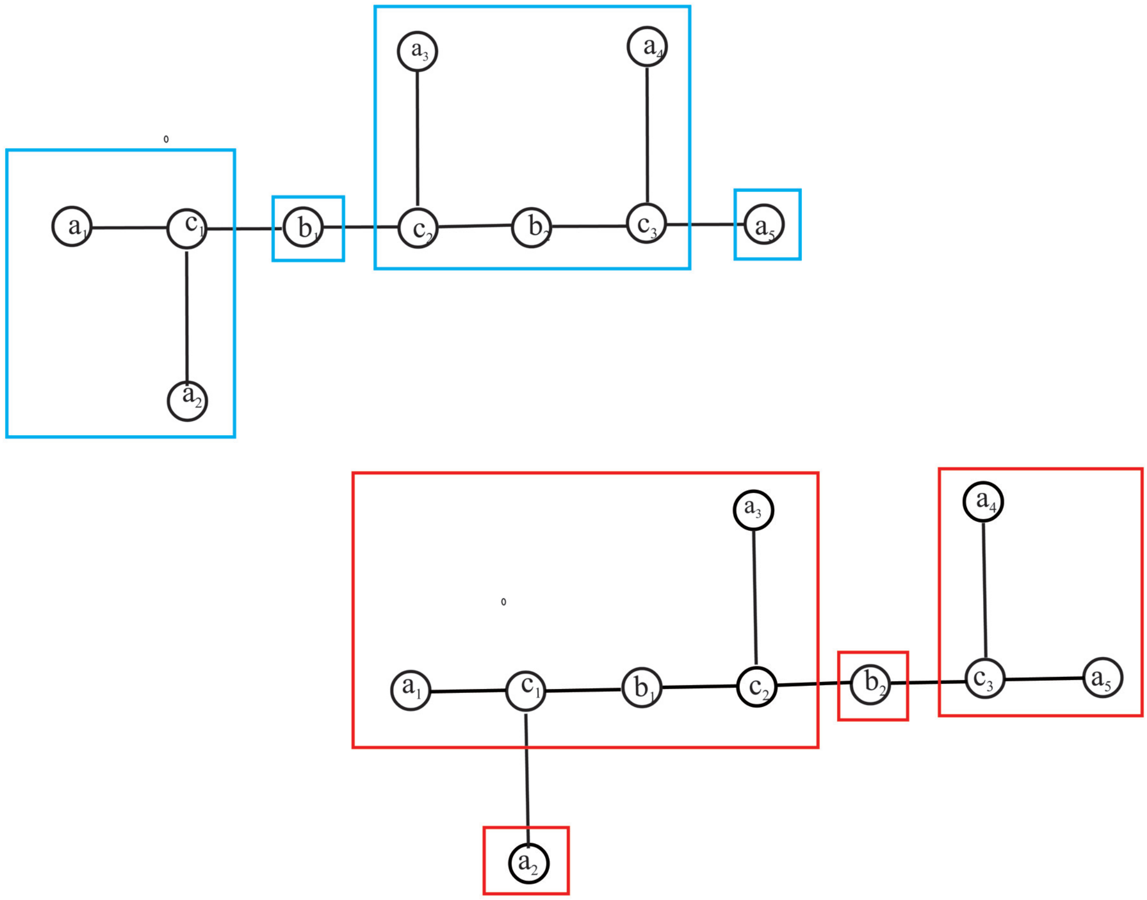

Figure 13: $D\Gamma_w \setminus S$ corresponds to a loop saturated assembly graph Γ_w depicted at Figure 12 and set of path covers.

Example 4.3. Consider $\beta_1 = (a_2, c_1, b_1, c_2)$ and $\beta_2 = (b_2, c_3, a_4)$. Then $\beta_1 \circ \beta_2 = \beta_1 c_2 \circ b_2 \beta_2 = \beta_3 = (a_2, c_1, b_1, c_2, b_2, c_3, a_4)$.

Theorem 4.4. Let T be a tree such that $T = T_1 \circ_p T_2$. Then

1. $\mu(T) = \mu(T_1) + \mu(T_2)$ or
2. $\mu(T) = \mu(T_1) + \mu(T_2) - 1$, i.e., $\mu(T_1) + \mu(T_2) - 1 \leq \mu(T) \leq \mu(T_1) + \mu(T_2)$.

Proof. The proof follows from Theorem 2.7 and Theorem 3.8. ■

Lemma 4.5. Let T be a tree that corresponds to an edge-unique simple assembly graph Γ_w where $|A_2| = 0$. Then $|A_1| = |A_3| - 2$.

Proof. See the partition $T = A_1 \cup A_2 \cup A_3$ below. Let $|T| = n$. Then $n = |A_1| + |A_3|$.(*)

Because $\sum_{i=1}^{n} d_i = 2n-2$ where $(d_1, d_2, \ldots, d_n)$ is a degree sequence of a tree T,

$$\sum_{v\in A_1} deg(v) + \sum_{u\in A_3} deg(u) = 2n-2. \qquad (**)$$

This implies $3|A_1| + |A_3| = 2n-2$. From (*) and (**), $|A_1| = \dfrac{n-2}{2}$ and $|A_3| = \dfrac{n+2}{2}$. Hence, $|A_1| = |A_3| - 2$. ■

Theorem 4.6. Let T be a tree of order $n \geq 2$. Then

$$\left\lfloor \frac{m+1}{2} \right\rfloor \leq \mu(T) \leq (m-1)$$

where m is the number of pedant vertices in T.

Proof. Let $T = (V, E)$ and partition $V = A_1 \cup\ A_2 \cup A_3$ where

$$A_1 = \{v \in V : deg(v) = 3\}$$

$$A_2 = \{u \in V : deg(u) = 2\}$$

$$A_3 = \{w \in V : deg(w) = 1\}$$

Set $A_1 = B_1 \cup B_2$ with $B_1 = \{(v_1, v_2) : v_1 \text{ and } v_2 \text{ are 2-nhbd vertices}\}$ and $B_2 = A_1 - B_1$. If $A_1 = \emptyset$ then T is a path with two pedants,i.e., $m = 2$ and $\mu(T) = m - 1 = 1$. Suppose $A_1 \neq \emptyset$. $(\exists v) \in A_1$. Pick this $v \in A_1$. This implies adjacent vertices $r, s \in A_3$ exist such that either $\beta = \beta_1 \circ \beta_2 = \beta_1 k \circ v\beta_2$ where $\beta_1 = (r, \ldots, k)$ and $\beta_2 = (v, \ldots, s)$ or $\chi = \chi_1 \circ \chi_2 = \chi_1 u \circ p\chi_2$ where $\chi_1 = (r, \ldots, v, \ldots, u)$ and $\chi_2 = (p, \ldots, s)$ are paths in T for some $k, u \in A_2$ and $p \in A_1$. See Figure 12. Consequently, for each $v \in A_1$ we can choose a unique $t \in A_3$ such that $\eta = (t, \ldots, v)$ is a path in T where $t \in \{r, s\} \subseteq A_3$ and $|\eta| \geq 2$. $(\star)$

Let $\Phi = \{F_i : F_i = \{\phi_{i1}, \phi_{i2}, \ldots, \phi_{it_i}\}, i, t_i \in \mathbb{N}\}$ where F_i is a path cover of T.

1. Suppose $A_2 = \emptyset$. Then $|T| = |A_1| + |A_3|$ because $|A_2| = 0$, and $(|A_1|, |A_3|) = (m-2, m)$, Lemma 4.5 and $(\star)$. This implies each $\phi_{ij} \in F_i \in \Phi$ for each i and $1 \leq j \leq t_i$ has size $1 \leq |\phi_{ij}| \leq m$ and

$$\min\{|F_i| : F_i \in \Phi\} = \mu(T) \geq \left\{ \begin{array}{ll} \frac{m}{2}, & if\ m\ is\ even \\ \frac{m+1}{2}, & if\ m\ is\ odd \end{array} \right|$$

$$\Rightarrow \mu(T) \geq \left\lfloor \frac{m+1}{2} \right\rfloor \text{ for any } m.$$

Because there is no a path ϕ of T that includes at least three pedant vertices, $|F_i|$ is minimum if each $\phi_{ij} \in F_i \in \Phi$ but at most one contains two adjacent vertices $a, b \in A_3$, and if any two pair of vertices $v_1, v_2 \in A_1$ are 2-*nhbd* that is $|A_1| = |B_1| + 1$ then $\mu(T)$ is maximum .

2. Suppose $A_2 \neq \emptyset$. **First** count the number of all possible disjoint paths ϕ_{ij} each contains two adjacent pedants and set F_i, **second** because $|A_2| \neq 0$, count the number of disjoint paths that contains the set of vertices $u \in A_2$ which are not covered by the former set of paths F_i for some i.

$$\Rightarrow min\{|F_i| : F_i \in \Phi\} = \mu(T) \leq \left\{ \begin{array}{ll} \overbrace{\frac{m}{2}}^{\text{First}} + \overbrace{\frac{m}{2} - 1}^{\text{Second}}, & if\ m\ is\ even \\ \overbrace{\frac{m-1}{2}}^{\text{First}} + \overbrace{\frac{m-1}{2}}^{\text{Second}}, & if\ m\ is\ odd \end{array} \right|$$

$$\Rightarrow \mu(T) \leq (m-1)\ for\ any\ m.$$

From 1 and 2,

$$\left\lfloor \frac{m+1}{2} \right\rfloor \leq \mu(T) \leq (m-1).$$

■

Lemma 4.7. If $T = D\Gamma_w \setminus S$ correspondence to a loop saturated graph Γ_w then $\mu(T)$ attains an upper bound.

Proof. Suppose T correspondence to a loop saturated graph Γ_w. Then each pair of vertices $v_i, v_{i+1} \in A_1$ labelled in the order are 2-*nhbd* vertices. Extending Lemma 4.5 to Γ_w results the same relation $|A_3| = |A_1| + 2$ and $|A_2| = |A_1| - 1$. See also Figure 11.

$$\left\{ \begin{array}{ll} 3\text{An}(\Gamma_w) - 2 = |\Gamma_w|, & from\ [6,\ 7]\ (\star) \\ \mu(T) = \text{An}(\Gamma_w), & Theorem\ 3.8\ (\star\star) \end{array} \right|$$

From $|T| = |\Gamma_w|$, $(\star)$ and $(\star\star)$, $|A_3| = |A_1| + 2$, $|A_2| = |A_1| - 1$ and $|A_3| = m$, we have $3\mu(T) - 2 = |T| = |A_1| + |A_2| + |A_3| = (m-2) + (m-3) + m = 3m - 5$. Hence, $\mu(T) = m - 1$. ■

5. Conclusion

Regarding future work, the goal is of course to extend my work to find digraph representation of any simple assembly graph and find relationship between minimum path cover of a digraph and minimum Hamiltonian set of polygonal paths. It would also be of interest to consider a corresponding relationship with permutation graphs.

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