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The R-automorphism of the ring $R[\{\alpha_i\}_{i=1}^n]$

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Abstract

Let R be a commutative ring with identity. This article is devoted to the study of the R-automorphisms of the ring $R[\{\alpha_i\}_{i=1}^n]$, where we provide necessary and sufficient conditions for any R-endomorphism φ to be one-to-one and onto.

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1. Introduction

Throughout this article, all rings are commutative with identity. Given a ring R, the set consisting of all the elements of the form $a_0 + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$, where $a_i \in R$ and $\alpha_i \notin R$ with $\alpha_i\alpha_j = 0$ for all $1 \leq i \leq n$, is denoted by $R[\{\alpha_i\}_{i=1}^n]$. Two elements $(a_0 + a_1\alpha_1 + \cdots + a_n\alpha_n)$ and $(b_0 + b_1\alpha_1 + \cdots + b_n\alpha_n)$ of $R[\{\alpha_i\}_{i=1}^n]$ are equal if and only if $a_i = b_i$ for $i = 0, 1, 2, \ldots, n$. Addition and multiplication of elements of $R[\{\alpha_i\}_{i=1}^n]$ are defined by $(a_0 + a_1\alpha_1 + \cdots + a_n\alpha_n) + (b_0 + b_1\alpha_1 + \cdots + b_n\alpha_n) = (a_0 + b_0) + (a_1 + b_1)\alpha_1 + \cdots + (a_n + b_n)\alpha_n$ and $(a_0 + a_1\alpha_1 + \cdots + a_n\alpha_n)(b_0 + b_1\alpha_1 + \cdots + b_n\alpha_n) = a_0b_0 + (a_0b_1 + a_1b_0)\alpha_1 + \cdots + (a_0b_n + a_nb_0)\alpha_n$. Then the set $R[\{\alpha_i\}_{i=1}^n]$ forms a commutative ring under operations of addition and multiplication given above. Note that, this ring is a generalization for the ring $R[\alpha]$ which can be described using Nagata's principle of idealization as the ring $R(+)R = R \oplus R$ (direct sum), with product $(r_0, r_1)(t_0, t_1) = (r_0t_0, r_0t_1 + r_1t_0)$, see [7].

The purpose of this article is to describe the R-automorphisms of $R[\{\alpha_i\}_{i=1}^n]$. Gilmer in [5] aroused the idea, where he studied and characterized the R automorphisms of the polynomial ring $R[x]$. In other words, these automorphisms of $R[X]$ that fix elements of R . The area has been under active investigation since the 1960s. The aim of the present article is to investigate the analogous problem in the ring $R[\{\alpha_i\}_{i=1}^n]$. in a similar manner, if φ is an R-endomorphism (the endomorphism of $R[\{\alpha_i\}_{i=1}^n]$ that fixes elements of R), then we try to find necessary and sufficient conditions in order that φ be one-to-one and/or onto. Further work on this group of R-automorphisms was carried by others, see [2], [6], [4] and [3]. In this article, R-automorphisms of the ring $R[\{\alpha_i\}_{i=1}^n]$ are introduced and fully characterized.

For undefined notions and terminology, the reader is referred to [8] and [1].

2. The R-automorphism of the ring $R[\{\alpha_i\}_{i=1}^n]$

Definition 2.1. An endomorphism φ of $R[\{\alpha_i\}_{i=1}^n]$ is called an R-endomorphism if for any $r \in R$, $\varphi(r) = r$. It is clear that an R-endomorphism φ of $R[\{\alpha_i\}_{i=1}^n]$ is completely determined by $\varphi(\alpha_i)$ for $1 \leq i \leq n$. That is, if $\varphi(\alpha_i) = v_i$ for each $1 \leq i \leq n$, then $\varphi(\sum_{i=1}^n a_i\alpha_i) = \sum_{i=1}^n a_i v_i$ for any $\sum_{i=1}^n a_i\alpha_i \in R[\{\alpha_i\}_{i=1}^n]$; Furthermore, if $v_i = \sum_{k=0}^n v_{ik}\alpha_k \in R[\{\alpha_i\}_{i=1}^n]$, then the mapping $\varphi_V : R[\{\alpha_i\}_{i=1}^n] \longrightarrow R[\{\alpha_i\}_{i=1}^n]$ defined by $\varphi_V(\sum_{i=1}^n a_i\alpha_i) = \sum_{i=1}^n a_i v_i$, where φ_V sends α_i onto v_i, is an R-endomorphism.

The question that arises here is what the R-automorphisms of the ring $R[\{\alpha_i\}_{i=1}^n]$ are. Accordingly, it is enough to find sufficient and efficient conditions on $v_1, v_2, \ldots, v_n$ in order that φ_V should be one to one and onto.

Lemma 2.2. Let R be a ring and let φ_V be an R-endomorphism of the ring $R[\{\alpha_i\}_{i=1}^n]$. If φ_V is an R-automorphism, then $\varphi_V(\alpha_i) = v_i = \sum_{k=0}^n v_{ik}\alpha_k$, for $1 \leq i \leq n$, such that

1. $2v_{i0} = 0$, for all $1 \leq i \leq n$,
2. v_{i0} is nilpotent with nilpotency 2, and
3. The $n \times n$ matrix $V = \left[v_{i,j}\right]$, (with $1 \leq i, j \leq n$) is invertible in the ring $M_n(R)$.

Proof. Suppose that φ_V is an R-automorphism. Now, since $\alpha_i\alpha_j = 0$ for any $i, j \in \{1, 2, \ldots, n\}$, then we have $\varphi_V(\alpha_i)\varphi_V(\alpha_j) = 0$. This leads to the following system of equations

$$
\begin{aligned}
v_{i0}v_{j0} &= 0 && (0^*)\\
v_{i0}v_{j1} + v_{i1}v_{j0} &= 0 && (1^*)\\
v_{i0}v_{j2} + v_{i2}v_{j0} &= 0 && (2^*)\\
\vdots \quad \vdots \quad &\vdots = \vdots && (\vdots)\\
v_{i0}v_{jn} + v_{in}v_{j0} &= 0 && (n^*)
\end{aligned}
$$

Since φ_V is an R-automorphism φ_V^{-1} is defined and also an R-automorphism with $\varphi_V^{-1}(\alpha_i) = v_i' = \sum_{k=0}^{n} v_{ik}'\alpha_k$, for $1 \leq i \leq n$. Now, since $\varphi_V(\alpha_i) = \sum_{k=0}^{n} v_{ik}\alpha_k$ then we have

$$\begin{aligned}\alpha_i &= v_{i0} + \sum_{k=1}^{n} v_{ik}\varphi_V^{-1}(\alpha_k)\\ &= v_{i0} + \sum_{k=1}^{n} v_{ik}\left(\sum_{t=0}^{n} v_{kt}'\alpha_t\right)\end{aligned}$$

Consequently, we have the following system of equations

$$\begin{aligned} v_{i0} + v_{i1}v_{10}' + v_{i2}v_{20}' + \cdots + v_{in}v_{n0}' &= 0 && (0^{*}*)\\ v_{i1}v_{11}' + v_{i2}v_{21}' + \cdots + v_{in}v_{n1}' &= 0 && (1^{*}*)\\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots &= \vdots && (\vdots)\\ v_{i1}v_{1i}' + v_{i2}v_{2i}' + \cdots + v_{in}v_{ni}' &= 1 && (\mathrm{i}^{*}*)\\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots &= \vdots && (\vdots)\\ v_{i1}v_{1n}' + v_{i2}v_{2n}' + \cdots + v_{in}v_{nn}' &= 0 && (\mathrm{n}^{*}*)\end{aligned}$$

Multiply the equation (i**) by v_{i0} to get

$$v_{i0} = v_{i0}v_{i1}v_{1i}' + v_{i0}v_{i2}v_{2i}' + \cdots + v_{i0}v_{in}v_{ni}'$$

By the (*) equations we get

$$\begin{aligned} v_{i0} &= -v_{i1}v_{i0}v_{1i}' - v_{i2}v_{i0}v_{2i}' - \cdots - v_{in}v_{i0}v_{ni}'\\ &= -v_{i0}(v_{i1}v_{1i}' + v_{i2}v_{2i}' + \cdots + v_{in}v_{ni}')\\ &= -v_{i0}\end{aligned}$$

Hence $2v_{i0} = 0$, for all $1 \leq i \leq n$.

On the other hand, the equations (**) produce the following

$$\begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n}\\ v_{21} & v_{22} & \cdots & v_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ v_{n1} & v_{n2} & \cdots & v_{nn}\end{bmatrix}\begin{bmatrix} v_{11}' & v_{12}' & \cdots & v_{1n}'\\ v_{21}' & v_{22}' & \cdots & v_{2n}'\\ \vdots & \vdots & \ddots & \vdots\\ v_{n1}' & v_{n2}' & \cdots & v_{nn}'\end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}$$

Therefore, the matrix

$$\begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n}\\ v_{21} & v_{22} & \cdots & v_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ v_{n1} & v_{n2} & \cdots & v_{nn}\end{bmatrix} \text{ is invertible in the ring } M_n(R)$$

Furthermore, since $\alpha_i\alpha_j = 0$ for $i, j \in \{1, 2, ..., n\}$, then we have

$$0 = \alpha_i^2 = \left(v_{i0} + \sum_{k=1}^{n} v_{ik}\varphi_V^{-1}(\alpha_k)\right)^2 = v_{i0}^2$$

■

Lemma 2.3. Let R be a ring and let φ_V be an R-endomorphism of the ring $R[\{\alpha_i\}_{i=1}^n]$ with $\varphi_V(\alpha_i) = \sum_{k=0}^{n} v_{ik}\alpha_k$. Then φ_V is one to one if

1. $2v_{i0} = 0$, for all $1 \leq i \leq n$,
2. v_{i0} is nilpotent with nilpotency 2, and
3. The $n \times n$ matrix $V = \left[v_{i,j}\right]$, (with $1 \leq i, j \leq n$) is invertible in the ring $M_n(R)$.

Proof. Suppose that $\varphi_V : R[\{\alpha_i\}_{i=1}^n] \longrightarrow R[\{\alpha_i\}_{i=1}^n]$ is an R-endomorphism with $\varphi_V(\alpha_i) = \sum_{k=0}^{n} v_{ik}\alpha_k$. Let $a = a_0 + \sum_{i=1}^{n} a_i\alpha_i \in R[\{\alpha_i\}_{i=1}^n]$ such that $\varphi_V(a) = 0$. Then

$$\begin{aligned} 0 &= \varphi_V\left(a_0 + \sum_{i=1}^{n} a_i\alpha_i\right) \\ &= a_0 + \sum_{i=1}^{n} a_i\varphi_V(\alpha_i) \\ &= a_0 + \sum_{i=1}^{n} a_i\left(\sum_{k=0}^{n} v_{ik}\alpha_k\right) \end{aligned}$$

As a result, we have the following system of equations

$$a_0 + a_1v_{10} + a_2v_{20} + \cdots + a_nv_{n0} = 0 \quad (1)$$
$$a_1v_{11} + a_2v_{21} + \cdots + a_nv_{n1} = 0 \quad (2)$$
$$\vdots \quad (\vdots)$$
$$a_1v_{1n} + a_2v_{2n} + \cdots + a_nv_{nn} = 0 \quad (\text{n})$$

Hence,

$$\begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}^T \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since the matrix V is invertible, we deduce that $a_1 = a_2 = \cdots = a_n = 0$. Substituting this in equation (1) produces $a_0 = a_1 = a_2 = \cdots = a_n = 0$. Therefore, φ_V is one to one. ■

Lemma 2.4. Let R be a ring and let φ_V be an R-endomorphism of the ring $R[\{\alpha_i\}_{i=1}^n]$ with $\varphi_V(\alpha_i) = \sum_{k=0}^{n} v_{ik}\alpha_k$. Then φ_V is onto if

1. $2v_{i0} = 0$, for all $1 \le i \le n$,
2. v_{i0} is nilpotent with nilpotency 2, and
3. The $n \times n$ matrix $V = \left[v_{i,j}\right]$, (with $1 \le i, j \le n$) is invertible in the ring $M_n(R)$.

Proof. Suppose that $\varphi_V : R[\{\alpha_i\}_{i=1}^n] \longrightarrow R[\{\alpha_i\}_{i=1}^n]$ is an R-endomorphism that satisfies the assumptions. Let $a = a_0 + \sum_{i=1}^{n} a_i\alpha_i \in R[\{\alpha_i\}_{i=1}^n]$. So, I need $b = b_0 + \sum_{i=1}^{n} b_i\alpha_i \in R[\{\alpha_i\}_{i=1}^n]$ such that $\varphi_V(b) = a$. Hence,

$$b_0 + b_1v_{10} + b_2v_{20} + \cdots + b_nv_{n0} = a_0 \tag{1}$$

$$b_1v_{11} + b_2v_{21} + \cdots + b_nv_{n1} = a_1 \tag{2}$$

$$\vdots \qquad (\vdots)$$

$$b_1v_{1n} + b_2v_{2n} + \cdots + b_nv_{nn} = a_n \tag{n}$$

Thus, we have

$$\begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}^T \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Since the matrix V is invertible, we deduce that

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \left(\begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}^T\right)^{-1} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Therefore, b can be choosen as $b = \left(a_0 - \sum_{i=1}^{n} c_i v_{i0}\right) + \sum_{i=1}^{n} c_i \alpha_i$. In conclusion, φ_V is onto. ■

We must point out that the conditions $v_{i0}^2 = 2v_{i0} = 0$ are needed for φ_V to be an R-endomorphism.

We now state the main result in this article which is obtained in Lemma 2.3 and Lemma 2.4.

Theorem 2.5. Let R be a ring and let φ_V be an R-endomorphism of the ring $R[\{\alpha_i\}_{i=1}^n]$ with $\varphi_V(\alpha_i) = \sum_{k=0}^{n} v_{ik}\alpha_k$. Then φ_V is an R-automorphism if and only if

1. $2v_{i0} = 0$, for all $1 \leq i \leq n$,
2. v_{i0} is nilpotent with nilpotency 2, and
3. The $n \times n$ matrix $V = \left[v_{i,j}\right]$, (with $1 \leq i, j \leq n$) is invertible in the ring $M_n(R)$.

The following corollary is immediatly obtained by Theorem 2.5.

Corollary 2.6. Let R be a reduced (or of characteristic 2) ring and let φ_V be an R-endomorphism of the ring $R[\{\alpha_i\}_{i=1}^n]$ with $\varphi_V(\alpha_i) = \sum_{k=0}^{n} v_{ik}\alpha_k$. Then φ_V is an R-automorphism if and only if

1. $v_{i0} = 0$, for all $1 \leq i \leq n$, and
2. The $n \times n$ matrix $V = \left[v_{i,j}\right]$, (with $1 \leq i, j \leq n$) is invertible in the ring $M_n(R)$.

The set of all R-automorphisms of the ring $R[\{\alpha_i\}_{i=1}^n]$ will be denoted by $\mathrm{Aut}_R R[\{\alpha_i\}_{i=1}^n]$. Then, $\mathrm{Aut}_R R[\{\alpha_i\}_{i=1}^n]$ is a group under the binary operation (composition of maps).

For the next corollary $GL(n, R)$ is the group of all $n \times n$ invertible matrices with entries in the ring R, and $I = \left\{a \in R : a^2 = 2a = 0\right\}$ is an ideal of R.

Corollary 2.7. Let R be a finite ring. Then $\mathrm{Aut}_R R[\{\alpha_i\}_{i=1}^n]$ is finite, and $\left|\mathrm{Aut}_R R[\{\alpha_i\}_{i=1}^n]\right| = |I|^n \left|GL(n, R)\right|$.

Corollary 2.8. For any positive integer m we have

$$\left|\mathrm{Aut}_{\mathbb{Z}_m}\mathbb{Z}_m[\{\alpha_i\}_{i=1}^n]\right| = \begin{cases} 2^n \left|GL(n, \mathbb{Z}_m)\right| & \text{if } m = 2^k t \text{ and } k \geq 2 \\ \left|GL(n, \mathbb{Z}_m)\right| & \text{otherwise} \end{cases}$$

Proof. By Corollary 2.7, it is enough to prove that for $I = \left\{\overline{a} \in \mathbb{Z}_m : \overline{a}^2 = 2\overline{a} = \overline{0}\right\}$, then

$$|I| = \begin{cases} 2 & \text{if } m = 2^k t \text{ and } k \geq 2 \\ 1 & \text{otherwise} \end{cases}$$

If $\overline{0} \neq \overline{a} \in I$, then since $2\overline{a} = \overline{0}$, we conclude $2|m$. Thus, $m = 2^{k_1} p_2^{k_2} p_3^{k_3} \cdots p_r^{k_r}$ is the prime factorization of m. Also, since $\overline{a}^2 = \overline{0}$, we get $a = 2^{d_1} p_2^{d_2} p_3^{d_3} \cdots p_r^{d_r}$, where $\left\lceil \frac{k_i}{2} \right\rceil \leq d_i \leq k_i$ for $i = 1, 2, \ldots r$. If $k_1 = 1$, then $d_1 = 1$. Which contradicts that $2\overline{a} = \overline{0}$. Hence $k_1 \geq 2$. Now, again because $2\overline{a} = \overline{0}$, we have $2^{d_1+1} p_2^{d_2} p_3^{d_3} \cdots p_r^{d_r} = 2^{k_1} p_2^{k_2} p_3^{k_3} \cdots p_r^{k_r}$. For this purpose and since $\mathbb{Z}$ is a unique factorization domain, we deduce that $d_1 = k_1 - 1$, and $d_i = k_i$ for $i = 2, \ldots r$. Therefore, $a = 2^{k_1-1} p_2^{k_2} p_3^{k_3} \cdots p_r^{k_r}$. So, if $m = 2^k t$ and $k \geq 2$, then $I = \left\{\overline{0}, \overline{2^{k_1-1} p_2^{k_2} p_3^{k_3} \cdots p_r^{k_r}}\right\}$. otherwise, $I = \left\{\overline{0}\right\}$. ■

3. Open Problems

We end this article by presenting open problems, some of which could possibly be tackled in the near future.

Problem 3.1. Let R be a ring. Then

1. For which rings R, $\mathrm{Aut}_R R[\{\alpha_i\}_{i=1}^n]$ is cyclic group?
2. For which rings R, $\mathrm{Aut}_R R[\{\alpha_i\}_{i=1}^n]$ is Abelian group?

Problem 3.2. Let H be a subgroup of the group $\mathrm{Aut}_R R[\{\alpha_i\}_{i=1}^n]$. Then $R[\{\alpha_i\}_{i=1}^n]^H = \{a = a_0 + a_1\alpha_1 + \cdots + a_n\alpha_n \in R[\{\alpha_i\}_{i=1}^n] : \varphi(a) = a \text{ for all } \varphi \in H\}$ is a subring of $R[\{\alpha_i\}_{i=1}^n]$. Then

1. What is the structure of the subring $R[\{\alpha_i\}_{i=1}^n]^H$?
2. When is $R[\{\alpha_i\}_{i=1}^n]^H = R$?
3. For which subgroups H, have we $R[\{\alpha_i\}_{i=1}^n]^{Aut_R R[\{\alpha_i\}_{i=1}^n]} = R[\{\alpha_i\}_{i=1}^n]^H$?

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