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# An analytical solution for three symmetrical collinear straight cracks with coalesced yield zones: A complex variable approach

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**Abstract**

Theoretical analysis have been conducted for load bearing capacity of an infinite plate weakened by three collinear straight quasi-static cracks with coalesced yield zones. Analytical expressions for remotely applied stress and crack-tip opening displacement (CTOD) are derived, when developed yield zones are subjected to linear/non-linear stress distributions. The problem is solved using complex variable method and a qualitative study is carried out to investigate the load bearing capacity of the plate with respect to yield zone lengths and CTODs etc. Results are obtained for various fracture parameters numerically and reported graphically. Analytical results are validated with existing published work of various researchers.

**AMS subject classification:** 74R05, 74R10.
**Keywords:** Crack opening displacement, Dugdale model, Mode-I type deformation, Stress intensity factor, Yield zone.

## 1. Introduction

The study of residual strength of structures in the presence of multiple cracks is extremely important for their safety and integrity. So that design of structures may be modified in future to avoid such defects. Few examples are shown in [1] and [2] to study the residual strength of the panel reduces due to multiple site damage. As far as modelling of such a situation is concern, Dugdale[3] gave a model to evaluate the residual strength of an infinite plate containing a crack under general yielding conditions. The model has been widely used due to its mathematical simplicity for single slit problem. The model was modified [4] for various metal configurations and different types of mechanical loading, under the conditions that the minimum stresses acting on the crack is yield stress of the body. Dugdale model was further modified for linearly varying stress distribution [5] and for the case when rims of the developed plastic zones were subjected to a parabolic stress distribution to arrest the crack from further opening [6]. Moreover, coalesced case of two equal collinear cracks with coalesced yield zones under quadratically stress distribution has been studied in [7]. The analytical results of Dugdale model was compared with the results obtained by using finite element method [8] when the cracked plate is under applied strain.

The problem of two unequal and equal collinear straight cracks were considered in [9] to extend the idea of Dugdale for multiple cracks. Closed form solution was obtained using complex variable method, to evaluate load-bearing capacity of an infinite plate containing two cracks under general yielding conditions. Fourier transform method was used to solve a mixed boundary value problem of two collinear cracks in [10]. Strip yield model was further studied for two equal collinear straight cracks in [11] to derived analytical expression for plastic zone length. Recently, two collinear cracks problem was studied in [12] analytically using Föppl integral equation and then using Gauss Chebyshev quadrature method. Recently, complex variable method is used to solve three collinear straight cracks problem under general yielding conditions [13].

When stresses applied at infinite boundary increase to a limit, such that the developed plastic zones at the adjacent tips of two cracks are coalesced. The situation is more complicated because the coalescence of small micro cracks is responsible for the formation of a big macro crack. Coalescence conditions of plastic zones were evaluated in [14] for multiple cracks and a parametric analysis also carried out to study plastic zone length and crack-tip opening displacement [15]. The coalescing process of microscopic cracks have been discussed in [16] using Dugdale model for quasi-brittle materials. Recently, conditions for coalescence of two closely spaced collinear cracks has been investigated in [17]. Weight function approach has been adopted to study the strip yield model for multiple crack problem in [18]. Complex variable method is used to solve two asymmetrical cracks with coalesced yield zones [19] and for four collinear straight cracks with coalesced yield zones [20]. Stress intensity factors and crack opening displacements are given in [22] for various geometry of cracks and different mechanical loading conditions.

The objective of this paper is to study the load bearing capacity of an infinite plate and CTODs in case of three collinear straight cracks with coalesced yield zones. Developed yield zones (due to stresses applied at the infinite boundary of the plate) are subjected to linear/non-linear stress distribution to arrest the cracks from further opening. The problem is solved using complex variable method. Stress intensity factors (SIFs), yield zone size, CTODs are expressed in explicit form. A comparative study is carried out to evaluate the load-bearing capacity of an infinite plate, when yield zones are subjected to different stress profiles. Theoretical results are obtained in the paper for applied stresses, yield zone size, CTODs and compared with previously published work as a limiting case. A qualitative study is presented to measure the load carrying capacity of an infinite plate with three equal cracks with coalesced yield zones. Results are graphically reported and analyzed.

## Nomenclature

| | |
|---|---|
| $C_i (i = 0, 1)$ | Constants |
| $E$ | Young's modulus |
| $L_i (i = 1, 2, 3)$ | cracks |
| $P_n(z)$ | polynomial of degree $n$ |
| $\pm a_1, \pm b_1, \pm c_1$ | crack tips |
| $\pm a$ | tips of the developed yield plastic zones |
| $\Gamma'$ | $-\frac{1}{2}(N_1 - N_2)e^{-2i\alpha}$, $N_1$ and $N_2$ are the values of principal stresses at infinity, $\alpha$ be the angle between $N_1$ and the $ox$-axis |
| $p_i (i = 1, 2, 3, 4)$ | developed plastic/yield zones |
| $p(t), q(t)$ | applied stresses on the yield zones |
| $u, v$ | components of displacement in $x$ and $y$ directions, repectively |
| $z = x + iy$ | complex variable |
| $\delta(t)$ | crack-tip-opening displacement at the crack tip $t$ |
| $\gamma$ | Poisson's ratio |
| $\mu$ | shear modulus |
| $\kappa$ | $= \frac{3-\gamma}{1+\gamma}$ for the plane-stress, $= 3 - 4\gamma$ for the plane-strain |
| $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ | components of stress |
| $\sigma_\infty$ | remotely applied stress at infinite boundary |
| $\sigma_0$ | yield stress of the plate |
| $\Omega(z) = \omega'(z), \Phi(z) = \phi'(z)$ | complex potential functions |

## 2. Mathematical formulation

Mushkelishvili [21] expressed the components of stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ and displacements $(u, v)$ in terms of two complex potential functions $\Phi(z)$ and $\Omega(z)$ of a complex variable $z$ for two-dimensional in-plane problems, as

$$\sigma_{xx} + \sigma_{yy} = 2[\Phi(z) + \overline{\Phi(z)}], \tag{2.1}$$

$$\sigma_{yy} - i\sigma_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}, \tag{2.2}$$

$$2\mu(u + iv) = \kappa\phi(z) - \omega(\bar{z}) - (z - \bar{z})\overline{\phi'(z)}. \tag{2.3}$$

Consider an infinite elastic-perfectly plastic plate containing $n$ quasi-static collinear straight cracks $L_i (i = 1, 2, \ldots, n)$. Rims of these $n$ cracks are subjected to the stress distribution $\sigma^{\pm}_{yy}$, $\sigma^{\pm}_{xy}$, where superscripts (+) and (−) refer to upper and lower faces of the crack reached from $y > 0$ or $y < 0$ planes. Eqs. (2.1) and (2.2) be expressed in terms of dual Hilbert problems

$$\Phi^+(t) + \Omega^-(t) = \sigma_{yy}^+ - i\sigma_{xy}^+, \tag{2.4}$$

$$\Phi^-(t) + \Omega^+(t) = \sigma_{yy}^- - i\sigma_{xy}^-, \qquad on \quad \bigcup_{i=1}^{n} L_i \tag{2.5}$$

under the assumption $\lim_{y\to 0} y\Phi'(t+iy) = 0$.

Solution of the problems expressed in Eqs. (2.4) and (2.5) may be directly written using [21], as

$$\Phi(z) = \Phi_0(z) + \frac{P_n(z)}{X(z)} - \frac{1}{2}\overline{\Gamma}', \tag{2.6}$$

$$\Omega(z) = \Omega_0(z) + \frac{P_n(z)}{X(z)} + \frac{1}{2}\overline{\Gamma}', \tag{2.7}$$

where

$$\Phi_0(z) = \frac{1}{2\pi i X(z)} \int\limits_{\bigcup_{i=1}^{n} L_i} \frac{X^+(t)p(t)}{t-z} dt + \frac{1}{2\pi i} \int\limits_{\bigcup_{i=1}^{n} L_i} \frac{q(t)}{t-z} dt, \tag{2.8}$$

$$\Omega_0(z) = \frac{1}{2\pi i X(z)} \int\limits_{\bigcup_{i=1}^{n} L_i} \frac{X^+(t)p(t)}{t-z} dt - \frac{1}{2\pi i} \int\limits_{\bigcup_{i=1}^{n} L_i} \frac{q(t)}{t-z} dt, \tag{2.9}$$

$$p(t) = \frac{1}{2}(\sigma_{yy}^+ + \sigma_{yy}^-) - \frac{i}{2}(\sigma_{xy}^+ + \sigma_{xy}^-), \quad q(t) = \frac{1}{2}(\sigma_{yy}^- - \sigma_{yy}^-) - \frac{i}{2}(\sigma_{xy}^- - \sigma_{xy}^-), \tag{2.10}$$

$$X(z) = \prod_{k=1}^{n} \sqrt{z-a_k}\sqrt{z-b_k}, \quad P_n(z) = C_0 z^n + C_1 z^{n-1} + \cdots + C_n. \tag{2.11}$$

$C_0$ is determined using loading condition at infinite boundary of the plate and other constants $C_i (i = 1, 2, .., n)$ are calculated using the condition of single-valuedness of displacements around the rims of cracks

$$2(\kappa+1)\int_{L_i} \frac{P_n(z)}{X(z)} dz + \kappa \int\limits_{L_i} [\Phi_0^+(z) - \Phi_0^+(z)] dz + \int\limits_{L_i} [\Omega_0^+(z) - \Omega_0^-(z)] dz = 0. \tag{2.12}$$

Stress intensity factor for mode-I type deformation at each crack tip $z = z_1$ is calculated using the formula given in [11],

$$K_I = 2\sqrt{2\pi} \lim_{z\to z_1} \sqrt{z-z_1}\,\Phi(z). \tag{2.13}$$

Crack-tip opening displacement at crack tip $z = z_1$ is written in terms of displacement components using [16], as

$$\delta^i(z_1) = [v_A^+(z_1) + (v^+)_B^i(z_1)] - [v_A^-(z_1) + (v^-)_B^i(z_1)]. \quad i = I, II, III. \quad (2.14)$$

## 3. Statement of the problem

Consider a thin infinite plate made of isotropic elastic-perfectly plastic material, weakened by three collinear straight cracks. As shown in Fig.1 cracks are located on real axis of a Cartesian coordinate system *oxy* occupy the intervals $(-a_1, -b_1)$, $(-c_1, c_1)$ and $(b_1, a_1)$ and denoted by $L_1$, $L_2$ and $L_3$, respectively. Normal stress distribution $\sigma_{yy} = \sigma_\infty$ is applied at the infinite boundary of the plate, such that the cracks are open in mode-I type deformation and yield zones develop at each tip of the cracks. Remotely applied stress at infinite boundary is increased to an extent such that the developed yield zones at the adjacent tip of three cracks are coalesced. In order to detain the cracks from further opening the rims of the yield zones are subjected to a stress distribution, which is assumed linear/non-linear in nature to study the behaviour of load bearing capacity of the plate. The entire configuration is schematically depicted in Fig. 1.

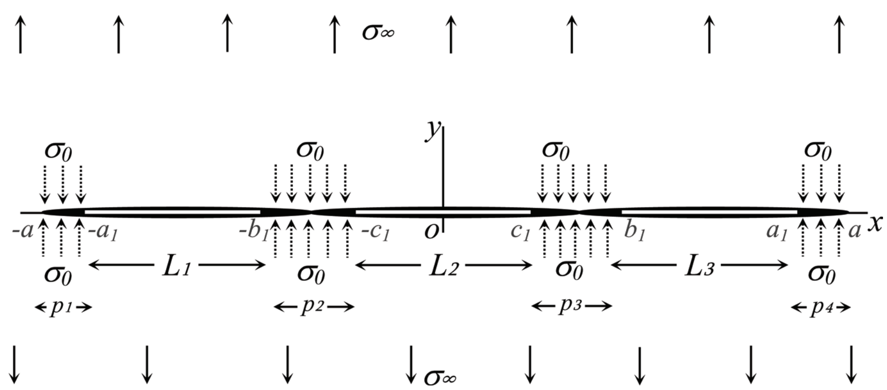



Figure 1: Configuration of the main problem

## 4. Solution of the problem

The problem stated in Section-3 is solved using methodology given in Section-2 after converting it into two component problems. First problem is the opening case (termed as *auxiliary problem-A*), when stresses applied at the infinite boundary open faces of the cracks in mode-I type deformation. In second component problem (termed as *auxiliary problem-B*) yield zones are subjected to normal stress distribution to arrest the cracks from further opening. Three different cases of *auxiliary problem-B* for three different types of stress profiles are considered, because some structure fails at a stress level which is well below the yield stress of the plate as mentioned in [23]. Therefore, it an attempt to create such stress profiles which is below the yield stress of the plate (see Case-II and Case-III).

**Case-I:** Constant yield stress distribution, $\sigma_{yy} = \sigma_0$;

**Case-II:** Linearly varying stress distribution, $\sigma_{yy} = \frac{|t|}{a}\sigma_0$;

**Case-III:** Quadratically varying stress distribution, $\sigma_{yy} = \frac{t^2}{a^2}\sigma_0$.

### 4.1. Statement and solution of *auxiliary problem-A*

Consider an infinite homogeneous isotropic elastic perfectly-plastic plate containing a single crack of length $2a_1$, occupies the interval $C(-a_1, a_1)$ on $ox$-axis as shown in Fig. 2. Uniform stress distribution, $\sigma_\infty$, applied at infinite boundary of the plate, causes the opening of cracks in mode-I type deformation. Consequently, yield zones $p_1 : (-a, -a_1)$ and $p_4 : (a_1, a)$ develop ahead each tip of the crack. The boundary conditions of the problem are

$$\sigma_{yy} = \sigma_\infty, \sigma_{xy} = 0, \quad when \quad y \to \infty, \tag{4.1}$$

$$\sigma_{yy} = 0, \sigma_{xy} = 0, \quad when \quad y \to 0. \tag{4.2}$$

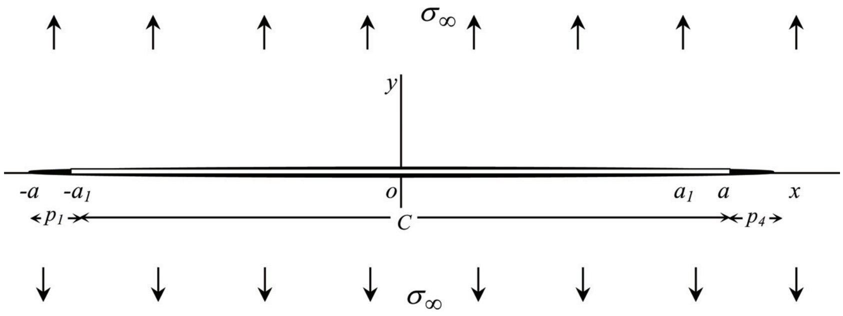



Figure 2: Configuration of the *auxiliary problem-A*

Under the boundary conditions given in Eqs. (4.1) to (4.2), the desired complex potential function for the *auxiliary problem-A* be written as (or may be taken directly from [21]),

$$\Phi_A(z) = \frac{\sigma_\infty}{2}\Big(\frac{z}{\sqrt{z^2 - a^2}} - \frac{1}{2}\Big). \tag{4.3}$$

Opening mode stress intensity factor is obtained by substituting $\Phi_A(z)$ from Eq. (4.3) into Eq. (2.13)

$$(K_I)_A = \sigma_\infty\sqrt{\pi a}. \tag{4.4}$$

Components of displacement due to applied stress, $\sigma_\infty$, are obtained by substituting $\Phi_A(z)$ from Eq. (4.3) into Eq. (2.3)

$$v_A^{\pm}(a_1) \;=\; \pm\frac{2\sigma_\infty}{E}\sqrt{a^2-a_1^2}. \tag{4.5}$$

### 4.2. Statement and solution of *auxiliary problem-B*

Consider, an infinite homogeneous isotropic elastic perfectly-plastic plate occupying *xoy*-plane is weakened by three collinear cracks $L_1$, $L_2$, $L_3$ occupy the intervals $(-a_1, -b_1)$, $(-c_1, c_1)$, $(b_1, a_1)$, respectively. Developed yield zones are denoted by $p_1$, $p_2$, $p_3$, $p_4$ and occupy the intervals $(-a, -a_1)$, $(-b_1, -c_1)$, $(c_1, b_1)$, $(a_1, a)$, respectively. Consider three different cases of *auxiliary problem-B* according to three different stress profiles acting on the rims of yield zones in order to arrest further opening of cracks.

#### 4.2.1 Case-I of *auxiliary problem-B*, when $\sigma_{yy} = \sigma_0$

Consider the case when developed yield zones $p_1$, $p_2$, $p_3$, $p_4$ are subjected to constant yield stress distribution $\sigma_0$ as shown in Fig. 3. The boundary conditions of the problem are

$$\sigma_{yy} = 0, \sigma_{xy} = 0, \quad when \quad y \to \infty, \tag{4.6}$$

$$\sigma_{yy} = \sigma_0, \sigma_{xy} = 0, \quad when \quad x \in \bigcup_{i=1}^{4} p_i,\, y \to 0. \tag{4.7}$$

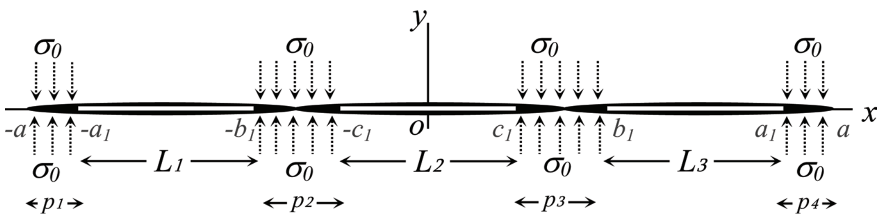



Figure 3: Configuration of the *auxiliary problem-B* (Case-I: $\sigma_{yy} = \sigma_0$))

Solution of the problem discussed in this section may be written using boundary conditions given in Eqs. (4.6) and (4.7). Hence, complex potential function $\Phi_B^I(z)$ is written as

$$\Phi_B^I(z) \;=\; \frac{\sigma_0}{\pi}\left(\frac{-z}{\sqrt{z^2-a^2}}H_1 + \cot^{-1}\frac{a_1\sqrt{z^2-a^2}}{z\sqrt{a^2-a_1^2}} + \tan^{-1}\frac{b_1\sqrt{z^2-a^2}}{z\sqrt{a^2-b_1^2}} - \tan^{-1}\frac{c_1\sqrt{z^2-a^2}}{z\sqrt{a^2-c_1^2}}\right), \tag{4.8}$$

where

$$H_1 = \cos^{-1}\frac{a_1}{a} + \sin^{-1}\frac{b_1}{a} - \sin^{-1}\frac{c_1}{a}.$$

The superscript I indicates that the function refers to *auxiliary problem-B (Case-I)*.

Mode-I type stress intensity factor is obtained by substituting $\Phi_B^I(z)$ from Eq. (4.8) into Eq. (2.13) and after some mathematical calculation one may find,

$$(K_I)_B^I = -2\sigma_0 H_1\sqrt{\frac{a}{\pi}}. \tag{4.9}$$

Yield zone length may be obtained by superimposing the solutions for two component problems *auxiliary problem-A* and *auxiliary problem-B (Case-I)*. Hence, stress intensity factors for both the component problems given in Eqs. (4.4) and (4.9) must vanishes. Which yields,

$$\left(\frac{\sigma_\infty}{\sigma_0}\right)_B^I = \frac{2}{\pi}H_1. \tag{4.10}$$

The displacement component $v$ due to yield stress distribution $\sigma_0$ acting on the rims of yield zones is obtained by substituting Eq. (4.8) into Eq. (2.3). Hence,

$$(v_B^{\pm}(a_1))^I = \mp\frac{4\sigma_0}{E\pi}\left(H_1\sqrt{a^2-a_1^2} + H_2 + a_1\ln\frac{a_1}{a}\right), \tag{4.11}$$

where

$$H_2 = -a_1\tanh^{-1}\frac{b_1\sqrt{a^2-a_1^2}}{a_1\sqrt{a^2-b_1^2}} + b_1\tanh^{-1}\frac{\sqrt{a^2-a_1^2}}{\sqrt{a^2-b_1^2}}$$
$$- c_1\tanh^{-1}\frac{\sqrt{a^2-a_1^2}}{\sqrt{a^2-c_1^2}} + a_1\tanh^{-1}\frac{c_1\sqrt{a^2-a_1^2}}{a_1\sqrt{a^2-c_1^2}}.$$

Substituting results from Eqs. (4.5) and (4.11) into Eq. (2.14) to derive analytical expressions for crack-tip opening displacement at crack tip $x = a_1$. Hence,

$$\delta^I(a_1) = -\frac{8\sigma_0}{E\pi}\left(H_2 + a_1\ln\frac{a_1}{a}\right). \tag{4.12}$$

### 4.2.2 Case-II of *Auxiliary problem-B*, when $\sigma_{yy} = \frac{|t|}{a}\sigma_0$

In place of constant yield stress distribution consider a linearly varying yield stress distribution $\frac{t}{a}\sigma_0$ acting over the rims of the developed yield zones, where $t$ is any point on the rims of the yield zones and $\sigma_0$ is the yield stress of the plate. The entire configuration is depicted in Fig. 4. Accordingly, the boundary conditions of the problem are

$$\sigma_{yy} = 0, \sigma_{xy} = 0, \qquad when \quad y \to \infty, \tag{4.13}$$

$$\sigma_{yy} = \frac{|t|}{a}\sigma_0, \sigma_{xy} = 0, \quad when \quad x \in \bigcup_{i=1}^{4} p_i, y \to 0, \tag{4.14}$$

$$\sigma_{yy} = 0, \sigma_{xy} = 0, \quad when \quad x \in \bigcup_{i=1}^{3} L_i, y \to 0. \tag{4.15}$$

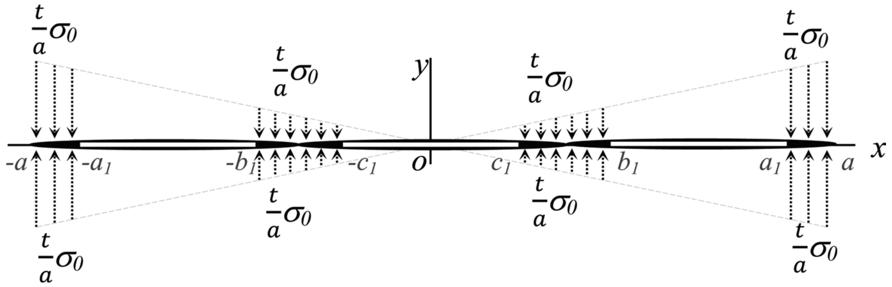



Figure 4: Configuration of the *auxiliary problem-B* $\left(\text{Case-II: } \sigma_{yy} = \frac{|t|}{a}\sigma_0\right)$

The desired complex potential function and stress intensity factor for this case analogously obtained as in Case-I, using methodology explained in Section-2, under the boundary conditions given in Eqs. (4.13), (4.14), (4.15)

$$\begin{aligned} \Phi_B^{II}(z) &= -\frac{\sigma_0}{\pi}\Big[\frac{z}{\sqrt{z^2-a^2}}\Big(\frac{\sqrt{a^2-a_1^2}}{a} - \frac{\sqrt{a^2-b_1^2}}{a} + \frac{\sqrt{a^2-c_1^2}}{a}\Big) \\ &\quad -\frac{z}{a}\Big(\cot^{-1}\frac{\sqrt{z^2-a^2}}{\sqrt{a^2-a_1^2}} - \cot^{-1}\frac{\sqrt{z^2-a^2}}{\sqrt{a^2-b_1^2}} + \cot^{-1}\frac{\sqrt{z^2-a^2}}{\sqrt{a^2-c_1^2}}\Big)\Big]. \end{aligned} \tag{4.16}$$

Opening mode stress intensity factor, $K_1^{II}$

$$(K_I)_B^{II} = \frac{2\sigma_0}{\sqrt{a\pi}}\Big( -\sqrt{a^2-a_1^2} + \sqrt{a^2-b_1^2} - \sqrt{a^2-c_1^2}\Big). \tag{4.17}$$

According to Dugdale's cohesive zone hypothesis, $(K_I)_A + (K_I)_B^{II} = 0$, which yields the nonlinear equation to determine the size of yield zone

$$\Big(\frac{\sigma_\infty}{\sigma_0}\Big)_B^{II} = \frac{2}{a\pi}\Big(\sqrt{a^2-a_1^2} - \sqrt{a^2-b_1^2} + \sqrt{a^2-c_1^2}\Big). \tag{4.18}$$

Components of displacement for the case may be obtained by substituting Eq. (4.16) into Eq. (2.3), hence

$$(v^{\pm}(a_1))_B^{II} = \mp\frac{2\sigma_0}{E\pi a}\Big(H_3 + \big(\sqrt{a^2-a_1^2} - \sqrt{a^2-b_1^2} + \sqrt{a^2-c_1^2}\big)\sqrt{a^2-a_1^2}\Big), \tag{4.19}$$

where

$$H_3 = -a_1^2 \tanh^{-1}\frac{\sqrt{a^2-b_1^2}}{\sqrt{a^2-a_1^2}} + b_1^2 \tanh^{-1}\frac{\sqrt{a^2-a_1^2}}{\sqrt{a^2-b_1^2}} + a_1^2 \tanh^{-1}\frac{\sqrt{a^2-c_1^2}}{\sqrt{a^2-a_1^2}} - c_1^2 \tanh^{-1}\frac{\sqrt{a^2-a_1^2}}{\sqrt{a^2-c_1^2}}.$$

Crack-tip opening displacement for linearly varying stress distribution is obtained using Eqs. (2.14), (4.5) and (4.19). Finally, CTOD at the crack tip $a_1$ is

$$\delta^{II}(a_1) = \frac{4\sigma_0}{\pi E a}\Big[\Big(\sqrt{a^2-a_1^2} - \sqrt{a^2-b_1^2} + \sqrt{a^2-c_1^2}\Big)\sqrt{a^2-a_1^2} - H_3\Big]. \tag{4.20}$$

### 4.2.3 Case-III: $\sigma_{yy} = \frac{t^2}{a^2}\sigma_0$ *for* auxiliary problem-B

Consider a significantly different stress profile (quadratically varying) $\frac{t^2}{a^2}\sigma_0$ which is applied on the rims of yield zones to arrest further opening the cracks as shown in Fig. 5. The said stress distribution is well below the yield stress distribution of the plate[23]. Hence, boundary conditions of the problem are

$$\sigma_{yy} = 0, \sigma_{xy} = 0, \qquad when \quad y \to \infty, \tag{4.21}$$

$$\sigma_{yy} = \frac{t^2}{a^2}\sigma_0, \sigma_{xy} = 0, \quad when \quad x \in \bigcup_{i=1}^{4} p_i, y \to 0, \tag{4.22}$$

$$\sigma_{yy} = 0, \sigma_{xy} = 0, \quad when \quad x \in \bigcup_{i=1}^{3} L_i, y \to 0. \tag{4.23}$$

The appropriate complex potential function $\Phi_B^{III}(z)$ can be written in the following form using methodology given in Section-2 and Eqs. (4.21)-(4.23),

$$\Phi_B^{III}(z) = \frac{z\sigma_0}{2\pi a^2\sqrt{z^2-a^2}}\Big[a^2 H_4 + (a^2-2z^2)H_1 - 2z\sqrt{a^2-z^2}G(z)\Big], \tag{4.24}$$

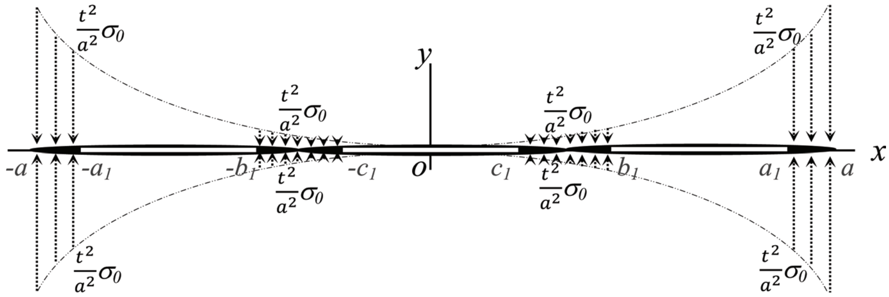



Figure 5: Configuration of the *auxiliary problem-B* $\left(\text{Case-III: } \sigma_{yy} = \dfrac{t^2}{a^2}\sigma_0\right)$

where

$$\begin{aligned} H_4 &= \frac{1}{a^2}\Big( - a_1\sqrt{a^2-a_1^2} + b_1\sqrt{a^2-b_1^2} - c_1\sqrt{a^2-c_1^2}\Big), \\ G(z) &= \frac{-i\pi}{2} - \tanh^{-1}\frac{a_1\sqrt{a^2-z^2}}{z\sqrt{a^2-a_1^2}} + \tanh^{-1}\frac{b_1\sqrt{a^2-z^2}}{z\sqrt{a^2-b_1^2}} - \tanh^{-1}\frac{c_1\sqrt{a^2-z^2}}{z\sqrt{a^2-c_1^2}}. \end{aligned}$$

Opening mode stress intensity factor for this case is evaluated by substituting $\Phi_B^{III}(z)$ from Eq. (4.24) to Eq. (2.13)

$$(K_I)_B^{III} = \sigma_0\sqrt{\frac{a}{\pi}}\Big(H_4 - H_1\Big). \tag{4.25}$$

A non-linear equation is obtained to evaluate yield zone length using $(K_I)_A + (K_I)_B^{III} = 0$,

$$\left(\frac{\sigma_\infty}{\sigma_0}\right)_B^{III} = \frac{1}{\pi}\Big(H_1 - H_4\Big). \tag{4.26}$$

Due to nonlinear stress distribution acting on the rims of yield zones, the crack-tip opening displacement is expressed in the following form

$$\begin{aligned} \delta^{III}(a_1) &= \frac{8\sigma_0}{3a^2E\pi}\Big[(a^2-a_1^2)^{\frac{3}{2}}H_1 - a^2H_4\sqrt{a^2-a_1^2} - a_1^3\ln\frac{a_1}{a} \\ &\qquad + H_5(b_1) - H_5(c_1)\Big]. \end{aligned} \tag{4.27}$$

where

$$H_5(x) = a_1^3\tanh^{-1}\frac{x\sqrt{a^2-a_1^2}}{a_1\sqrt{a^2-x^2}} - x^3\tanh^{-1}\frac{\sqrt{a^2-a_1^2}}{\sqrt{a^2-x^2}}.$$

## 5. Validation of solution

To verify the accuracy, theoretical results obtained above for various quantities of interest complex potential functions, stress intensity factors, applied stresses, displacement components and CTODs reduce to the analytical results obtained by various researchers previously in [6],[11], [16] etc.

Analytical expressions derived for various quantities in Eqs. (4.8), (4.10), (4.11) and (4.12) agree with the analytical expressions investigated in [6], [11] and [16] for single Dugdale crack of length $2a_1$ taking $b_1 = c_1$. Eqs. (4.16), (4.18), (4.19), (4.20), (4.24), (4.26) and (4.27) reduce to the solution given in [6] by substituting $b_1 = c_1$ for respective cases of linearly and quadratically varying stress distributions.

## 6. Illustrative examples

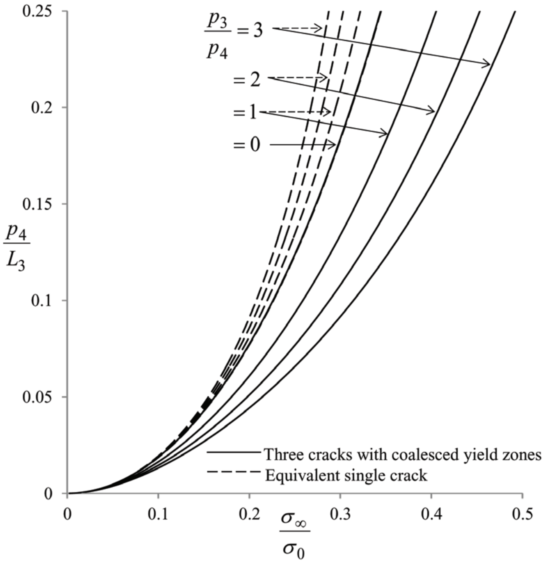
For the safety and security of structures, it is necessary to know the load bearing capacity of that structure. To accomplish this task, a qualitative study is carried out to determine load carrying capacity of an infinite plate weakened by three cracks with coalesced yield zones.

In the following sections, numerical results are presented for yield zone length, load required ratio and crack-tip opening displacement for different lengths of coalesced yield zones. Numerical results so obtained are compared with the results of an equivalent single crack under same mechanical loading conditions.

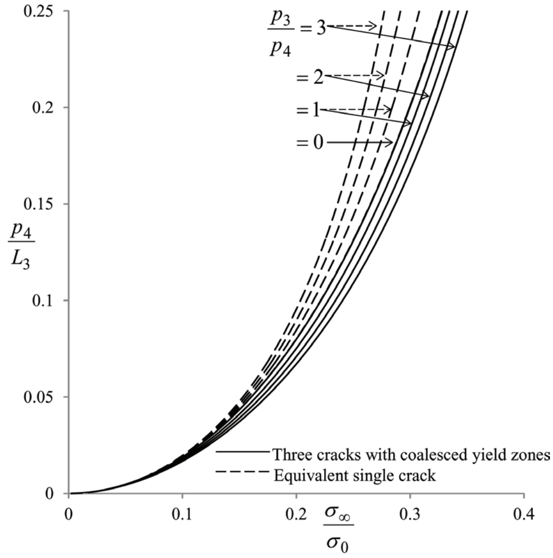
### 6.1. Yield zone length

Figure 6 shows the variation between normalized yield zone length, $\frac{p_4}{L_3}$ to applied load ratio, $\frac{\sigma_\infty}{\sigma_0}$, when uniform stress distribution $\sigma_0$ is applied on the rims of the yield zones. It has been seen from the figure that as load applied at the infinite boundary of the plate increases yield zone length increases, as expected. Also, when $p_3$ is three times the length of $p_4$ infinite plate bear more load at it boundary in comparison to the case of equal values of $p_3$ and $p_4$. Moreover, when $p_3 = 0$ the results are similar to the results of a single crack of length $2a_1$. Hence, it clear that bigger coalesced yield zone length means higher load bearing capacity of the plate in the presence of three straight cracks with coalesced yield zones.

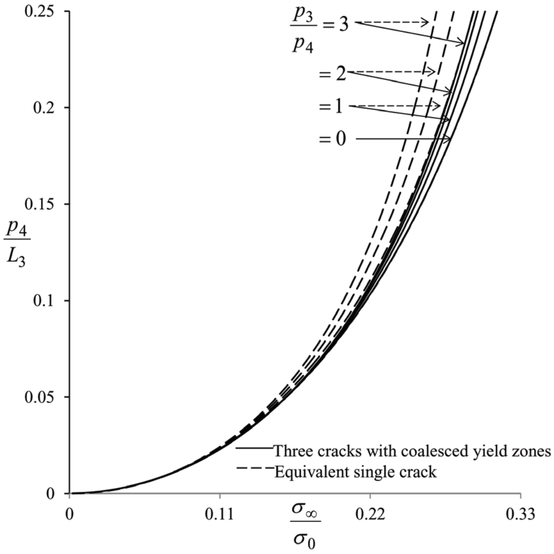
Same variation has been plotted for linearly varying stress distribution $\frac{|t|}{a}\sigma_0$ in Fig. 6. The plate bear less load due to a stress distribution, which is below the yield stress of the material acting on the rims of yield zones. In other words, length of each yield zone is bigger in case of linearly varying yield stress distribution in comparison to constant yield stress distribution. As a result, strength of the plate decreases rapidly in case of linearly varying yield stress distribution. The results are compared with the results of an equivalent single crack under same loading conditions. It has been observed that the plate bear more load if yield zone $p_3$ is bigger than $p_4$.




(a) $\sigma_0$(Case-I)



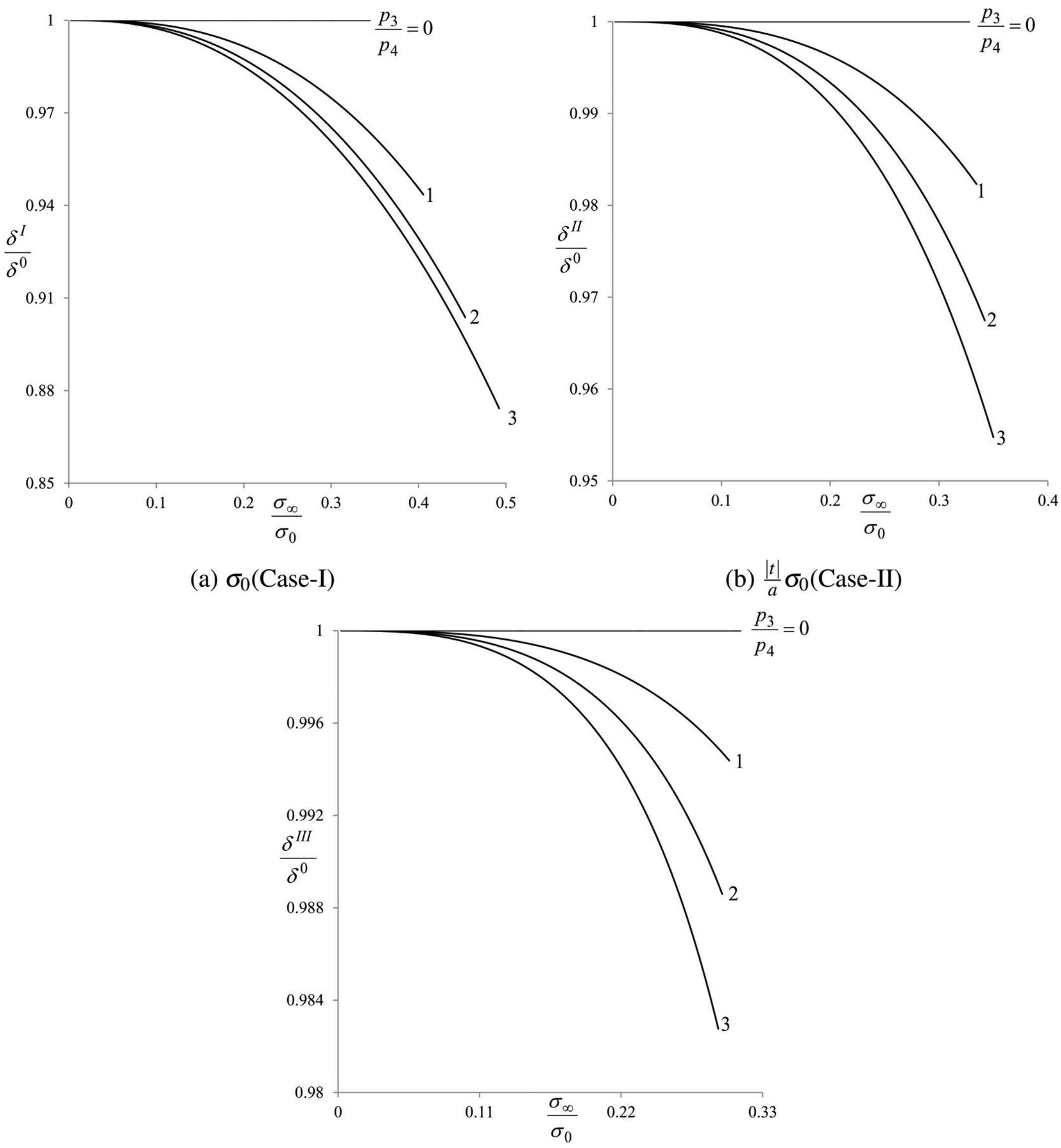

(b) $\frac{|t|}{a}\sigma_0$(Case-II)




(c) $\frac{t^2}{a^2}\sigma_0$(Case-III)

Figure 6: Variation between $\frac{\sigma_\infty}{\sigma_0}$ and normalized yield zone length $\frac{p_4}{L_3}$ for different values of $\frac{p_3}{p_4}$ and stress profiles.

Quadratically varying stress distribution is assumed on the rims of the yield zones to study bearing capacity of the plate. It has been observed from Fig. 6 that as load applied at the infinite boundary of the plate increases yield zone length at each crack tip increases. Comparatively less load (due to quadratic in nature) is applied on the rims




(a) $\sigma_0$(Case-I)

(b) $\frac{|t|}{a}\sigma_0$(Case-II)

(c) $\frac{t^2}{a^2}\sigma_0$(Case-III)

Figure 7: Variation between $\dfrac{\sigma_\infty}{\sigma_0}$ and normalized crack-tip opening displacement for various stress profiles.

of coalesced yield zones, therefore, entire configuration is approximately behave like an equivalent single crack.

## 6.2. Crack-tip opening displacement

Crack-tip opening displacement at each original crack tip is evaluated numerically and normalized with the crack-tip opening displacement of an equivalent single crack under similar mechanical loading conditions. Depending upon the stress profiles applied on the rims of yield zones three different cases have been discussed in this section.

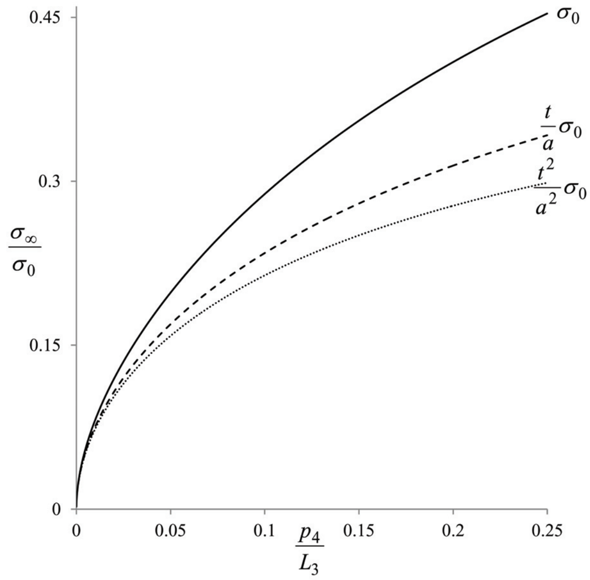



Figure 8: Normalized yield zone length $\frac{p_4}{L_3}$ to applied stress $\frac{\sigma_\infty}{\sigma_0}$.

Figure 7 shows the variation of normalized CTOD $\frac{\delta_I}{\delta_0}$ with increasing values of applied load ratio $\frac{\sigma_\infty}{\sigma_0}$, where $\delta_0$ is the CTOD of a single crack under same mechanical loading conditions. When $p_3 = 0$ the outer cracks shown in Fig. 1 shows same opening as an equivalent single crack. Its again validates the analytical expressions for CTODs. Furthermore, when $p_3 = p_4$ opening of three cracks with coalesced yield zones at the outer tips is less as compared to the opening of a single crack, as expected. Moreover, less opening of cracks (on comparing the results with single crack) is seen at outer crack tips of three cracks with coalesced yield zones (shown in Fig. 1) on increasing length of coalesced yield zones.

Same variation has been plotted at the outer tip for linearly varying stress distribution in Fig. 7. Bigger value of $p_3$ in comparison to $p_4$ effect the opening of cracks, means $\delta_{II}$ is less than $\delta_0$. For quadratically varying stress distribution same variation is plotted in Fig. 7. Insignificant difference is seen between $\delta_{III}$ and $\delta_0$ due a very small load applied on the rims of coalesced yield zones.

## 7. Comparative study

Presence of cracks beset strength of the structures due to different yield zone lengths under different mechanical loading conditions. A comparison is shown in Fig. 8 for applied

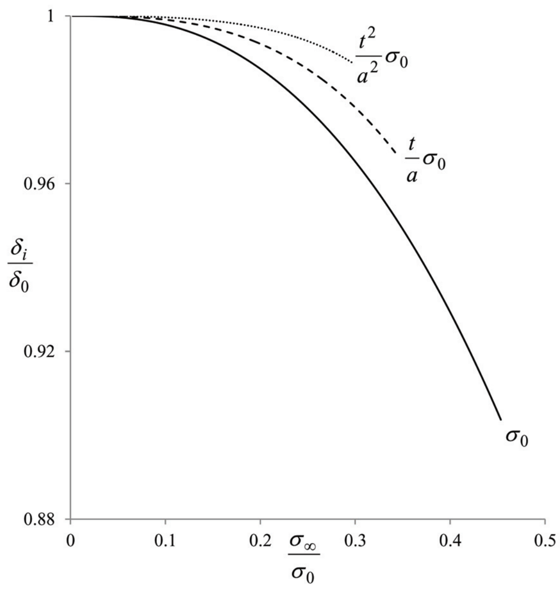



Figure 9: $\frac{\sigma_\infty}{\sigma_0}$ to $\frac{\delta_i}{\delta_0}$, $i = I, II, III$.

load ratio and for CTODs in Fig. 9 between three different types of stress distributions applied on the rims of yield zones.

As expected, yield zone length at each crack tip is bigger in case of quadratic stress distribution in comparison to constant and linearly varying stress distribution. Hence, it may be concluded that the structures fail at a stress, which is below the yield stress of the martial when linear and parabolic stress distributions are acting on the rims of yield zones.

## 8. Conclusion

A crack arrest model of three equal cracks with coalesced yield zones was formulated and the solution is obtained by complex variable method. Three different cases were discussed depending on three different types of stress distributions applied on the rims of the yield zones. Analytical expressions are derived for complex potential function, displacement components and crack-tip opening displacements. Numerical results were obtained for three equal cracks for different length of coalesced yield zones. In addition, a comparative study is carried out to find the difference in load bearing capacity of an infinite plate under constant, linear and quadratic stress distribution.

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