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An Application of δ - Quasi Monotone Sequence

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ABSTRACT

A result concerning absolute indexed Summability factor of an infinite series using δ - Quasi monotone sequence has been established.

KEY WORDS: Quasi-increasing, Quasi - f - power increasing, δ - Quasi monotone, index absolute Summability, summability factor.

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1. INTRODUCTION

A positive sequence (a_n) is said to be almost increasing if there exists a positive sequence (b_n) and two positive constants A and B such that

(1.1) $Ab_n \le a_n \le Bb_n$, for all n .

The sequence (a_n) is said to be quasi-β-power increasing, if there exists a constant K depending upon β with $K \ge 1$ such that

(1.2) $K n^{\beta} a_n \ge m^{\beta} a_m$,

for all $n \ge m$. In particular, if $\beta = 0$, then (a_n) is said to be quasi-increasing sequence. It is clear that every almost increasing sequence is a quasi-β-power increasing sequence for any non-negative β . But the converse is not true as $(n^{-\beta})$ is quasi-β-power increasing but not almost increasing.

Let $f = (f_n)$ be a positive sequence of numbers. Then the positive sequence (a_n) is said to be quasi- f -power increasing, if there exists a constant K depending upon f with $K \ge 1$ such that

(1.3) $K f_n a_n \ge f_m a_m$,

for $n \ge m \ge 1([7])$. Clearly, if (α_n) is a quasi- f -power increasing sequence, then the $(\alpha_n f_n)$ is a quasi- increasing sequence.

Let $\delta = (\delta_n)$ be a positive sequence of numbers. Then a positive sequence (a_n) is said to be δ - quasi monotone [1], if $a_n \to 0$, $a_n > 0$ ultimately and $\Delta d_n \ge -\delta_n$, where $\Delta d_n = d_n - d_{n+1}$.

Let $\sum a_n$ be an infinite series with sequence of partial sums $\{s_n\}$. Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty, \text{ as } n \to \infty .$$

Then the sequence-to-sequence transformation

(1.4) $T_n = \dfrac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu$, $P_n \ne 0$,

defines the $(\overline{N}, p_n)$- mean of the sequence (s_n) generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n\right|_k$, $k \ge 1$ [2], if

(1.5) $\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|T_n - T_{n-1}|^k < \infty$.

The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n; \delta\right|_k, k \geq 1, \delta \geq 0,$ if

(1.6) $\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_n - T_{n-1}|^k < \infty$.

For any real number γ, the series $\sum a_n$ is said to be summable by the summabilty method $\left|\bar{N}, p_n; \delta, \gamma\right|_k, k \geq 1, \delta \geq 0$, if

(1.6) $\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\gamma(\delta k+k-1)}|t_n - t_{n-1}|^k < \infty$.

For $\gamma = 1$, the summability method $\left|\bar{N}, p_n; \delta, \gamma\right|_k, k \geq 1, \delta \geq 0,$ any real γ, reduces to the method $\left|\bar{N}, p_n; \delta\right|_k, k \geq 1, \delta \geq 0$

2. KNOWN THEOREMS

Dealing with quasi-β-power increasing sequence Bor and Debnath **[3]** have established the following theorem:

2.1. THEOREM

Let (X_n)be a quasi-β-power increasing sequence for $0 < \beta < 1$ and (λ_n)be a real sequence.

If the conditions

(2.1.1) $\sum_{n=1}^{m}\frac{P_n}{n} = O(P_m)$,

(2.1.2) $\lambda_n X_n = O(1)$,

(2.1.3) $\sum_{n=1}^{m}\frac{|t_n|^k}{n} = O(X_m)$,

(2.1.4) $\sum_{n=1}^{m}\frac{p_n|t_n|^k}{P_n} = O(X_m)$

and

(2.1.5) $\sum_{n=1}^{m} nX_n|\Delta^2\lambda_n| < \infty$

are satisfied, where $t_n$is the $(C,1)$ mean of the sequence (na_n).Then the series $\sum a_n\lambda_n$ is summable $\left|\bar{N}, p_n\right|_k, k \geq 1$.

Subsequently Leindler **[4]** established a similar result reducing certain condition of Bor. He established:

2.2. THEOREM

Let the sequence (X_n) be a quasi-β-power increasing sequence for $0 < \beta < 1$ **,** and the real sequence (λ_n) satisfies the conditions

(2.2.1) $\sum_{n=1}^{m} \lambda_n = O(m)$

and

(2.2.2) $\sum_{n=1}^{m} |\Delta \lambda_m| = O(m)$.

Furthere, suppose the conditions (2.1.3), (2.1.4) and

(2.2.3) $\sum_{n=1}^{m} n X_n(\beta) |\Delta |\Delta \lambda_n|| < \infty$,

hold, where $X_n(\beta) = \max(n^{\beta} X_n, \log n)$ .Then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k, k \geq 1$.

Recently, extending the above results to quasi- f -power increasing sequence, Sulaiman **[8]** have established the following theorem:

2.3. THEOREM

Let $f = (f_n) = (n^{\beta} \log^{\gamma} n), 0 \leq \beta < 1, \gamma \geq 0$ be a sequence. Let (X_n) be a quasi- f -power sequence and (λ_n) a sequence of constants satisfying the conditions

(2.3.1) $\lambda_n \to 0 \; as \, n \to \infty$,

(2.3.2) $\sum_{n=1}^{\infty} n X_n |\Delta |\Delta \lambda_n|| < \infty$,

(2.3.3) $|\lambda_n| X_n = O(1)$,

(2.3.4) $\sum_{n=1}^{\infty} \frac{1}{n X_n^{k-1}} |t_n|^k = O(X_m)$

and

(2.3.5) $\sum_{n=1}^{\infty} \frac{p_n}{P_n} \frac{1}{X_n^{k-1}} |t_n|^k = O(X_m)$,

where t_n is the $(C,1)$ mean of the sequence $(n a_n)$ .Then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k, k \geq 1$.

Very recently, Misra et al [5] established the following theorem:

2.4. THEOREM

Let $f=(f_n)=\left(n^{\beta}(\log n)^{\gamma}\right)$ be a sequence and (X_n)be a quasi- f -power sequence. Let (λ_n) a sequence of constants such that

(2.4.1) $\lambda_n \to 0,\ as\ \ n\to\infty$,

(2.4.2) $\sum_{n=1}^{\infty} nX_n|\Delta|\Delta\lambda_n\|<\infty$,

(2.4.3) $|\lambda_n|X_n=O(1)$,

(2.4.4) $\sum_{n=\nu+1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{P_{n-1}}=O\left(\frac{P_m}{p_m}\right)^{\delta k-1}$,

(2.4.5) $\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{|t_\nu|^k}{X_\nu^{k-1}}=O(X_m)$,

(2.4.6) $\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{|t_n|^k}{nX_n^{k-1}}=O(X_m)$.

Then the series $\sum a_n\lambda_n$ is summable $\left|\overline{N},p_n;\delta\right|_k, k\geq 1,\delta\geq 0.$

Using δ - quasi monotone sequence, Misra et al [6] extended theorem-2.4, establishing the following result:

2.5. THEOREM

Let $f=(f_n)=\left(n^{\beta}(\log n)^{\gamma}\right)$ be a sequence and (X_n)be a quasi- f -power sequence. Suppose also that there exists a sequence of numbers (A_n) such that it is $\delta-$quasi – monotone with

(2.5.1) $\sum n\delta_n X_n<\infty$

(2.5.2) $\Delta A_n\leq\delta_n$ for all n.

Let (λ_n) a sequence of constants such that

(2.5.3) $\lambda_n \to 0,\ as\ \ n\to\infty$,

(2.5.4) $|\lambda_n|X_n=O(1)$,

and

(2.5.5) $|\Delta\lambda_n|\leq|A_n|$ for all n.

Then the series $\sum a_n\lambda_n$ is summable $\left|\overline{N},p_n;\sigma\right|_k, k\geq 1,\sigma\geq 0.$ if ,

(2.5.6) $\sum_{n=\nu+1}^{m}\left(\frac{P_n}{p_n}\right)^{\sigma k-1}\frac{1}{P_{n-1}}=O\left(\frac{P_\nu}{p_\nu}\right)^{\sigma k-1}$,

(2.5.7) $\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\sigma k-1}\frac{|t_n|^k}{X_n^{k-1}}=O(X_m)$,

(2.5.8) $\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\sigma k}\frac{|t_n|^k}{nX_n^{k-1}}=O(X_m),$

where (t_n) is the nth $(C,1)$mean of the sequence $(n\,a_n)$.

In what follows in this paper, using δ - quasi monotone sequence, we prove the following theorem.

3. THEOREM

Let $f=(f_n)=\left(n^{\beta}(\log n)^{\gamma}\right)$ be a sequence and (X_n)be a quasi- f -power sequence. Suppose also that there exists a sequence of numbers (A_n) such that it is $\delta-$quasi – monotone with

(3.5.1) $\sum n\delta_n X_n<\infty$

(3.5.2) $\Delta A_n\le\delta_n$ for all n.

Let (λ_n) a sequence of constants such that

(3.5.3) $\lambda_n\to 0,\ as\ \ n\to\infty$,

(3.5.4) $|\lambda_n|X_n=O(1)$,

and

(3.5.5) $|\Delta\lambda_n|\le|A_n|$ for all n.

Then the series $\sum a_n\lambda_n$ is summable $\left|\overline{N},p_n;\sigma,\mu\right|_k, k\ge 1,\sigma\ge 0.$ if
,

(3.5.6) $\sum_{n=\nu+1}^{m}\left(\frac{P_n}{p_n}\right)^{\sigma k-1}\frac{1}{P_{n-1}}=O\left(\frac{P_\nu}{p_\nu}\right)^{\sigma k-1}$,

(3.5.7) $\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)-k}\frac{|t_n|^k}{X_n^{k-1}}=O(X_m)$,

(3.5.8) $\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)-k+1}\frac{|t_n|^k}{nX_n^{k-1}}=O(X_m)$,

where (t_n) is the nth $(C,1)$mean of the sequence $(n\,a_n)$.

In order to prove the theorem we require the following lemma.

4. LEMMA

Let $f=(f_n)=\left(n^{\beta}(\log n)^{\gamma}\right),0\le\beta<1,\gamma\ge 0$ be a sequence and (X_n) be a quasi - f - power increasing sequence. Let (A_n) be a sequence of numbers such that it is $\delta-$ quasi – monotone satisfying (3.1) and (3.2). then

(4.1) $n\,X_n|A_n|=O(1)$

and

(4.2) $\sum_{n=1}^{m} X_n |A_n| < \infty$, as $m \to \infty$.

4.1. PROOF OF THE LEMMA

As $A_n \to 0$ and $n^{\beta}(\log n)^{\gamma} X_n$ is non-decreasing, we have

$$n X_n |A_n| = n^{1-\beta} (\log n)^{-\gamma} \left(n^{\beta}(\log n)^{\gamma} X_n\right) \sum_{\upsilon=n}^{\infty} \Delta | A_\upsilon |$$

$$= O(1) n^{1-\beta} (\log n)^{\gamma} n \sum_{\upsilon=n}^{\infty} \nu^{\beta}(\log \nu)^{\gamma} X_\upsilon | \Delta A_\upsilon |$$

$$= O(1) \sum_{\upsilon=n}^{\infty} \nu^{1-\beta}(\log \nu)^{-\gamma} \nu^{\beta}(\log \nu)^{\gamma} X_\upsilon | \Delta A_\upsilon |$$

$$= O(1) \sum_{\upsilon=n}^{\infty} \nu X_\upsilon | \Delta A_\upsilon | .$$

$$= O(1) \sum_{\upsilon=n}^{\infty} \nu X_\upsilon | \delta_\upsilon |$$

$$= O(1)$$

This establishes (4.1).

Next

$$\sum_{n=1}^{m} X_n | A_n | = \sum_{n=1}^{m-1} \left(\sum_{r=1}^{n} X_r \right) \Delta | A_n | + | A_m | \left(\sum_{r=1}^{m} X_r \right)$$

$$= O(1) \sum_{n=1}^{m-1} \left(\sum_{r=1}^{n} r^{-\beta}(\log r)^{-\gamma} r^{\beta}(\log r)^{\gamma} X_r \right) \Delta | A_n |$$

$$+ O(1) \left(\sum_{r=1}^{m} r^{-\beta}(\log r)^{-\gamma} r^{\beta}(\log r)^{\gamma} X_r \right) | A_m |$$

$$= O(1) \sum_{n=1}^{m-1} \left(n^{\beta}(\log n)^{\gamma} X_n\right) \Delta | A_n | \sum_{r=1}^{n} r^{-\beta-\in}(\log r)^{-\gamma} r^{\in}$$

$$+ O(1) m^{\beta} X_m | A_m | (\log m)^{\gamma} \sum_{r=1}^{m} r^{-\beta-\in}(\log r)^{-\gamma} r^{\in}, \in < 1-\beta.$$

$$= O(1) \sum_{n=1}^{m-1} \left(n^{\beta}(\log n)^{\gamma} X_n\right) \Delta | A_n | n^{\in} (\log n)^{-\gamma} \sum_{r=1}^{n} r^{-\beta-\in}$$

$$+ O(1) m^{\beta} X_m | A_m | (\log m)^{\gamma} m^{\in} (\log m)^{-\gamma} \sum_{r=1}^{m} r^{-\beta-\in}$$

$$= O(1) \sum_{n=1}^{m} n^{\beta+\in} X_n \Delta | A_n | \left(\int_1^n u^{-\beta-\in} du \right) + O(1) m^{\beta+\in} X_m | A_m | \left(\int_1^m u^{-\beta-\in} du \right)$$

$$= O(1) \sum_{n=1}^{m} n X_n \Delta |A_n| + O(1) m X_m | A_m |$$

$$= O(1) .$$

This establishes (4.2).

5. PROOF OF THE THEOREM

Let (T_n) be the sequence of $(\overline{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} a_n \lambda_n$, then

$$T_n = \frac{1}{P_n}\sum_{\upsilon=0}^{n} p_\upsilon \sum_{r=0}^{\upsilon} a_r\,\lambda_r$$

$$= \frac{1}{P_n}\sum_{\upsilon=0}^{n}\left(P_n - P_{\upsilon-1}\right)a_\nu\,\lambda_\nu$$

Hence for $n \geq 1$

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{\upsilon=1}^{n} P_{\upsilon-1} a_\nu\,\lambda_\nu$$

$$= \frac{p_n}{P_n P_{n-1}}\sum_{\upsilon=1}^{n} \nu\, a_\nu \left(\frac{1}{\nu} P_{\nu-1}\,\lambda_\nu\right)$$

$$= \frac{(n+1)}{n}\frac{p_n}{P_n} t_n \lambda_n + \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1} p_{\nu-1} t_\nu \lambda_\nu \frac{\nu+1}{\nu} + \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1} P_\nu t_\nu \frac{\nu+1}{\nu}\Delta\lambda_\nu$$

$$+ \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1} P_\nu t_\nu \frac{\lambda_{\nu+1}}{\nu}$$

$= T_{n1} + T_{n2} + T_{n3} + T_{n\,4}$ (say).

In order to prove the theorem, using Minkowski's inequality it is enough to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)} \left|T_{n\,j}\right| < \infty \quad , j = 1,2,3,4 .$$

Applying H$\ddot{o}$lder's inequality, we have

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)} \left|T_{n1}\right|^k = \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)} \left|\frac{n+1}{n}\frac{p_n}{P_n} t_n\lambda_n\right|^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)-k} \frac{\left|t_n\right|^k}{X_n^{k-1}}\left(X_n\left|\lambda_n\right|\right)^{k-1}\left|\lambda_n\right|$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)-k} \frac{\left|t_n\right|^k}{X_n^{k-1}}\left|\lambda_n\right|$$

$$= O(1)\sum_{n=1}^{m-1}\left(\sum_{\upsilon=1}^{n}\left(\frac{P_\nu}{p_\nu}\right)^{\mu(\sigma k+k-1)-k} \frac{\left|t_\nu\right|^k}{X_\nu^{k-1}}\right)\Delta\left|\lambda_n\right| + O(1)\sum_{\upsilon=1}^{m}\left(\frac{P_\upsilon}{p_\upsilon}\right)^{\mu(\sigma k+k-1)-k} \frac{\left|t_\nu\right|^k}{X_\nu^{k-1}}\left|\lambda_m\right|$$

$$\leq O(1)\sum_{n=1}^{m-1} X_n\left|A_n\right| + 0(1) X_m\left|\lambda_m\right|$$

$= O(1)$.

Next

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)}\left|T_{n2}\right|^k=\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n-1}p_{\nu-1}\,t_\nu\lambda_\nu\,\frac{\nu+1}{\nu}\right|^k$$

$$=O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)-k}\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}p_{\nu-1}\left|t_\nu\right|^k\left|\lambda_\nu\right|^k\left(\sum_{\nu=1}^{n-1}\frac{p_\nu}{P_{n-1}}\right)^{k-1}$$

$$=O(1)\sum_{\nu=1}^{m}p_{\nu-1}\left|t_\nu\right|^k\left|\lambda_\nu\right|^k\sum_{n=\nu+1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)-k}\frac{1}{P_{n-1}}.$$

$$=O(1)\sum_{\nu=1}^{n}\left(\frac{P_\nu}{p_\nu}\right)^{\mu(\sigma k+k-1)-k}\left|t_\nu\right|^k\left|\lambda_\nu\right|^k$$

$= O(1)$, as in the case of T_{n1}.

Next

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\sigma k+k-1}\left|T_{n3}\right|^k=\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{\upsilon=1}^{n-1}P_\upsilon t_\upsilon\,\frac{\nu+1}{\nu}\Delta\lambda_\upsilon\right|^k$$

$$=O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)-k}\frac{1}{P_{n-1}^k}\sum_{\upsilon=1}^{n-1}P_\nu^k\,\frac{\left|t_\upsilon\right|^k}{X_\nu^{k-1}}\left|\Delta\lambda_\upsilon\right|\left(\sum_{\upsilon=1}^{n-1}X_\upsilon\left|\Delta\lambda_\nu\right|\right)^{k-1}$$

$$\leq O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)-k}\frac{1}{P_{n-1}^k}\sum_{\upsilon=1}^{n-1}P_\nu^k\,\frac{\left|t_\upsilon\right|^k}{X_\nu^{k-1}}\left|A_\upsilon\right|\left(\sum_{\upsilon=1}^{n-1}X_\upsilon\left|A_\nu\right|\right)^{k-1}$$

$$=O(1)\sum_{\upsilon=1}^{m}P_\nu^k\,\frac{\left|t_\upsilon\right|^k}{X_\nu^{k-1}}\left|A_\upsilon\right|\sum_{n=\upsilon+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)-k}\frac{1}{P_{n-1}^k}.$$

$$=O(1)\sum_{\upsilon=1}^{m}\left(\frac{P_\nu}{p_\nu}\right)^{\mu(\sigma k+k-1)-k+1}\frac{1}{\nu}\frac{\left|t_\upsilon\right|^k}{X_\nu^{k-1}}\left(\nu\left|A_\nu\right|\right)$$

$$=O(1)\sum_{\upsilon=1}^{m-1}\sum_{r=1}^{\upsilon}\left(\frac{P_r}{p_r}\right)^{\mu(\sigma k+k-1)-k+1}\frac{1}{r}\frac{\left|t_r\right|^k}{X_r^{k-1}}\Delta\left(\nu\left|A_\nu\right|\right)+0\left(m\left|A_m\right|\right)\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{\mu(\sigma k+k-1)-k+1}\frac{1}{r}\frac{\left|t_r\right|^k}{X_r^{k-1}}\quad.$$

$$=O(1)\sum_{\upsilon=1}^{m-1}\left(-\left|A_\upsilon\right|+(\nu+1)\left|\Delta\right|\left|A_\upsilon\right|\right)X_\nu+0\left(mX_m\left|A_m\right|\right).$$

$$=O(1)\sum_{\upsilon=1}^{n}X_\upsilon\left|A_\upsilon\right|+0(1)\sum_{\upsilon=1}^{n}\nu X_\upsilon\left|\Delta A_\upsilon\right|+0\left(mX_m\left|A_m\right|\right)$$

$$\leq O(1)\sum_{\upsilon=1}^{n}X_\upsilon\left|A_\upsilon\right|+0(1)\sum_{\upsilon=1}^{n}\nu\,X_\upsilon\delta_\nu+0\left(mX_m\left|A_m\right|\right)$$

$= O(1)$.

Finally

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)}\left|T_{n4}\right|^k=\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n-1}P_\nu t_\nu\frac{\lambda_{\nu+1}}{\nu}\right|^k$$

$$=O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)-k}\frac{1}{P_{n-1}^k}\sum_{\nu=1}^{n-1}\frac{P_\nu}{\nu}\left|t_\nu\right|^k\left|\lambda_\nu\right|^k\left(\sum_{\nu=1}^{n-1}\frac{P_\nu}{\nu}\right)^{k-1}$$

$$=O(1)\sum_{\nu=1}^{m}\frac{P_\nu}{\nu}\left|t_\nu\right|^k\left|\lambda_\nu\right|^k\sum_{n=\nu+1}^{m}\left(\frac{P_n}{p_n}\right)^{\mu(\sigma k+k-1)-k}\frac{1}{P_{n-1}}$$

$$=O(1)\sum_{\nu=1}^{m}\left(\frac{P_\nu}{p_\nu}\right)^{\mu(\sigma k+k-1)-k+1}\frac{\left|t_\nu\right|^k}{X_\nu^{k-1}}\left(X_\nu\left|\lambda_\nu\right|\right)^{k-1}\left|\lambda_\nu\right|$$

$$=O(1)\sum_{\nu=1}^{m}\left(\frac{P_\nu}{p_\nu}\right)^{\mu(\sigma k+k-1)-k+1}\frac{\left|t_\nu\right|^k\left|\lambda_\nu\right|}{\nu X_\nu^{k-1}}$$

$$=O(1)\sum_{\nu=1}^{m-1}\left(\sum_{r=1}^{\nu}\left(\frac{P_r}{p_r}\right)^{\mu(\sigma k+k-1)-k+1}\frac{\left|t_r\right|^k}{rX_r^{k-1}}\right)\left|\Delta\left|\lambda_\nu\right|\right|+O(1)\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{\mu(\sigma k+k-1)-k+1}\frac{\left|t_r\right|^k}{rX_r^{k-1}}$$

$$\leq O(1)\sum_{\nu=1}^{m-1}\left(\sum_{r=1}^{\nu}\left(\frac{P_r}{p_r}\right)^{\mu(\sigma k+k-1)-k+1}\frac{\left|t_r\right|^k}{rX_r^{k-1}}\right)\left|A_\nu\right|+O(1)\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{\mu(\sigma k+k-1)-k+1}\frac{\left|t_r\right|^k}{rX_r^{k-1}}$$

$$=O(1)\sum_{\nu=1}^{m}X_\nu\left|A_\nu\right|+O(1)X_m\left|\lambda_m\right|$$

$$=O(1).$$

This completes the proof of the theorem.

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