Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 5 (2015), pp. 2787-2811

http://www.ripublication.com/gjpam.htm

M/G/1 Feedback Queue with Three Types of Service

S. Vanitha
SSN College of Engineering,
Department of Mathematics.
E-mail: vanithasudhakar05@gmail.com

Abstract

We consider an M/G/1 feedback queue with optional server vacations based on Bernoulli schedule and a single vacation policy. We assume that the server provides three types of service, type 1 with probability p_1, type 2 with probability p_2 and type 3 with probability p_3 with the service times following general distribution and each arriving customer may choose any of the three types of service. However after the completion of each type 1, type 2 service or type 3 service, the customer can feedback to the tail of the original queue with probability p to repeat the service until it is successful or may depart the system with probability q if service happens to be successful. The feedback customer also has the option to choose either type 1, type 2 or type 3 service with probability p_1, p_2 and p_3 respectively. Again at the completion of each type of service, the server can take a vacation with probability θ or may continue to stay in the system with probability $1 - \theta$. Further, we assume that whenever, the server takes a vacation, it is always a single vacation with exponentially distributed vacation period. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results are obtained explicitly. Also the mean queue length and the mean waiting time are computed.

AMS subject classification: 60K25, 60K30.
Keywords: $M/G/1$ queue with Bernoulli feedback, Bernoulli schedule server vacation, Single vacation Policy, Steady state, Stability condition, Mean number in the system, Mean waiting time.

1. Introduction

In many examples such as production system, bank services, computer and communication networks, besides feedback the system have vacation. Vacation queues with different vacation policies including Bernoulli schedules, assuming a single vacation policy or multiple vacation policy have been studied by many researchers.

Levy and Yechiali [12], Fuhrman [9], Doshi [7] and [8], Keilson and Servi [11], Baba [1], Cramer [6], Borthakur and Chaudhury [3], Madan [13], [14] and [15], Choi and Park [5], Takagi [18] and [19], Rosenberg and Yechiali [17], Chaudhury [4], Badamchi Zadeh and Shankar [2] and many others have studied vacation queues with different vacation policies. Madan and Chaudhury [16] have studied a single server queue with two phase of heterogeneous service under Bernoulli schedule and a general vacation time. In this system, without feedback, the server after completing the service can take vacation with probability θ or remain in the system with probability $1 - \theta$.

Madan and Anabosi [15] have studied a single server queue with optional server vacations based on Bernoulli schedules and a single vacation policy. In this system, without feedback, the server provides two types of heterogeneous exponential service and a customer may choose either type of service. Moreover, the server after completing the service can take vacation with probability θ or remain in system with probability $1 - \theta$.

In this paper, we consider a single server vacation queueing model with feedback, in which the server provides three types of service and each arriving customer has the option of choosing any of the three types of service. Further, we assume Bernoulli schedule server vacations, which means that on completion of each type of service the server may take a vacation or may continue staying in the system. When the server is under vacation, the customers arriving during the vacation period have to wait in the queue until the server comes back. We further assume that the three types of service follow general distribution and whenever the server takes a vacation, it is a single vacation with exponentially distributed vacation period.

The rest of the paper is organized as follows. The mathematical description of our model is in section 2 and equations governing the model are given in section 3. The time dependent solution have been obtained in section 4 and the corresponding steady state results have been derived explicitly in section 5. Mean queue length and mean waiting time are computed in section 6 and in section 7 respectively.

2. Mathematical description of the model

We assume the following to describe the queueing model of our study.

- Customers arrive at the system one by one according to a Poisson stream with arrival rate $\lambda(> 0)$.
- The server provides three types of service, type 1, type 2 and type 3 with the service times having general distribution. Let $B_1(v)$ and $b_1(v)$ respectively be the distribution and the density function of the type 1 service.
- The service time of type 2 are assumed to be general with the distribution function $B_2(v)$ and the density function $b_2(v)$.

- The service time of type 3 are assumed to be general with the distribution function $B_3(v)$ and the density function $b_3(v)$.

- Just before the service of a customer starts he may choose type 1 service with probability p_1 or type 2 service with probability p_2, or type 3 service with probability p_3 where $p_1 + p_2 + p_3 = 1$.

- Further, $\mu_i(x)\mathrm{d}x$ is the probability of completion of the i-th type service given that elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, \; i = 1, 2, 3, \tag{2.1}$$

  and therefore,

$$b_i(v) = \mu_i(v)e^{-\int\limits_0^v \mu_i(x)\mathrm{d}x}, \; i = 1, 2, 3. \tag{2.2}$$

- After the completion of each type of service, the server may take a vacation with probability θ or may continue staying in the system with probability $1 - \theta$.

- The vacation periods are exponentially distributed with mean vacation time $\dfrac{1}{\beta}$.

- On returning from vacation the server instantly starts serving the customer at the head of the queue, if any.

- Moreover, after the completion of each type 1 or type 2 or type 3 service, if the customer is dissatisfied with its service, he can immediately join the tail of the original queue as a feedback customer for receiving another service with probability p. Otherwise the customer may depart forever from the system with probability $q = 1 - p$. After joining the tail of the queue, the feedback customer again has the option to choose either type 1 service with probability p_1 or type 2 service with probability p_2 or type 3 with probability p_3. Further, we do not distinguish the new arrival with feedback.

- The customer both newly arrived and those that are fed back are served in the order in which they join the tail of the original queue.

- The customers are served according to the first come, first served rule.

- The inter-arrival times, the service times of each type of service and the vacation times are independent of each other.

3. Equations governing the system

We define

$P_n^{(1)}(x,t)$ = Probability that at time t, there are $n(\geq 0)$ customers in the queue excluding one customer in the first type of service and the elapsed service time for this customer is x. Consequently $P_n^{(1)}(t) = \int_0^\infty P_n^{(1)}(x,t)\mathrm{d}x$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the first type of service irrespective of the value of x.

$P_n^{(2)}(x,t)$ = Probability that at time t, there are $n(\geq 0)$ customers in the queue excluding one customer in the second type of service and the elapsed service time for this customer is x. Consequently $P_n^{(2)}(t) = \int_0^\infty P_n^{(2)}(x,t)\mathrm{d}x$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the second type of service irrespective of the value of x.

$P_n^{(3)}(x,t)$ = Probability that at time t, there are $n(\geq 0)$ customers in the queue excluding one customer in the third type of service and the elapsed service time for this customer is x. Consequently $P_n^{(3)}(t) = \int_0^\infty P_n^{(3)}(x,t)\mathrm{d}x$ denotes the probability that at time t there are n customers in the queue excluding the one customer in the third type of service irrespective of the value of x.

$V_n(t)$ = Probability that at time t, there are $n(\geq 0)$ customers in the queue and the server is on vacation.

$Q(t)$ = Probability that at time t, there is no customer in the queue or in service and the server is idle but available in the system.

The model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + (\lambda + \mu_1(x))P_n^{(1)}(x,t) = \lambda P_{n-1}^{(1)}(x,t) \quad n = 1, 2, \ldots, \tag{3.1}$$

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t) + \frac{\partial}{\partial t}P_0^{(1)}(x,t) + (\lambda + \mu_1(x))P_0^{(1)}(x,t) = 0, \tag{3.2}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t)+\frac{\partial}{\partial t}P_n^{(2)}(x,t)+(\lambda+\mu_2(x))P_n^{(2)}(x,t) = \lambda P_{n-1}^{(2)}(x,t) \quad n=1,2,\ldots, \tag{3.3}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t)+\frac{\partial}{\partial t}P_0^{(2)}(x,t)+(\lambda+\mu_2(x))P_0^{(2)}(x,t) = 0, \tag{3.4}$$

$$\frac{\partial}{\partial x}P_n^{(3)}(x,t)+\frac{\partial}{\partial t}P_n^{(3)}(x,t)+(\lambda+\mu_3(x))P_n^{(3)}(x,t) = \lambda P_{n-1}^{(3)}(x,t) \quad n=1,2,\ldots, \tag{3.5}$$

$$\frac{\partial}{\partial x}P_0^{(3)}(x,t)+\frac{\partial}{\partial t}P_0^{(3)}(x,t)+(\lambda+\mu_3(x))P_0^{(3)}(x,t) = 0, \tag{3.6}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_0(t) &= -(\lambda+\beta)V_0(t)+\theta q\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)\mathrm{d}x \\ &\quad +\theta q\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)\mathrm{d}x + +\theta q\int_0^\infty P_0^{(3)}(x,t)\mu_3(x)\mathrm{d}x,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}V_n(t) &= -(\lambda+\beta)V_n(t)+\lambda V_{n-1}(t)+\theta p\int_0^\infty P_{n-1}^{(1)}(x,t)\mu_1(x)\mathrm{d}x \\ &\quad +\theta p\int_0^\infty P_{n-1}^{(2)}(x,t)\mu_2(x)\mathrm{d}x+\theta p\int_0^\infty P_{n-1}^{(3)}(x,t)\mu_3(x)\mathrm{d}x \\ &\quad +\theta q\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)\mathrm{d}x+\theta q\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)\mathrm{d}x \\ &\quad +\theta q\int_0^\infty P_n^{(3)}(x,t)\mu_3(x)\mathrm{d}x,\ n=1,2,\ldots,\end{aligned} \tag{3.8}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}Q(t) &= -\lambda Q(t)+\beta V_0(t)+(1-\theta)q\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)\mathrm{d}x \\ &\quad +(1-\theta)q\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)\mathrm{d}x+(1-\theta)q\int_0^\infty P_0^{(3)}(x,t)\mu_3(x)\mathrm{d}x\ ,\end{aligned} \tag{3.9}$$

Equations (3)–(11) are to be solved subject to the following boundary conditions:

$$\begin{aligned}
P_0^{(1)}(0,t) &= Q(t)\lambda p_1 + p_1\beta V_1(t) + p_1(1-\theta)p\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)\mathrm{d}x \\
&+p_1(1-\theta)p\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)\mathrm{d}x + p_1(1-\theta)p\int_0^\infty P_0^{(3)}(x,t)\mu_3(x)\mathrm{d}x \\
&+p_1(1-\theta)q\int_0^\infty P_1^{(1)}(x,t)\mu_1(x)\mathrm{d}x + p_1(1-\theta)q\int_0^\infty P_1^{(2)}(x,t)\mu_2(x)\mathrm{d}x \\
&+p_1(1-\theta)q\int_0^\infty P_1^{(3)}(x,t)\mu_3(x)\mathrm{d}x, \qquad (3.10)
\end{aligned}$$

$$\begin{aligned}
P_n^{(1)}(0,t) &= p_1\beta V_{n+1}(t) + p_1(1-\theta)p\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)\mathrm{d}x \\
&+p_1(1-\theta)p\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)\mathrm{d} + p_1(1-\theta)p\int_0^\infty P_n^{(3)}(x,t)\mu_3(x)\mathrm{d}x \\
&+p_1(1-\theta)q\int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)\mathrm{d}x + p_1(1-\theta)q\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)\mathrm{d}x \\
&+p_1(1-\theta)q\int_0^\infty P_{n+1}^{(3)}(x,t)\mu_3(x)\mathrm{d}x,\ n=1,2,\ldots, \qquad (3.11)
\end{aligned}$$

$$\begin{aligned}
P_0^{(2)}(0,t) &= Q(t)\lambda p_2 + p_2\beta V_1(t) + p_2(1-\theta)p\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)\mathrm{d}x \\
&+p_2(1-\theta)p\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)\mathrm{d}x + p_2(1-\theta)p\int_0^\infty P_0^{(3)}(x,t)\mu_3(x)\mathrm{d}x \\
&+p_2(1-\theta)q\int_0^\infty P_1^{(1)}(x,t)\mu_1(x)\mathrm{d}x + p_2(1-\theta)q\int_0^\infty P_1^{(2)}(x,t)\mu_2(x)\mathrm{d}x, \\
&+p_2(1-\theta)q\int_0^\infty P_1^{(3)}(x,t)\mu_3(x)\mathrm{d}x \qquad (3.12)
\end{aligned}$$

$$
\begin{aligned}
P_n^{(2)}(0,t) &= p_2\beta V_{n+1}(t) + p_2(1-\theta)p\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)\mathrm{d}x \\
&\quad + p_2(1-\theta)p\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)\mathrm{d}x + p_2(1-\theta)p\int_0^\infty P_n^{(3)}(x,t)\mu_3(x)\mathrm{d}x \\
&\quad + p_2(1-\theta)q\int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)\mathrm{d}x + p_2(1-\theta)q\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)\mathrm{d}x \\
&\quad + p_2(1-\theta)p\int_0^\infty P_n^{(3)}(x,t)\mu_3(x)\mathrm{d}x,\ n=1,2,\ldots
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
P_0^{(3)}(0,t) &= Q(t)\lambda p_3 + p_3\beta V_1(t) + p_3(1-\theta)p\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)\mathrm{d}x \\
&\quad + p_3(1-\theta)p\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)\mathrm{d}x + p_3(1-\theta)p\int_0^\infty P_0^{(3)}(x,t)\mu_3(x)\mathrm{d}x \\
&\quad + p_3(1-\theta)q\int_0^\infty P_1^{(1)}(x,t)\mu_1(x)\mathrm{d}x + p_3(1-\theta)q\int_0^\infty P_1^{(2)}(x,t)\mu_2(x)\mathrm{d}x, \\
&\quad + p_3(1-\theta)q\int_0^\infty P_1^{(3)}(x,t)\mu_3(x)\mathrm{d}x
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
P_n^{(3)}(0,t) &= p_3\beta V_{n+1}(t) + p_3(1-\theta)p\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)\mathrm{d}x \\
&\quad + p_3(1-\theta)p\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)\mathrm{d}x + p_3(1-\theta)p\int_0^\infty P_n^{(3)}(x,t)\mu_3(x)\mathrm{d}x \\
&\quad + p_3(1-\theta)q\int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)\mathrm{d}x + p_3(1-\theta)q\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)\mathrm{d}x \\
&\quad + p_3(1-\theta)p\int_0^\infty P_n^{(3)}(x,t)\mu_3(x)\mathrm{d}x,\ n=1,2,\ldots
\end{aligned}
\tag{3.15}
$$

We assume that initially there is no customer in the system, the server is not under vacation and the server is idle. So the initial conditions are

$$V_0(0) = V_n(0) = 0,\ Q(0) = 1 \text{ and } P_n^j(0) = 0 \text{ for } n = 0, 1, 2, \dots, j = 1, 2, 3. \tag{3.16}$$

4. Generating functions of the queue length: The time-dependent solution

In this section we obtain the transient solution for the above set of differential-difference equations.

We define the probability generating functions,

$$P_q^{(1)}(x, z, t) = \sum_{n=0}^{\infty} z^n P_n^{(1)}(x, t),\ \ P_q^{(1)}(z, t) = \sum_{n=0}^{\infty} z^n P_n^{(1)}(t), \tag{4.1}$$

$$P_q^{(2)}(x, z, t) = \sum_{n=0}^{\infty} z^n P_n^{(2)}(x, t),\ \ P_q^{(2)}(z, t) = \sum_{n=0}^{\infty} z^n P_n^{(2)}(t), \tag{4.2}$$

$$P_q^{(3)}(x, z, t) = \sum_{n=0}^{\infty} z^n P_n^{(3)}(x, t),\ \ P_q^{(3)}(z, t) = \sum_{n=0}^{\infty} z^n P_n^{(3)}(t), \tag{4.3}$$

$$V(z, t) = \sum_{n=0}^{\infty} z^n V_n(t), \tag{4.4}$$

which are convergent inside the 'circle' given by $|z| \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\overline{f}(s) = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t,\ \Re(s) > 0. \tag{4.5}$$

Taking the Laplace transforms of equations $(3) - (13)$ and using (18), we obtain

$$\frac{\partial}{\partial x}\overline{P}_n^{(1)}(x, s) + (s + \lambda + \mu_1(x))\overline{P}_n^{(1)}(x, s) = \lambda\overline{P}_{n-1}^{(1)}(x, s) \quad n = 1, 2, \dots, \tag{4.6}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(1)}(x, s) + (s + \lambda + \mu_1(x))\overline{P}_0^{(1)}(x, s) = 0, \tag{4.7}$$

$$\frac{\partial}{\partial x}\overline{P}_n^{(2)}(x, s) + (s + \lambda + \mu_2(x))\overline{P}_n^{(2)}(x, s) = \lambda\overline{P}_{n-1}^{(2)}(x, s) \quad n = 1, 2, \dots \tag{4.8}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(2)}(x,s)+(s+\lambda+\mu_2(x))\overline{P}_0^{(2)}(x,s) = 0, \tag{4.9}$$

$$\frac{\partial}{\partial x}\overline{P}_n^{(3)}(x,s)+(s+\lambda+\mu_3(x))\overline{P}_n^{(3)}(x,s) = \lambda\overline{P}_{n-1}^{(3)}(x,s) \quad n=1,2,\dots \tag{4.10}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(3)}(x,s)+(s+\lambda+\mu_3(x))\overline{P}_0^{(3)}(x,s) = 0, \tag{4.11}$$

$$\begin{aligned}(s+\lambda)\overline{Q}(s) &= 1+\beta\overline{V}_0(s)+(1-\theta)q\int_0^\infty \overline{P}_0^{(1)}(x,s)\mu_1(x)\mathrm{d}x \\ &\quad +(1-\theta)q\int_0^\infty \overline{P}_0^{(2)}(x,s)\mu_2(x)\mathrm{d}x \\ &\quad +(1-\theta)q\int_0^\infty \overline{P}_0^{(3)}(x,s)\mu_3(x)\mathrm{d}x,\end{aligned} \tag{4.12}$$

$$\begin{aligned}(s+\lambda+\beta)\overline{V}_0(s) &= \theta q\int_0^\infty \overline{P}_0^{(1)}(x,s)\mu_1(x)\mathrm{d}x+\theta q\int_0^\infty \overline{P}_0^{(2)}(x,s)\mu_2(x)\mathrm{d}x \\ &\quad +\theta q\int_0^\infty \overline{P}_0^{(3)}(x,s)\mu_3(x)\mathrm{d}x,\end{aligned} \tag{4.13}$$

$$\begin{aligned}(s+\lambda+\beta)\overline{V}_n(s) &= \lambda\overline{V}_{n-1}(s)+\theta p\int_0^\infty \overline{P}_{n-1}^{(1)}(x,s)\mu_1(x)\mathrm{d}x \\ &\quad +\theta p\int_0^\infty \overline{P}_{n-1}^{(2)}(x,s)\mu_2(x)\mathrm{d}x+\theta p\int_0^\infty \overline{P}_{n-1}^{(3)}(x,s)\mu_2(x)\mathrm{d}x \\ &\quad +\theta q\int_0^\infty \overline{P}_n^{(1)}(x,s)\mu_1(x)\mathrm{d}x+\theta q\int_0^\infty \overline{P}_n^{(2)}(x,s)\mu_2(x)\mathrm{d}x \\ &\quad +\theta q\int_0^\infty \overline{P}_n^{(3)}(x,s)\mu_3(x)\mathrm{d}x,\ n=1,2,\dots,\end{aligned} \tag{4.14}$$

$$
\begin{aligned}
\overline{P}_0^{(1)}(0,s) &= \overline{Q}(s)\lambda p_1 + p_1\beta\overline{V}_1(s)\\
&+p_1(1-\theta)p\int_0^\infty \overline{P}_0^{(1)}(x,s)\mu_1(x)\mathrm{d}x\\
&+p_1(1-\theta)p\int_0^\infty \overline{P}_0^{(2)}(x,s)\mu_2(x)\mathrm{d}x\\
&+p_1(1-\theta)p\int_0^\infty \overline{P}_0^{(3)}(x,s)\mu_3(x)\mathrm{d}x\\
&+p_1(1-\theta)q\int_0^\infty \overline{P}_1^{(1)}(x,s)\mu_1(x)\mathrm{d}x\\
&+p_1(1-\theta)q\int_0^\infty \overline{P}_1^{(2)}(x,s)\mu_2(x)\mathrm{d}x\\
&+p_1(1-\theta)q\int_0^\infty \overline{P}_1^{(3)}(x,s)\mu_3(x)\mathrm{d}x, \qquad (4.15)
\end{aligned}
$$

$$
\begin{aligned}
\overline{P}_n^{(1)}(0,s) &= p_1\beta\overline{V}_{n+1}(s)\\
&+p_1(1-\theta)p\int_0^\infty \overline{P}_n^{(1)}(x,s)\mu_1(x)\mathrm{d}x\\
&+p_1(1-\theta)p\int_0^\infty \overline{P}_n^{(2)}(x,s)\mu_2(x)\mathrm{d}x\\
&+p_1(1-\theta)p\int_0^\infty \overline{P}_n^{(3)}(x,s)\mu_3(x)\mathrm{d}x\\
&+p_1(1-\theta)q\int_0^\infty \overline{P}_{n+1}^{(1)}(x,s)\mu_1(x)\mathrm{d}x\\
&+p_1(1-\theta)q\int_0^\infty \overline{P}_{n+1}^{(2)}(x,s)\mu_2(x)\mathrm{d}x,\\
&+p_1(1-\theta)q\int_0^\infty \overline{P}_{n+1}^{(3)}(x,s)\mu_3(x)\mathrm{d}x \; n=1,2,\ldots, \qquad (4.16)
\end{aligned}
$$

$$\overline{P}_0^{(2)}(0,s) = \overline{Q}(s)\lambda p_2 + p_2\beta\overline{V}_1(s) + p_2(1-\theta)p\int_0^\infty \overline{P}_0^{(1)}(x,s)\mu_1(x)\mathrm{d}x$$
$$+ p_2(1-\theta)p\int_0^\infty \overline{P}_0^{(2)}(x,s)\mu_2(x)\mathrm{d}x + p_2(1-\theta)p\int_0^\infty \overline{P}_0^{(3)}(x,s)\mu_3(x)\mathrm{d}x$$
$$+ p_2(1-\theta)q\int_0^\infty \overline{P}_1^{(1)}(x,s)\mu_1(x)\mathrm{d}x + p_2(1-\theta)q\int_0^\infty \overline{P}_1^{(2)}(x,s)\mu_2(x)\mathrm{d}x$$
$$+ p_2(1-\theta)q\int_0^\infty \overline{P}_1^{(3)}(x,s)\mu_3(x)\mathrm{d}x, \tag{4.17}$$

$$P_n^{(2)}(0,s) = p_2\beta\overline{V}_{n+1}(s) + p_2(1-\theta)p\int_0^\infty \overline{P}_n^{(1)}(x,s)\mu_1(x)\mathrm{d}x$$
$$+ p_2(1-\theta)p\int_0^\infty \overline{P}_n^{(2)}(x,s)\mu_2(x)\mathrm{d}x + p_2(1-\theta)p\int_0^\infty \overline{P}_n^{(3)}(x,s)\mu_3(x)\mathrm{d}x$$
$$+ p_2(1-\theta)q\int_0^\infty \overline{P}_{n+1}^{(1)}(x,s)\mu_1(x)\mathrm{d}x + p_2(1-\theta)q\int_0^\infty \overline{P}_{n+1}^{(2)}(x,s)\mu_2(x)\mathrm{d}x$$
$$+ p_2(1-\theta)q\int_0^\infty \overline{P}_{n+1}^{(3)}(x,s)\mu_3(x)\mathrm{d}x,\ n = 1,2,\ldots, \tag{4.18}$$

$$\overline{P}_0^{(3)}(0,s) = \overline{Q}(s)\lambda p_3 + p_3\beta\overline{V}_1(s) + p_3(1-\theta)p\int_0^\infty \overline{P}_0^{(1)}(x,s)\mu_1(x)\mathrm{d}x$$
$$+ p_3(1-\theta)p\int_0^\infty \overline{P}_0^{(2)}(x,s)\mu_2(x)\mathrm{d}x + p_3(1-\theta)p\int_0^\infty \overline{P}_0^{(3)}(x,s)\mu_3(x)\mathrm{d}x$$
$$+ p_3(1-\theta)q\int_0^\infty \overline{P}_1^{(1)}(x,s)\mu_1(x)\mathrm{d}x + p_3(1-\theta)q\int_0^\infty \overline{P}_1^{(2)}(x,s)\mu_2(x)\mathrm{d}x$$
$$+ p_3(1-\theta)q\int_0^\infty \overline{P}_1^{(3)}(x,s)\mu_3(x)\mathrm{d}x, \tag{4.19}$$

$$
\begin{aligned}
P_n^{(3)}(0,s) = p_3\beta\overline{V}_{n+1}(s) + p_3(1-\theta)p\int_0^\infty \overline{P}_n^{(1)}(x,s)\mu_1(x)\mathrm{d}x \\
+ p_3(1-\theta)p\int_0^\infty \overline{P}_n^{(2)}(x,s)\mu_2(x)\mathrm{d}x + p_3(1-\theta)p\int_0^\infty \overline{P}_n^{(3)}(x,s)\mu_3(x)\mathrm{d}x \\
+ p_3(1-\theta)q\int_0^\infty \overline{P}_{n+1}^{(1)}(x,s)\mu_1(x)\mathrm{d}x + p_3(1-\theta)q\int_0^\infty \overline{P}_{n+1}^{(2)}(x,s)\mu_2(x)\mathrm{d}x \\
+ p_3(1-\theta)q\int_0^\infty \overline{P}_{n+1}^{(3)}(x,s)\mu_3(x)\mathrm{d}x,\ n = 1,2,\ldots. \qquad (4.20)
\end{aligned}
$$

We multiply equations (24) and (25) by suitable powers of z, sum over n and use (19) and simplify. We thus have after algebraic simplifications

$$
\frac{\partial}{\partial x}\overline{P}_q^{(1)}(x,z,s) + (s+\lambda-\lambda z+\mu_1(x))\overline{P}_q^{(1)}(x,z,s) = 0. \qquad (4.21)
$$

Performing similar operations on equations (26)–(29) and using (20) and (21), we have

$$
\frac{\partial}{\partial x}\overline{P}_q^{(2)}(x,z,s) + (s+\lambda-\lambda z+\mu_2(x))\overline{P}_q^{(2)}(x,z,s) = 0, \qquad (4.22)
$$

$$
\frac{\partial}{\partial x}\overline{P}_q^{(3)}(x,z,s) + (s+\lambda-\lambda z+\mu_3(x))\overline{P}_q^{(3)}(x,z,s) = 0, \qquad (4.23)
$$

Multiplying equations (31) and (32) by suitable powers of z, summing over n and using (22), leads to the following after simplification:

$$
\begin{aligned}
(s+\lambda+\beta-\lambda z)\overline{V}(z,s) &= \theta(q+pz)\int_0^\infty \overline{P}_q^{(1)}(x,z,s)\mu_1(x)\mathrm{d}x \\
&+\theta(q+pz)\int_0^\infty \overline{P}_q^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x \\
&+\theta(q+pz)\int_0^\infty \overline{P}_q^{(3)}(x,z,s)\mu_3(x)\mathrm{d}x. \qquad (4.24)
\end{aligned}
$$

Next, we multiply both sides of equation (33) by z, multiply both sides of equation (34) by z^{n+1}, sum over n from 1 to ∞, add the two results and use (19)–(22). Thus we obtain after mathematical adjustments

$$\begin{aligned} z\overline{P}_q^{(1)}(0,z,s) &= p_1\lambda\overline{Q}(s)[z-1] + p_1\left[1-s\overline{Q}(s)\right] + \beta p_1\overline{V}(z,s) \\ &+p_1(1-\theta)(q+pz)\int_0^\infty \overline{P}_q^{(1)}(x,z,s)\mu_1(x)\mathrm{d}x \\ &+p_1(1-\theta)(q+pz)\int_0^\infty \overline{P}_q^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x \\ &+p_1(1-\theta)(q+pz)\int_0^\infty \overline{P}_q^{(3)}(x,z,s)\mu_3(x)\mathrm{d}x. \end{aligned} \tag{4.25}$$

Similarly multiplying equations (35)–(38) by suitable powers of z and then adding, we obtain,

$$\begin{aligned} z\overline{P}_q^{(2)}(0,z,s) &= p_2\lambda\overline{Q}(s)[z-1] + p_2\left[1-s\overline{Q}(s)\right] + \beta p_2\overline{V}(z,s) \\ &+p_2(1-\theta)(q+pz)\int_0^\infty \overline{P}_q^{(1)}(x,z,s)\mu_1(x)\mathrm{d}x \\ &+p_2(1-\theta)(q+pz)\int_0^\infty \overline{P}_q^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x \\ &+p_2(1-\theta)(q+pz)\int_0^\infty \overline{P}_q^{(3)}(x,z,s)\mu_3(x)\mathrm{d}x, \end{aligned} \tag{4.26}$$

$$\begin{aligned} z\overline{P}_q^{(3)}(0,z,s) &= p_3\lambda\overline{Q}(s)[z-1] + p_3\left[1-s\overline{Q}(s)\right] + \beta p_3\overline{V}(z,s) \\ &+p_3(1-\theta)(q+pz)\int_0^\infty \overline{P}_q^{(1)}(x,z,s)\mu_1(x)\mathrm{d}x \\ &+p_3(1-\theta)(q+pz)\int_0^\infty \overline{P}_q^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x \\ &+p_3(1-\theta)(q+pz)\int_0^\infty \overline{P}_q^{(3)}(x,z,s)\mu_3(x)\mathrm{d}x. \end{aligned} \tag{4.27}$$

Integrating equations (39), (40) and (41) between 0 and x, we obtain

$$\overline{P}_q^{(1)}(x,z,s) = \overline{P}_q^{(1)}(0,z,s)\, e^{-(s+\lambda-\lambda z)x-\int_0^x \mu_1(t)dt} \tag{4.28}$$

$$\overline{P}_q^{(2)}(x,z,s) = \overline{P}_q^{(2)}(0,z,s)\, e^{-(s+\lambda-\lambda z)x-\int_0^x \mu_2(t)dt} \tag{4.29}$$

$$\overline{P}_q^{(3)}(x,z,s) = \overline{P}_q^{(3)}(0,z,s)\, e^{-(s+\lambda-\lambda z)x-\int_0^x \mu_3(t)dt} \tag{4.30}$$

where $\overline{P}_q^{(1)}(0,z,s)$, $\overline{P}_q^{(2)}(0,z,s)$ and $\overline{P}_q^{(3)}(0,z,s)$ are given by equations (43), (44) and (45). After integrating equation (46) with respect to x, we have,

$$\overline{P}_q^{(1)}(z,s) = \overline{P}_q^{(1)}(0,z,s)\left[\frac{1-\overline{B}_1(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], \tag{4.31}$$

where

$$\overline{B}_1(s+\lambda-\lambda z) = \int_0^\infty e^{-(s+\lambda-\lambda z)x} dB_1(x). \tag{4.32}$$

Now from equation (46), after some simplifications and using equation (2), we obtain,

$$\int_0^\infty \overline{P}_q^{(1)}(x,z,s)\mu_1(x)\mathrm{d}x = \overline{P}_q^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z). \tag{4.33}$$

We now integrate equation (47) with respect to x, to get

$$\overline{P}_q^{(2)}(z,s) = \overline{P}_q^{(2)}(0,z,s)\left[\frac{1-\overline{B}_2(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], \tag{4.34}$$

where

$$\overline{B}_2(s+\lambda-\lambda z) = \int_0^\infty e^{-(s+\lambda-\lambda z)x} dB_2(x). \tag{4.35}$$

We see that by virtue of equation (47), we have,

$$\int_0^\infty \overline{P}_q^{(2)}(x,z,s)\mu_2(x)\mathrm{d}x = \overline{P}_q^{(2)}(0,z,s)\overline{B}_2(s+\lambda-\lambda z). \tag{4.36}$$

We now integrate equation (48) with respect to x, to get

$$\overline{P}_q^{(3)}(z,s) = \overline{P}_q^{(3)}(0,z,s)\left[\frac{1-\overline{B}_3(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], \tag{4.37}$$

where

$$\overline{B}_3(s+\lambda-\lambda z)=\int_0^\infty e^{-(s+\lambda-\lambda z)x}dB_3(x). \tag{4.38}$$

We see that by virtue of equation (48), we have,

$$\int_0^\infty \overline{P}_q^{(3)}(x,z,s)\mu_3(x)\mathrm{d}x=\overline{P}_q^{(3)}(0,z,s)\overline{B}_3(s+\lambda-\lambda z). \tag{4.39}$$

We now substitute the value of $\overline{V}(z,s)$ from equation (42) into equation (43), (44) and (45) and also making use of equations (51), (54) and (57),we obtain after simplifications

$$\begin{aligned} z\overline{P}_q^{(1)}(0,z,s) &= p_1\lambda\overline{Q}(s)[z-1]+p_1\left[1-s\overline{Q}(s)\right]+\beta p_1\overline{V}(z,s) \\ &\quad +p_1(1-\theta)(q+pz)\overline{P}_q^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z) \\ &\quad +p_1(1-\theta)(q+pz)\overline{P}_q^{(2)}(0,z,s)\overline{B}_2(s+\lambda-\lambda z) \\ &\quad +p_1(1-\theta)(q+pz)\overline{P}_q^{(3)}(0,z,s)\overline{B}_3(s+\lambda-\lambda z), \end{aligned} \tag{4.40}$$

$$\begin{aligned} z\overline{P}_q^{(2)}(0,z,s) &= p_2\lambda\overline{Q}(s)[z-1]+p_2\left[1-s\overline{Q}(s)\right]+\beta p_2\overline{V}(z,s) \\ &\quad +p_2(1-\theta)(q+pz)\overline{P}_q^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z) \\ &\quad +p_2(1-\theta)(q+pz)\overline{P}_q^{(2)}(0,z,s)\overline{B}_2(s+\lambda-\lambda z) \\ &\quad +p_2(1-\theta)(q+pz)\overline{P}_q^{(3)}(0,z,s)\overline{B}_3(s+\lambda-\lambda z), \end{aligned} \tag{4.41}$$

$$\begin{aligned} z\overline{P}_q^{(3)}(0,z,s) &= p_3\lambda\overline{Q}(s)[z-1]+p_3\left[1-s\overline{Q}(s)\right]+\beta p_3\overline{V}(z,s) \\ &\quad +p_3(1-\theta)(q+pz)\overline{P}_q^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z) \\ &\quad +p_3(1-\theta)(q+pz)\overline{P}_q^{(2)}(0,z,s)\overline{B}_2(s+\lambda-\lambda z) \\ &\quad +p_3(1-\theta)(q+pz)\overline{P}_q^{(3)}(0,z,s)\overline{B}_3(s+\lambda-\lambda z). \end{aligned} \tag{4.42}$$

Further, from equation (42), the above equations (58), (59) and (60) can now be written as

$$\begin{aligned} &\left(g_1-\frac{\beta p_1\theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}\right)\overline{P}_q^{(1)}(0,z,s) \\ &\qquad =\left[p_1(1-\theta)(q+pz)\overline{B}_2(s+\lambda-\lambda z)\right. \\ &\qquad\quad \left.+\frac{\beta p_1\theta(q+pz)\overline{B}_2(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}\right]\overline{P}_q^{(2)}(0,z,s) \end{aligned}$$

$$+\left[p_1(1-\theta)(q+pz)\overline{B}_3(s+\lambda-\lambda z) + \frac{\beta p_1\theta(q+pz)\overline{B}_3(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}\right]\overline{P}_q^{(3)}(0,z,s) + p_1\lambda\overline{Q}(s)[z-1] + p_1(1-s\overline{Q}(s)) \;, \tag{4.43}$$

$$\left(g_2 - \frac{\beta p_2\theta(q+pz)\overline{B}_2(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}\right)\overline{P}_q^{(2)}(0,z,s) = \left[p_2(1-\theta)(q+pz)\overline{B}_1(s+\lambda-\lambda z) + \frac{\beta p_2\theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}\right]\overline{P}_q^{(1)}(0,z,s) + \left[p_2(1-\theta)(q+pz)\overline{B}_3(s+\lambda-\lambda z) + \frac{\beta p_2\theta(q+pz)\overline{B}_3(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}\right]\overline{P}_q^{(3)}(0,z,s) + p_2\lambda\overline{Q}(s)[z-1] + p_2(1-s\overline{Q}(s)) \;. \tag{4.44}$$

$$\left(g_3 - \frac{\beta p_3\theta(q+pz)\overline{B}_3(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}\right)\overline{P}_q^{(3)}(0,z,s) = \left[p_3(1-\theta)(q+pz)\overline{B}_1(s+\lambda-\lambda z) + \frac{\beta p_3\theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}\right]\overline{P}_q^{(1)}(0,z,s) + \left[p_3(1-\theta)(q+pz)\overline{B}_2(s+\lambda-\lambda z) + \frac{\beta p_3\theta(q+pz)\overline{B}_2(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}\right]\overline{P}_q^{(2)}(0,z,s) + p_3\lambda\overline{Q}(s)[z-1] + p_3(1-s\overline{Q}(s)) \;. \tag{4.45}$$

where

$$g_1 = z - p_1(1-\theta)(q+pz)\overline{B}_1(s+\lambda-\lambda z) \;, \tag{4.46}$$
$$g_2 = z - p_2(1-\theta)(q+pz)\overline{B}_2(s+\lambda-\lambda z), \tag{4.47}$$
$$g_3 = z - p_3(1-\theta)(q+pz)\overline{B}_3(s+\lambda-\lambda z). \tag{4.48}$$

Next, we write equations (61), (62) and (63) in matrix form as

$$\begin{bmatrix} h_1(z) & -u_2(z) & -k_3(z) \\ -k_1(z) & h_2(z) & -u_3(z) \\ -u_1(z) & -k_2(z) & h_3(z) \end{bmatrix} \begin{bmatrix} \overline{P}_q^{(1)}(0,z,s) \\ \overline{P}_q^{(2)}(0,z,s) \\ \overline{P}_q^{(3)}(0,z,s) \end{bmatrix}$$
$$= \begin{bmatrix} p_1\lambda\overline{Q}(s)(z-1) + p_1(1-s\overline{Q}(s)) \\ p_2\lambda\overline{Q}(s)(z-1) + p_2(1-s\overline{Q}(s)) \\ p_3\lambda\overline{Q}(s)(z-1) + p_3(1-s\overline{Q}(s)) \end{bmatrix}. \tag{4.49}$$

where

$$h_1(z) = g_1 - \frac{\beta p_1\theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}, \tag{4.50}$$

$$h_2(z) = g_2 - \frac{\beta p_2\theta(q+pz)\overline{B}_2(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}, \tag{4.51}$$

$$h_3(z) = g_3 - \frac{\beta p_3\theta(q+pz)\overline{B}_3(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}, \tag{4.52}$$

$$k_1(z) = p_2(1-\theta)(q+pz)\overline{B}_1(s+\lambda-\lambda z) + \frac{\beta p_2\theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}, \tag{4.53}$$

$$k_2(z) = p_3(1-\theta)(q+pz)\overline{B}_2(s+\lambda-\lambda z) + \frac{\beta p_1\theta(q+pz)\overline{B}_2(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}, \tag{4.54}$$

$$k_3(z) = p_1(1-\theta)(q+pz)\overline{B}_3(s+\lambda-\lambda z) + \frac{\beta p_1\theta(q+pz)\overline{B}_3(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}, \tag{4.55}$$

$$u_1(z) = p_3(1-\theta)(q+pz)\overline{B}_1(s+\lambda-\lambda z) + \frac{\beta p_3\theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}, \tag{4.56}$$

$$u_2(z) = p_1(1-\theta)(q+pz)\overline{B}_2(s+\lambda-\lambda z) + \frac{\beta p_1\theta(q+pz)\overline{B}_2(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}, \tag{4.57}$$

$$u_3(z) = p_2(1-\theta)(q+pz)\overline{B}_3(s+\lambda-\lambda z) + \frac{\beta p_2\theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)}{s+\lambda-\lambda z+\beta}. \tag{4.58}$$

We solve the system (67) simultaneously for $\overline{P}_q^{(1)}(0,z,s)$, $\overline{P}_q^{(2)}(0,z,s)$ and $\overline{P}_q^{(3)}(0,z,s)$

and obtain

$$\overline{P}_q^{(1)}(0,z,s) = \frac{\begin{vmatrix} p_1\lambda\overline{Q}(s)(z-1)+p_1(1-s\overline{Q}(s)) & -u_2(z) & -k_3(z) \\ p_2\lambda\overline{Q}(s)(z-1)+p_2(1-s\overline{Q}(s)) & h_2(z) & -u_3(z) \\ p_3\lambda\overline{Q}(s)(z-1)+p_3(1-s\overline{Q}(s)) & -k_2(z) & h_3(z) \end{vmatrix}}{\begin{vmatrix} h_1(z) & -u_2(z) & -k_3(z) \\ -k_1(z) & h_2(z) & -u_3(z) \\ -u_1(z) & -k_2(z) & h_3(z) \end{vmatrix}}, \quad (4.59)$$

$$\overline{P}_q^{(2)}(0,z,s) = \frac{\begin{vmatrix} h_1(z) & p_1\lambda\overline{Q}(s)(z-1)+p_1(1-s\overline{Q}(s)) & -k_3(z) \\ -k_1(z) & p_2\lambda\overline{Q}(s)(z-1)+p_2(1-s\overline{Q}(s)) & -u_3(z) \\ -u_1(z) & p_3\lambda\overline{Q}(s)(z-1)+p_3(1-s\overline{Q}(s)) & h_3(z) \end{vmatrix}}{\begin{vmatrix} h_1(z) & -u_2(z) & -k_3(z) \\ -k_1(z) & h_2(z) & -u_3(z) \\ -u_1(z) & -k_2(z) & h_3(z) \end{vmatrix}}, \quad (4.60)$$

$$\overline{P}_q^{(3)}(0,z,s) = \frac{\begin{vmatrix} h_1(z) & -u_2(z) & p_1\lambda\overline{Q}(s)(z-1)+p_1(1-s\overline{Q}(s)) \\ -k_1(z) & h_2(z) & p_2\lambda\overline{Q}(s)(z-1)+p_2(1-s\overline{Q}(s)) \\ -u_1(z) & -k_2 z & p_3\lambda\overline{Q}(s)(z-1)+p_3(1-s\overline{Q}(s)) \end{vmatrix}}{\begin{vmatrix} h_1(z) & -u_2(z) & -k_3(z) \\ -k_1(z) & h_2(z) & -u_3(z) \\ -u_1(z) & -k_2(z) & h_3(z) \end{vmatrix}} \quad (4.61)$$

Now from equation (42) and using equations (51), (54) and (57), we obtain,

$$\begin{aligned} \overline{V}_q(z,s) &= \left[\frac{\theta(q+pz)}{s+\lambda-\lambda z+\beta}\right]\Big[\overline{P}_q^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z) \\ &\quad + \overline{P}_q^{(2)}(0,z,s)\overline{B}_2(s+\lambda-\lambda z) + \overline{P}_q^{(3)}(0,z,s)\overline{B}_3(s+\lambda-\lambda z)\Big]. \end{aligned} \quad (4.62)$$

where $\overline{P}_q^{(1)}(0,z,s)$, $\overline{P}_q^{(2)}(0,z,s)$ and $\overline{P}_q^{(3)}(0,z,s)$ are given by equations (77), (78) and (79).

Thus $\overline{P}_q^{(1)}(z,s)$, $\overline{P}_q^{(2)}(z,s)$, $\overline{P}_q^{(3)}(z,s)$ and $\overline{V}(z,s)$ can be completely determined from equations (49), (52), (55) and (80) respectively.

5. The Steady State Results

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we supress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t). \tag{5.1}$$

In order to determine $\overline{P}_q^{(1)}(z,s)$, $\overline{P}_q^{(2)}(z,s)$, $\overline{P}_q^{(3)}(z,s)$ and $\overline{V}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (49), (52), (55) and (80) by using initial conditions (77), (78) and (79) . For that purpose, we shall use the normalizing condition

$$P_q^{(1)}(1) + P_q^{(2)}(1) + P_q^{(3)}(1) + V(1) + Q = 1. \tag{5.2}$$

Thus multiplying both sides of equations (49), (52), (55) and (80) by s, taking limit as $s \to 0$, applying property (81), simplifying and using equations (77), (78) and (79), we obtain,

$$P_q^{(1)}(z) = \frac{P_1(z)}{D(z)}, \tag{5.3}$$

$$P_q^{(2)}(z) = \frac{P_2(z)}{D(z)}, \tag{5.4}$$

$$P_q^{(3)}(z) = \frac{P_3(z)}{D(z)}, \tag{5.5}$$

$$V(z) = \left[\frac{\theta\lambda Q(q+pz)(z-1)}{(\lambda-\lambda z+\beta)}\right] \left[\frac{\overline{B}_1(\lambda-\lambda z)P_{11}(z) + \overline{B}_2(\lambda-\lambda z)P_{22}(z) + \overline{B}_3(\lambda-\lambda z)P_{33}(z)}{D(z)}\right] \tag{5.6}$$

where

$$\begin{aligned} P_1(z) = {} & Q\Big[p_1h_2(z)h_3(z) - p_1k_2(z)u_3(z) + p_2u_2(z)h_3(z) + p_3u_3(z)u_2(z) \\ & + p_2k_2(z)k_3(z) + p_3k_3(z)h_2(z)\Big]\left[\overline{B}_1(\lambda-\lambda z) - 1\right], \end{aligned} \tag{5.7}$$

$$\begin{aligned} P_2(z) = {} & Q\Big[p_2h_1(z)h_3(z) + p_3h_1(z)u_3(z) + p_1k_1(z)h_3(z) + p_1u_1(z)u_3(z) \\ & + p_3k_1(z)k_3(z) - p_2u_1(z)k_3(z)\Big]\left[\overline{B}_2(\lambda-\lambda z) - 1\right], \end{aligned} \tag{5.8}$$

$$\begin{aligned} P_3(z) = {} & Q\Big[p_3h_1(z)h_2(z) + p_2k_2(z)h_1(z) - p_3u_2(z)k_1(z) + p_2u_1(z)u_2(z) \\ & + p_1k_1(z)k_2(z) + p_1u_1(z)h_2(z)\Big]\left[\overline{B}_3(\lambda-\lambda z) - 1\right], \end{aligned} \tag{5.9}$$

$$D(z) = \begin{vmatrix} h_1(z) & -u_2(z) & -k_3(z) \\ -k_1(z) & h_2(z) & -u_3(z) \\ -u_1(z) & -k_2(z) & h_3(z) \end{vmatrix}, \tag{5.10}$$

$$\begin{aligned} P_{11}(z) = {} & p_1h_2(z)h_3(z) - p_1k_2(z)u_3(z) + p_2u_2(z)h_3(z) + p_3u_3(z)u_2(z) \\ & + p_2k_2(z)k_3(z) + p_3k_3(z)h_2(z), \end{aligned} \tag{5.11}$$

$$\begin{aligned} P_{22}(z) &= p_2h_1(z)h_3(z) + p_3h_1(z)u_3(z) + p_1k_1(z)h_3(z) + p_1u_1(z)u_3(z) \\ &\quad + p_3k_1(z)k_3(z) - p_2u_1(z)k_3(z), \qquad (5.12) \\ P_{33}(z) &= p_3h_1(z)h_2(z) + p_2k_2(z)h_1(z) - p_3u_2(z)k_1(z) + p_2u_1(z)u_2(z) \\ &\quad + p_1k_1(z)k_2(z) + p_1u_1(z)h_2(z). \qquad (5.13) \end{aligned}$$

It is easy to verify that for $z = 1$, the right hand sides of equations (83), (84), (85) and (86) are indeterminate of the form $\frac{0}{0}$. Hence, we apply L'Hopital's rule and obtain on simplifying

$$P_q^{(1)}(1) = \frac{p_1\lambda E(v_1)Q}{p_1[q-\lambda E(v_1)] + p_2[q-\lambda E(v_2)] + p_3[q-\lambda E(v_3)] - \frac{\theta\lambda}{\beta}}, \quad (5.14)$$

$$P_q^{(2)}(1) = \frac{p_2\lambda E(v_2)Q}{p_1[q-\lambda E(v_1)] + p_2[q-\lambda E(v_2)] + p_3[q-\lambda E(v_3)] - \frac{\theta\lambda}{\beta}}, \quad (5.15)$$

$$P_q^{(3)}(1) = \frac{p_3\lambda E(v_3)Q}{p_1[q-\lambda E(v_1)] + p_2[q-\lambda E(v_2)] + p_3[q-\lambda E(v_3)] - \frac{\theta\lambda}{\beta}}, \quad (5.16)$$

$$V(1) = \frac{\theta\lambda Q}{\beta\left[p_1[q-\lambda E(v_1)] + p_2[q-\lambda E(v_2)] + p_3[q-\lambda E(v_3)] - \frac{\theta\lambda}{\beta}\right]}. \quad (5.17)$$

Then using equations (94) to (97) into (82) and simplifying, we obtain

$$Q = \frac{p_1[q-\lambda E(v_1)] + p_2[q-\lambda E(v_2)] + p_3[q-\lambda E(v_3)] - \frac{\theta\lambda}{\beta}}{p_1\lambda E(v_1) + p_2\lambda E(v_2) + p_3\lambda E(v_3) + p_1[q-\lambda E(v_1)] + p_2[q-\lambda E(v_2) + p_3[q-\lambda E(v_3)] - \frac{\theta\lambda}{\beta}}. \quad (5.18)$$

We then substitute for Q from equation (98) into equations (49), (52), (55) and (80) to completely determine $\overline{P}_q^{(1)}(z,s)$, $\overline{P}_q^{(2)}(z,s)$, $\overline{P}_q^{(3)}(z,s)$ and $\overline{V}(z,s)$ in closed form.

Equation (98) also yields the steady state condition under which the steady state shall exist. This condition is given by

$$\lambda\left(p_1E(v_1) + p_2E(v_2) + p_3E(v_3) + \frac{\theta}{\beta}\right) < q. \quad (5.19)$$

And hence the utilization factor, ρ of the system is given by

$$\rho = \left[\frac{p_1\lambda E(v_1) + p_2\lambda E(v_2) + p_3\lambda E(v_3) + \frac{\theta\lambda}{\beta}}{q}\right]. \quad (5.20)$$

where $\rho < 1$ is the stability condition under which the steady state results. Equation (100) also give the probability that the server is idle.

6. The Mean Number in the System

Let L_q denote the mean number of customers in the queue. Then, we have, $L_q = \frac{\mathrm{d}}{\mathrm{d}z} P_q(z)$ at $z = 1$, where $P_q(z) = P_q^{(1)}(z) + P_q^{(2)}(z) + +P_q^{(3)}(z) + V(z)$ is obtained by adding equations (83), (84), (85) and (86). Since $P_q(z)$ is indeterminate of the form $\frac{0}{0}$ at the $z = 1$, we let

$$P_q(z) = \left[\frac{NR(z)}{DR(z)}\right], \tag{6.1}$$

where $N(z)$ and $D(z)$ respectively denote the numerator and the denominator of the right of equation (101), where

$$\begin{aligned} NR(z) &= Q\left\{(\lambda - \lambda z + \beta)\left\{P_{11}(z)\left[\overline{B}_1(\lambda - \lambda z) - 1\right]\right.\right. \\ &\quad +P_{22}(z)\left[\overline{B}_2(\lambda - \lambda z) - 1\right] + P_{33}(z)\left[\overline{B}_3(\lambda - \lambda z) - 1\right]\Big\} \\ &\quad +\lambda\theta(q + pz)(z - 1)\left\{\left[\overline{B}_1(\lambda - \lambda z)\right] P_{11}(z)\right. \\ &\quad \left.\left. + \left[\overline{B}_2(\lambda - \lambda z)\right] P_{22}(z) + \left[\overline{B}_3(\lambda - \lambda z)\right] P_{33}(z)\right\}\right\}, \qquad (6.2) \\ DR(z) &= (\lambda - \lambda z + \beta)D(z), \qquad (6.3) \end{aligned}$$

$h_1(z), h_2(z), h_3(z), k_1(z), k_2(z), k_3(z), u_1(z), u_2(z), u_3(z)$ are given by equations (68)–(76) and $D(z)$ is given by equation (90). Then we use the following well-known result in queueing theory (Kashyap and Chaudhry [10]). This is applied when $P_q(z)$ is indeterminate of the form $\frac{0}{0}$.

$$\begin{aligned} L_q = \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}P_q(z) = P_q^{'}(1) &= \lim_{z\to 1}\frac{D^{'}(z)N^{''}(z) - N^{'}(z)D^{''}(z)}{2(D^{'}(z))^2}, \\ &= \lim_{z\to 1}\frac{D^{'}(1)N^{''}(1) - N^{'}(1)D^{''}(1)}{2(D^{'}(1))^2}. \qquad (6.4) \end{aligned}$$

We carry out the required derivatives at $z = 1$, using the fact $\overline{B}_i(0) = 1$, $-\overline{B}_i^{'}(0) = E(v_i)$ and $\overline{B}_i^{''}(0) = E(v_i^2)$, $i = 1, 2, 3$, the second moment of the service time for the ith type of service. After a lot of algebraic simplifications, we obtain

$$N^{'}(1) = \lambda\beta\left[p_1E(v_1) + p_2E(v_2) + p_3E(v_3) + \frac{\theta}{\beta}\right]Q, \tag{6.5}$$

$$\begin{aligned} N^{''}(1) = {} & Q\left\{\lambda^2\beta\left[p_1E(v_1^2) + p_2E(v_2^2) + p_3E(v_3^2)\right]\right. \\ & +2\beta\lambda\left[p_1E(v_1) + p_2E(v_2) + p_3E(v_3)\right] \\ & -2\lambda^2\left[p_1E(v_1) + p_2E(v_2) + p_3E(v_3)\right] \\ & \left.+2\theta\lambda\left[(2-q) + p_1\lambda E(v_1) + p_2\lambda E(v_2) + p_3\lambda E(v_3)\right]\right\}, \end{aligned} \tag{6.6}$$

$$D^{'}(1) = \beta\left[q - p_1\lambda E(v_1) - p_2\lambda E(v_2) - p_3\lambda E(v_3) - \frac{\theta\lambda}{\beta}\right], \tag{6.7}$$

$$\begin{aligned} D^{''}(1) = {} & \beta\left\{2\left[q - p_1\lambda E(v_1) - p_2\lambda E(v_2) - p_3\lambda E(v_3) - \frac{\theta\lambda}{\beta}\right]\right. \\ & -\lambda^2\left[p_1E(v_1^2) + p_2E(v_2^2) + p_3E(v_3^2)\right] \\ & -2\lambda p\left[p_1E(v_1) + p_2E(v_2) + p_3E(v_3)\right] - \frac{2\lambda p\theta}{\beta} \\ & -\frac{2\lambda^2\theta}{\beta}\left[p_1E(v_1) + p_2E(v_2) + p_3E(v_3)\right] - \frac{2\lambda^2\theta}{\beta^2} \\ & \left.-\frac{2\lambda}{\beta}\left[q - p_1\lambda E(v_1) - p_2\lambda E(v_2) - p_3\lambda E(v_3) - \frac{\theta\lambda}{\beta}\right]\right\}. \end{aligned} \tag{6.8}$$

Using equations (105) to (108) into (104), we have obtained L_q in closed form, where Q has been found in equation (98).

Further, we find the average system size L using Little's formula. Thus we have

$$L = L_q + \rho \tag{6.9}$$

where L_q has been found in equation (104) and ρ is obtained from equation (100) as

$$\rho = 1 - Q. \tag{6.10}$$

7. The Mean Waiting Time

Let W_q and W denote the mean waiting time in the queue and the system respectively. Then using Little's formulas, we obtain,

$$W_q = \frac{L_q}{\lambda}, \tag{7.1}$$

$$W = \frac{L}{\lambda}. \tag{7.2}$$

where L_q and L have been found in equations (104) and (109).

8. Numerical Example

In order to see the effect of the parameters θ, p_1, p_2 and p_3 on some queue characteristics such as server's idle time Q, the system's utilization factor ρ, the average queue size L_q, the average system size L, the average waiting time in the queue W_q and the average waiting time in the system W_s, we choose the arbitrary values of the parameters λ, μ_1, μ_2, μ_3, p_1, p_2, p_3, θ and β such that the steady state condition (70) is satisfied. The following table has been computed by using the fixed values of $\mu_1 = 7$, $\mu_2 = 9$, $\mu_3 = 11$, $\lambda = 2$, $\beta = 9$ and varying value as of p_1, p_2, p_3 and θ.

The table shows that for a fixed pair of values p_1, p_2 and p_3 as θ increases then the server's idle time Q decreases, the system's utilization factor ρ increases and L_q, L_s, W_q, W_s all increase as it should be.

θ	p_1	p_2	p_3	Q	ρ	L_q	L_s	W_q	W_s
0.0	0.0	0.1	0.9	.962963	.309764	.119983	.449135	.069686	.224568
	0.1	0.2	0.7	.878307	.333814	.164679	.503747	.084966	.251874
	0.2	0.3	0.5	.793651	.357864	.207427	.561368	.101752	.280684
	0.3	0.4	0.3	.708995	.381914	.247785	.622351	.120218	.311176
	0.4	0.5	0.1	.624339	.405964	.285384	.687103	.140570	.343552
0.2	0.0	0.1	0.9	.555556	.383838	.268168	.652006	.134084	.326003
	0.1	0.2	0.7	.531506	.407888	.310793	.718682	.155397	.359341
	0.2	0.3	0.5	.507456	.431938	.357842	.789780	.178921	.394890
	0.3	0.4	0.3	.483406	.455989	.409901	.865890	.204951	.432945
	0.4	0.5	0.1	.459356	.480039	.467666	.947705	.233833	.473852
0.4	0.0	0.1	0.9	.481482	.457913	.432164	.890077	.216082	.445038
	0.1	0.2	0.7	.457432	.481963	.491937	.973899	.245968	.486950
	0.2	0.3	0.5	.433381	.506013	.558466	1.064479	.279233	.532239
	0.3	0.4	0.3	.409331	.530063	.632790	1.162852	.316395	.581426
	0.4	0.5	0.1	.385281	.554113	.716168	1.270280	.358084	.635140
0.6	0.0	0.1	0.9	.407407	.531987	.648073	1.180060	.324037	.590030
	0.1	0.2	0.7	.383357	.556037	.733527	1.289564	.366764	.644782
	0.2	0.3	0.5	.359307	.580087	.829872	1.409958	.414936	.704979
	0.3	0.4	0.3	.335257	.604137	.939092	1.543228	.469546	.771614
	0.4	0.5	0.1	.311207	.628187	1.063685	1.691871	.531843	.845936
0.8	0.0	0.1	0.9	.333333	.606061	.945178	1.551239	.472589	.775619
	0.1	0.2	0.7	.309283	.630111	1.071879	1.701990	.535940	.850995
	0.2	0.3	0.5	.285233	.654161	1.217540	1.871701	.608770	.935850
	0.3	0.4	0.3	.261183	.678211	1.386412	2.064622	.693206	1.032311
	0.4	0.5	0.1	.237133	.702261	1.584119	2.286380	.792059	1.143190

Acknowledgement

The author thanks the Management of SSN College of Engineering for providing the necessary requirements during the preparation of this paper.

References

[1] Y. Baba, On the $M^X/G/1$ queue with vacation time, *Operation Research Letters*, **5** (1986), 93–98.

[2] A. Badamchi Zadeh and G.H. Shankar, A two phase queue system with Bernoulli feedback and Bernoulli schedule server vacation, *Information and Management Sciences*, **19** (2008), 329–338.

[3] A. Borthakur and G. Chaudhury, On a batch arrival Possion queue with generalised vacation, *Sankhya Ser.B*, **59** (1997), 369–383.

[4] G. Chaudhury, An $M^X/G/1$ queueing system with a set up period and a vaction period, *Questa*, **36** (2000), 23–28.

[5] B.D. Choi and K.K. Park, The $M/G/1$ retrial queue with Bernoulli schedule, *Queueing Systems*, **7** (1990), 219–228.

[6] M. Cramer, Stationary distributions in a queueing system with vaction times and limited service, *Queueing Systems*, **4** (1989), 57–68.

[7] B.T. Doshi, Queueing systems with vacations-a survey, *Queueing Systems*, **1** (1986), 29–66.

[8] B.T. Doshi, Conditional and unconditional distributions for $M/G/1$ type queues with server vacation, *Questa*, **7** (1990), 229–252.

[9] S. Fuhrmann, A note on the $M/G/1$ queue with server vactions, *Questa*, **31** (1981), 13–68.

[10] B.R.K. Kashyap and M.L. Chaudhry, *An introduction to Queueing theory*, Kingston, Ontario, 1988.

[11] J. Keilson and L.D. Servi, Oscillating random walk models for $G/G/1$ vacation systems with Bernoulli schedules, *Journal of Applied Probability*, **23** (1986), 790–802.

[12] Y. Levi and U. Yechilai, An $M/M/s$ queue with server vactions, *Infor.*, **14** (1976), 153–163.

[13] K.C. Madan, On a $M^X/M^b/1$ queueing system with general vacation times, *International Journal of Information and Management Sciences*, **2** (1991), 51–61.

[14] K.C. Madan, An $M/G/1$ queue with optional deterministic server vacations, *Metron*, **7** (1999), 83–95.

[15] K.C. Madan and R.F. Anabosi, A single server queue with two types of service, Bernoulli schedule server vacations and a single vacation policy, *Pakistan Journal of Statistics*, **19** (2003), 331–342.

[16] K.C. Madan and G. Choudhury, A two stage arrival queueing system with a modified Bernoulli schedule vacation under N-policy, *Mathematical and Computer Modelling*, **42** (2005), 71–85.

[17] E. Rosenberg and U. Yechiali, The $M^X/G/1$ queue with single and multiple vacations under LIFO service regime, *Operation Research Letters*, **14** (1993), 171–179.

[18] H. Takagi, *Queueing Analysis: A foundation of performance evalution, Vol 1*, North Holland, Amsterdam, 1993.

[19] H. Takagi, Time dependent process of $M/G/1$ vacation models with exhaustive service, *Journal of Applied Probability*, **29** (1992), 418–429.