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Decay rate estimate of solution to damped wave equation with memory term in Fourier spaces

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Abstract

The viscoelastic equation with fading memory in bounded spaces has been deeply studied by several authors. Here, the energy decay results are established for weak-viscoelastic wave equation in $\mathbb{R}^n$, which depends on the behavior of both α and g. The main idea of the proof is to construct an appropriate Lyapunov function of the system obtained after taking the Fourier transform.

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1. Introduction and related results

The model here considered are well known ones and refer to materials with memory as they are termed in the wide literature which is concerned about their physical, mechanical behavior and the many interesting analytical problems. The physical characteristic property of such materials is that their behaviour depends on time not only through the present time but also through their past history.

Let us consider the weak-viscoelastic case in the following problem:

$$\begin{cases} u'' - \Delta u - \Delta u' + \alpha(t) \int_0^t g(t-s)\Delta u(s,x)ds = 0, x \in \mathbb{R}^n, t \in \mathbb{R}_*^+ \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^n), u'(0,x) = u_1(x) \in L^2(\mathbb{R}^n), x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 2$. It is well known that the presence of a viscoelastic term with and without the weighted function α doest not preclude the question of existence, but its effects are on the stability of the existing solution. For the existence, we refer the reader to works in [6], [7], [9], [13], [16], [17], [23] and [24].

The energy of u at time t is given by

$$E(t) = \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}\left(1 - \alpha(t)\int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}\alpha(t)(g \circ \nabla u). \tag{1.2}$$

and the following energy functional law holds.

$$\begin{aligned} E'(t) &= \frac{1}{2}\alpha(t)(g' \circ \nabla u) - \frac{1}{2}\alpha(t)g(t)\|\nabla u\|_2^2 - \|\nabla u'\|_2^2 \\ &\quad + \frac{1}{2}\alpha'(t)(g \circ \nabla u) - \frac{1}{2}\alpha'(t)\int_0^t g(s)ds\|\nabla u\|_2^2. \end{aligned} \tag{1.3}$$

The following notation will be used throughout this paper

$$(g \circ \Psi) = \int_0^t g(t-\tau)\,\|\Psi(t) - \Psi(\tau)\|_2^2\,d\tau, \text{ for any } \Psi \in L^\infty(0,T; L^2(\mathbb{R}^n)) \tag{1.4}$$

This type of problems is usually encountered in viscoelasticity in various areas of mathematical physics. It was first considered by Dafermos in [5], where the general decay was discussed. The problems related to (1.1) attracted a great deal of attention in the last decades and numerous results appeared on the existence and long time behavior of solutions but their results is by now rather developed, especially in any space dimension when it comes to nonlinear problems.

For the literature, in $\mathbb{R}^n$ we quote essentially the results of [3], [10], [11], [12], [14], [18]. In [11], authors showed that, for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the energy of solution of a linear Cauchy problem related of (1.1) is polynomial. The finite-speed propagation is used to compensate for the lack of Poincarés inequality. In [10], the author looked into a linear Cauchy viscoelastic equation with density. His study included the exponential and polynomial rates, where he used the spaces weighted by density to compensate for the lack of Poincaré's inequality. The same problem treated in [10], was considered in [12], where they consider a Cauchy problem for a viscoelastic wave equation. Under suitable conditions on the initial data and the relaxation function, they prove a polynomial decay result of solutions. Conditions used, on the relaxation function g and its derivative g' are different from the usual ones.

Ikehata in [6] considered, in the one-dimensional half space, the mixed problem of the equation

$$v_{tt} - v_{xx} + v_t = 0 \tag{1.5}$$

with a weighted initial data and presented a new decay estimates of solutions which also can be derived for the Cauchy problem in $\mathbb{R}^n$. Let us mention that a pioneer question on the long time asymptotic of strongly damped wave equations in [9]. Authors, studied the Cauchy problem for abstract dissipative equations in Hilbert spaces generalizing wave equations with strong damping terms in $\mathbb{R}^n$ or exterior domains

$$u_{tt}(t) + Au(t) + Au'(t) = 0, \qquad t \in (0, \infty). \tag{1.6}$$

$$u(0) = u_0, \qquad u_t(0) = u_1, \tag{1.7}$$

where $A : D(A) \subset H \to H$ is a nonnegative self-adjoint operator in $(H, \|.\|)$ with a dense domain $D(A)$. Using the energy method in the Fourier space and its generalization based on the spectral theorem for self-adjoint operators, their main result was a combination of solutions of diffusion and wave equations.

Recently, in [7], Ryo Ikehata considered the Cauchy problem in $\mathbb{R}^n$ for strongly damped wave equations (1.6) with $A = -\Delta$. He derived asymptotic profiles of its solutions with weighted $L^{1,1}(\mathbb{R}^n)$ data by using a method introduced in [6] and developed in [9]. The same author, extend his results in [8] when the initial data belongs to a weighted $L^{1,2}(\mathbb{R}^n)$ space.

2. Statement

We omit the space variable x of $u(x,t)$, $u'(x,t)$ and for simplicity reason denote $u(x,t) = u$ and $u'(x,t) = u'$, when there is no confusion. The constants c used throughout this paper are positive generic constants which may be different in various settings, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$.

In order to investigate the decay structure based on the memory and the weighted function, we also consider the following assumptions: $g, \alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are non-increasing differentiable functions of class C^1 satisfying:

$$1-\alpha(t)\int_0^t g(s)ds \geq k > 0, \quad g(0)=g_0>0 \tag{2.1}$$

$$\infty > \int\limits_0^\infty g(t)dt, \quad \alpha(t)>0, \tag{2.2}$$

In addition, there exists a non-increasing differentiable function $\beta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$\beta(t)>0, \quad g'(t)+\beta(t)g(t)\leq 0, \quad \forall t\geq 0, \quad \lim_{t\to\infty} = \frac{\alpha'(t)}{\beta(t)\alpha(t)} = 0. \tag{2.3}$$

We give some notations to be used below. Let F denote the Fourier transform in $L^2(\mathbb{R}^n)$ defined as follows:

$$F[f](\xi) = \widehat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \exp(-ix.\xi)f(x)dx. \tag{2.4}$$

Here ξ is the variable associated with the Fourier transform, where $i=\sqrt{-1}$, $x.\xi = \sum_{i=1}^n x_i\xi_i$ and denote its inverse transform by F^{-1}. The operator $-\Delta$ is defined by

$$-\Delta v(x) = F^{-1}\left(|\xi|^2 F(v)(\xi)\right)(x), v\in H^2(\mathbb{R}^n), x\in\mathbb{R}^n.$$

For $1\leq p\leq\infty$, we denote by $L^p(\mathbb{R}^n)$ the usual Lebesgue space on $\mathbb{R}^n$ with the norm $\|.\|_{L^p}$. For a nonnegative integer m, $H^m(\mathbb{R}^n)$ denotes the Sobolev space of $L^2(\mathbb{R}^n)$ functions on $\mathbb{R}^n$, equipped with the norm $\|.\|_{H^m}$. By direct calculations, we have the following technical Lemma which will play an important role in the sequel.

Lemma 2.1. ([23]) For any two functions $g\in C^1(\mathbb{R})$, $v\in W^{1,2}(0,T)$, it holds that

$$\begin{aligned} Re\left\{\alpha(t)\int\limits_0^t g(t-s)v(s)ds v'(t)\right\} &= -\frac{1}{2}\alpha(t)g(t)|v(t)|^2 + \frac{1}{2}\alpha(t)(g'\circ v)(t) \\ &- \frac{1}{2}\frac{d}{dt}\alpha(t)(g\circ v)(t) + \frac{1}{2}\frac{d}{dt}\alpha(t)\int_0^t g(s)ds|v(t)|^2 \\ &+ \frac{1}{2}\alpha'(t)(g\circ v)(t) - \frac{1}{2}\alpha'(t)\int\limits_0^t g(s)ds|v|^2 \end{aligned} \tag{2.5}$$

and

$$\left|\int_0^t g(t-s)(v(s)-v(t))ds\right|^2 \leq \int_0^t |g(s)|ds \int_0^t |g|(t-s)|v(t)-v(s)|^2 ds$$

Finally, we give the definition of *weak solutions* for the problem (1.1).

Definition 2.2. A weak solution of (1.1) is u such that

- $u \in C([0,T); H^1(\mathbb{R}^n)), \quad u' \in C^1([0,T); L^2(\mathbb{R}^n))$
- For all $v \in C_0^\infty([0,T]\times\mathbb{R}^n)$, u satisfies the generalized formulae:

$$\begin{aligned} 0 \;=\; & \int_0^T (u'', v)ds + \int_0^T \int_{\mathbb{R}^n} \nabla u \nabla v dxds + \int_0^T \int_{\mathbb{R}^n} \nabla u' \nabla v dxds \\ & - \int_0^T \int_{\mathbb{R}^n} \alpha(t) \int_0^s g(s-\tau)\nabla u(\tau)d\tau \nabla v(s)dxds, \end{aligned} \tag{2.6}$$

- u satisfies the initial conditions

$$u_0(x) \in H^1(\mathbb{R}^n), \quad u_1(x) \in L^2(\mathbb{R}^n).$$

We can now state and prove the asymptotic behavior of the solution of (1.1). Throughout this paper, let us set $\widehat{u}(t,\xi) = F(u(t,.))(\xi)$.

3. Main result

We show that our solution decays time asymptotically to zero and the rate of decay for the solution is fast and similar to both α and g.

Theorem 3.1. Assume u is the solution of (1.1), then the next general exponential estimate satisfies in the Fourier space

$$E(t) \leq W \exp\left(-\omega \int_0^t \alpha(s)\beta(s)ds\right), \qquad \forall t \geq 0. \tag{3.1}$$

for some positive constants W, ω.

Proof. We take the Fourier transform of both sides of (1.1). Then one has the reduced equation for $\xi \in \mathbb{R}^n$, $t \in \mathbb{R}_*^+$:

$$\begin{cases} \widehat{u}''(t,\xi) + |\xi|^2 \widehat{u}(t,\xi) - |\xi|^2 \alpha(t) \int_0^t g(t-s)\widehat{u}(s,\xi)ds + |\xi|^2 \widehat{u}'(t,\xi) = 0 \\ \widehat{u}(0,\xi) = \widehat{u}_0(\xi) \in H^1(\mathbb{R}^n), \widehat{u}'(0,\xi) = \widehat{u}_1(\xi) \in L^2(\mathbb{R}^n). \end{cases} \tag{3.2}$$

We apply the multiplier techniques in Fourier space in order to obtain useful estimates and prepare some functionals associated with the nature of our problem to introduce an appropriate Lyapunov functions.

First, to derive the equality for the physical energy, we multiply both sides of (3.2) by $\overline{\widehat{u}'}$. Then, taking the real part of the resulting identities, we obtain

$$E_1(t) = \frac{1}{2}|\widehat{u}'|^2 + \frac{1}{2}|\xi|^2(1-\alpha(t)\int_0^t g(s)ds)|\widehat{u}|^2 + \frac{1}{2}|\xi|^2\alpha(t)(g\circ\widehat{u})(t)$$

and

$$\begin{aligned} e_1(t) &= \frac{1}{2}|\xi|^2\left(\alpha(t)g(t)|\widehat{u}|^2 - \alpha(t)(g'\circ\widehat{u})(t) + 2|\widehat{u}'|^2\right) \\ &+\frac{1}{2}|\xi|^2\left(\alpha'(t)(g\circ\widehat{u})(t) - \alpha'(t)\int_0^t g(s)ds|\widehat{u}|^2\right). \end{aligned} \tag{3.3}$$

Then,

$$\frac{d}{dt}E_1(t) + e_1(t) = 0. \tag{3.4}$$

Second, the existence of the memory term forces us to make the first modification of the energy by multiplying (3.2) by $\left(-\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right)\right)$ and taking the real part, we have that

$$\begin{aligned} 0 = &-Re\left\{\widehat{u}''(t,\xi)\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right)\right\} \\ &-Re\left\{|\xi|^2\widehat{u}\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right)\right\} \\ &+\frac{1}{2}|\xi|^2\frac{d}{dt}\left(\left|\alpha(t)\int_0^t g(t-s)\widehat{u}(s)ds\right|^2\right) \\ &-Re\left\{|\xi|^2\widehat{u}'\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right)\right\} \end{aligned} \tag{3.5}$$

Since

$$\begin{aligned} &\frac{d}{dt}\left(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right) \\ &= \alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds + \alpha(t)\frac{d}{dt}\left(\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right) \\ &= \alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds + \alpha(t)g_0\overline{\widehat{u}} + \alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds. \end{aligned} \tag{3.6}$$

The first term in Eq.(3.5) takes the forme

$$\begin{aligned}
&- \quad Re\left\{\widehat{u}''(t,\xi)\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\
&= \quad -Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}'\\
&+ \quad Re\left\{\widehat{u}'\frac{d^2}{dt^2}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\
&= \quad -Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}' + \alpha(t)g_0|\widehat{u}'|^2\\
&+ \quad Re\left\{\widehat{u}'\Big(\alpha(t)\frac{d}{dt}\Big(\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big) + \alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}.
\end{aligned}$$

Denote by

$$E_2(t) = \frac{1}{2}\left(|\xi|^2\Big|\alpha(t)\int_0^t g(t-s)\widehat{u}(s)ds\Big|^2 - Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right),$$

and

$$\begin{aligned}
e_2(t) &= \alpha(t)g_0|\widehat{u}'|^2 - Re\left\{|\xi|^2\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\
&\quad +Re\left\{\alpha'(t)\widehat{u}'\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right\}\\
R_2(t) &= -Re\left\{|\xi|^2\widehat{u}\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\
&\quad +Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}.
\end{aligned}$$

Then,

$$\frac{d}{dt}E_2(t) + e_2(t) + R_2(t) = 0. \tag{3.7}$$

Next, to make the second modification of the energy which corresponds to the strong damping, we multiply (3.2) by $\overline{\widehat{u}}$ and taking the real part, we have

$$\begin{aligned}
0 &= (Re\{\widehat{u}'\overline{\widehat{u}}\})' - |\widehat{u}'|^2 + |\xi|^2|\widehat{u}|^2\\
&\quad -|\xi|^2 Re\left\{\alpha(t)\int_0^t g(t-s)\widehat{u}(s)\overline{\widehat{u}}(t)ds\right\} + \frac{1}{2}|\xi|^2(|\widehat{u}|^2)',
\end{aligned}$$

using results in Lemma 2.1, we get

$$\begin{aligned}
0 &= (Re\{\widehat{u}'\overline{\widehat{u}}\})' - |\widehat{u}'|^2 + |\xi|^2|\widehat{u}|^2 + \frac{1}{2}|\xi|^2(|\widehat{u}|^2)'\\
&\quad -|\xi|^2\left(\alpha(t)\int_0^t g(s)ds|\widehat{u}|^2 + Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\}\right).
\end{aligned}$$

Denote

$$E_3(t) = Re\{\widehat{u}'\overline{\widehat{u}}\} + \frac{1}{2}|\xi|^2|\widehat{u}|^2,$$

and

$$\begin{aligned} e_3(t) &= |\xi|^2\Big(1-\alpha(t)\int_0^t g(s)ds\Big)|\widehat{u}|^2. \\ R_3(t) &= -|\widehat{u}'|^2 - Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\} \end{aligned}$$

Then,

$$\frac{d}{dt}E_3(t) + e_3(t) + R_3(t) = 0. \tag{3.8}$$

Let us define for some constants $\varepsilon_1, \varepsilon_2 > 0$ to be chosen later

$$\begin{aligned} E_4(t) &= E_1(t) + \varepsilon_1\alpha(t)E_2(t) + \varepsilon_2\alpha(t)E_3(t) \\ &= \frac{1}{2}\left\{|\widehat{u}'|^2 + |\xi|^2\Big(1-\alpha(t)\int_0^t g(s)ds\Big)|\widehat{u}|^2 + |\xi|^2\alpha(t)(g\circ\widehat{u})(t)\right\} \\ &\quad + \frac{\varepsilon_1\alpha(t)}{2}\left(|\xi|^2\left|\alpha(t)\int_0^t g(t-s)\widehat{u}(s)ds\right|^2\right. \\ &\quad \left. - Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right) \\ &\quad + \varepsilon_2\alpha(t)\left(Re\left\{\widehat{u}'\overline{\widehat{u}}\right\} + \frac{1}{2}|\xi|^2|\widehat{u}|^2\right) \end{aligned}$$

and

$$\begin{aligned} e_4(t) &= e_1(t) + \varepsilon_1\alpha(t)e_2(t) + \varepsilon_2\alpha(t)e_3(t) \\ &= \frac{|\xi|^2}{2}\alpha(t)\left(g(t)|\widehat{u}|^2 - (g'\circ\widehat{u})(t) + 2\alpha^{-1}(t)|\widehat{u}'|^2\right) \\ &\quad + \frac{|\xi|^2}{2}\alpha'(t)\left((g\circ\widehat{u})(t) - \widehat{u}'\int_0^t g(s)ds|\widehat{u}|^2\right) \\ &\quad + \varepsilon_1\alpha(t)\left(\alpha(t)g_0|\widehat{u}'|^2 - Re\left\{|\xi|^2\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right) \\ &\quad + \varepsilon_1\alpha(t)\left(Re\left\{\alpha'(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\right\}\right) \\ &\quad + \varepsilon_2|\xi|^2\alpha(t)\left(1-\alpha(t)\int_0^t g(s)ds\right)|\widehat{u}|^2 \end{aligned}$$

and

$$\begin{aligned}
R_4(t) &= \varepsilon_1\alpha(t)R_2(t) + \varepsilon_2\alpha(t)R_3(t)\\
&= \varepsilon_1\alpha(t)\left(-Re\left\{|\xi|^2\widehat{u}\frac{d}{dt}\Big(\alpha(t)\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right)\\
&\quad+\varepsilon_1\alpha(t)\left(Re\left\{\widehat{u}'\frac{d}{dt}\Big(\alpha(t)\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\right)\\
&\quad+\varepsilon_2\alpha(t)\left(-|\widehat{u}'|^2 - Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\}\right)
\end{aligned}$$

At this point, we introduce the Lyapunov functions as

$$L_1(t) = \left\{|\widehat{u}'|^2 + k|\xi|^2|\widehat{u}|^2 + |\xi|^2\alpha(t)(g\circ\widehat{u})(t)\right\} \tag{3.9}$$

and

$$L_2(t) = \alpha(t)g(t)|\widehat{u}|^2 + \alpha(t)\beta(t)(g\circ\widehat{u})(t). \tag{3.10}$$

It is easy to verify that there exists positive constants $c_1(g_0), c_2(g_0)$ such that

$$c_1L_1(t) \le E_1(t) \le c_2L_1(t), \forall t > 0. \tag{3.11}$$

Thanks to Holder, Young's inequalities, one gets for some constant c_3

$$|\varepsilon_1E_2(t) + \varepsilon_2E_3(t)| \le c_3L_1(t),$$

which means that $L_1(t) \sim E(t)$. Using again (2.3), Holder and Young's inequalities and assumptions on g to obtain

$$\begin{aligned}
|R_4(t)| &= \varepsilon_1\alpha(t)R_2(t) + \varepsilon_2\alpha(t)R_3(t)\\
&\le \varepsilon_1\alpha(t)Re\left\{|\xi|^2\widehat{u}\alpha(t)\frac{d}{dt}\Big(\int_0^t g(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\
&\quad+\varepsilon_1\alpha(t)Re\left\{\widehat{u}'\alpha(t)\frac{d}{dt}\Big(\int_0^t g'(t-s)\overline{\widehat{u}}(s)ds\Big)\right\}\\
&\quad+\varepsilon_2\alpha(t)\left(|\widehat{u}'|^2 + Re\left\{\alpha(t)\int_0^t g(t-s)(\widehat{u}(s)-\widehat{u}(t))\overline{\widehat{u}}(s)ds\right\}\right)\\
&\le \varepsilon_1\alpha(t)|\widehat{u}'|^2 + c_4\varepsilon_1\alpha(t)|\xi|^2|\widehat{u}|^2 + c_5\varepsilon_1|\xi|^2L_2(t)\\
&\quad+\varepsilon_2\alpha(t)\left[|\widehat{u}'|^2 + c_6|\xi|^2\left(\lambda|\widehat{u}|^2 + c_\lambda\alpha(t)(g\circ\widehat{u})(t)\right)\right]\\
&\le (\varepsilon_1+\varepsilon_2)\alpha(t)|\widehat{u}'|^2 + (c_4\varepsilon_1 + \varepsilon_2c_6\lambda)\alpha(t)|\xi|^2|\widehat{u}|^2 + (c_5\varepsilon_1 + c_\lambda\varepsilon_2)|\xi|^2L_2(t).
\end{aligned}$$

Since $L_2(t) \le c_3\, e_1(t)$, one can easily check that there exists positive constants ε_1, ε_2, λ, c_4, c_5, c_6 such that

$$|R_4(t)| \le ce_4(t), c > 0. \tag{3.12}$$

By (3.4), (3.7) and (3.8), we get

$$\begin{aligned}\frac{d}{dt}E_4(t) &= \frac{d}{dt}E_1(t)+\varepsilon_1\alpha(t)\frac{d}{dt}E_2(t)+\varepsilon_2\alpha(t)\frac{d}{dt}E_3(t)\\ &\quad+\varepsilon_1\alpha'(t)E_2(t)+\varepsilon_2\alpha'(t)E_3(t).\end{aligned}$$

We use $\lim\limits_{t\to\infty}\dfrac{\alpha'(t)}{\alpha(t)}=0$ by (2.1)-(2.3) to choose $t_1>t_0$ and since $e_4(t)\geq cE_4(t)$, then (3.12) gives for some positive constant N

$$\frac{d}{dt}E_4(t)\leq -N\alpha(t)E_4(t)+c\alpha(t)(g\circ\widehat{u})(t). \tag{3.13}$$

Multiplying (3.13) by $\beta(t)$ and using (2.3), (3.10), we obtain

$$\begin{aligned}\beta(t)\frac{d}{dt}E_4(t) &\leq -N\beta(t)\alpha(t)E_4(t)+c\beta(t)\alpha(t)(g\circ\widehat{u})(t)\\ &\leq -N\beta(t)\alpha(t)E_4(t)-c\alpha(t)(g'\circ\widehat{u})(t)\\ &\leq -N\beta(t)\alpha(t)E_4(t)\\ &\quad -c|\xi|^2\alpha'(t)\int_0^t g(s)ds|\widehat{u}|^2-2c\frac{d}{dt}E_4(t),\quad \forall t>t_1.\end{aligned} \tag{3.14}$$

Since $\beta'(t)\leq 0$, we set $L(s)=(\beta(s)+2c)E_4(s)$ which is equivalent to $E_4(t)$, then

$$\begin{aligned}\frac{d}{dt}L(t) &\leq -N\beta(t)\alpha(t)E_4(t)-c|\xi|^2\alpha'(t)\int_0^t g(s)ds|\widehat{u}|^2\\ &\leq -\beta(t)\alpha(t)\left[N-\frac{2\alpha'(t)}{k\beta(t)\alpha(t)}\int_0^t g(s)ds\right]E_4(t),\quad \forall t>t_1.\end{aligned} \tag{3.15}$$

By (2.3), we can choose $t_2>t_1$ such that

$$\begin{aligned}\frac{d}{dt}L(t) &\leq -c\beta(t)\alpha(t)E_4(t)\\ &\leq -c\beta(t)\alpha(t)L(t),\quad \forall t>t_2.\end{aligned} \tag{3.16}$$

Integrating (3.16) over $[t_2,t]$ using equivalence between Lyapunov function and the energy function, it yields that

$$E(t)\leq W\exp(-\omega\int_0^t\alpha(s)\beta(s)ds),\ W,\omega>0.$$

■

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