Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 5 (2015), pp. 3911-3925

http://www.ripublication.com

Blow-Up For A Nonlocal Diffusion Equation With Exponential Reaction Term

Haijie Pei, Zhongping Li*, Yang Li

College of Mathematic and Information, China West Norm University,
Nanchong 637009, P. R. China
**Corresponding author. Li Zhongping (1980-), male, Sichuan DaZhou,*
professor of China West Normal University, major in differential equation.

Abstract

In this paper, we will study the initial-value problem for a nonlocal nonlinear diffusion equation in a bounded smooth domain with Neumann boundary conditions and an exponential reaction term**.** First we prove the existence, uniqueness and the valid of a comparison principle for solutions of the above problem. Then we will show that a nonnegative and nontrivial solution blows up in finite time if k>0and obtain the blow-up rate. Moreover we obtain an estimate for the life span and the blow-up set of solutions.

Keywords: Nonlocal diffusion; Blow-up; Life span; Blow-up rate; Blow-up set.

MR (2010) Subject Classification: 35B44; 35B51; 35F25.

1. Introduction and main results.

Let $J : R^N \to R$ be a nonnegative, smooth, symmetric radially function with $\int_{R^N} J(x)dx = 1$. We are concerned with the following nonlocal nonlinear diffusion problem with Neumann boundary conditions and an exponential reaction term in $\Omega\times[0,T)$,

$$\begin{cases} u_t(x,t)=\int_\Omega J(\frac{x-y}{u^\alpha(y,t)})u^{1-N\alpha}(y,t)dy-\int_\Omega J(\frac{x-y}{u^\alpha(x,t)})u^{1-N\alpha}(x,t)dy+e^{ku(x,t)}, \\ u(x,0)=u_0(x). \end{cases} \quad (1.\ 1)$$

Here $k>0$ and $\Omega\subset R^N$ is a bounded connected and smooth domain.

Those years, much more attention has been placed to nonlocal diffusion equations in the literature (see [1-3] and references therein). Equations of the form

$$u_t(x,t)=J*u-u(x,t)=\int_{R^N} J(x-y)(u(y,t)-u(x,t))dy, \quad (1.\ 2)$$

and variation of it has been widely used to model diffusion processes. Just as stated in [6], if $u(x,t)$ is thought as a density at the point x at time t, and $J(x-y)$ is thought as the probability distribution of jumping from position y to position x, then $(J*u)(x,t)$ is the rate at which individuals are arriving position x from all other places and $-u(x,t)=-\int_{R^N} J(y-x)u(x,t)dy$ is the rate at which they are leaving position x to travel to any other sites. To this consideration, in the existence of internal sources, leads immediately to the fact that the density $u(x,t)$ satisfies the nonlocal diffusion problem (1. 2), which is called nonlocal diffusion equation since the diffusion of the density $u(x,t)$ at a point x and time t does not only depend on $u(x,t)$, but also on all the values of $u(x,t)$ in a neighborhood of x through the convolution term $J*u$. More related works we refer readers to see [3, 7, 14].

Bogoya in [2] studied the following nonlocal nonlinear diffusion equations with Neumann boundary conditions in a higher space dimensions,

$$u_t(x,t)=\int_\Omega J(\frac{x-y}{u^\alpha(y,t)})u^{1-N\alpha}(y,t)dy-\int_\Omega J(\frac{x-y}{u^\alpha(x,t)})u^{1-N\alpha}(x,t)dy. \quad (1.\ 3)$$

In this model, the probability distribution of jumping from position y to position x is given by

$J(\frac{x-y}{u^{\alpha}(y,t)})\cdot\frac{1}{u^{N\alpha}(y,t)}$ for some $0<\alpha\leq\frac{1}{\mathrm{N}}$ when $u(y,t)>0$ and 0 otherwise.

The rate at which individuals are arriving position x from all other places is

$$\int_{R^N} J(\frac{x-y}{u^{\alpha}(y,t)})u^{1-N\alpha}(y,t)dy,$$

and the rate at which individuals are leaving position x to all other places is

$$-u(x,t)=-\int_{R^N} J(\frac{y-x}{u^{\alpha}(x,t)})u^{1-N\alpha}(x,t)dy.$$

As before, the consideration, in the absence of external sources, leads immediately to the fact that the density u will satisfy the equation (1. 3).

Motivated by these literatures, the purpose of this paper is to continue the study of the problem (1. 3) with an exponential reaction term. We will analysis some properties of the equation with Neumann boundary conditions.

Now we state our main results as follows.

Theorem 1. 1 For every nonnegative, nontrivial and bounded function $u_0\in C(\overline{\Omega})$, there exists a time $T>0$ and a unique solution $u\in C([0,T];C(\overline{\Omega}))$ for problem (1. 1). If the maximal existence time of the solution T is finite, then the solution blows up in $L^{\infty}(\overline{\Omega})$ -norm. That is $\lim_{t\to T^-}\sup\|u(.,t)\|_{L^{\infty}(\overline{\Omega})}=+\infty$. More -over we have following identity in Ω,

$$\int_{\Omega} u(x,t)dx=\int_{\Omega} u_0(x)dx+\int_0^t\int_{\Omega} e^{ku(x,s)}dxds.$$

Theorem 1. 2 Let $u_0\in C(\overline{\Omega})$ be nonnegative and nontrivial. Then we can get the following estimates for the blow-up time $T\leq\frac{e^{-\frac{k\int_{\Omega}u_0(x)dx}{|\Omega|}}}{k}$.

Theorem 1. 3 Let u be a solution of (1. 1) and blow-up at time T. Then we obtain

$$\lim_{t\to T^-}\frac{\|u(.,t)\|_{L^{\infty}(\overline{\Omega})}}{\ln[k(T-t)]}=-\frac{1}{k}.$$

Theorem1. 4 Let us consider the problem (1. 1) with k>1 in $\Omega = B_R = \{|x| < R\}$, $u(0,t)$ is bounded and $u_0 \in C^1(\overline{B_R})$ be a radial nonnegative function, with a unique maximum at the origin, that is

$u_0 = u_0(r) \geq 0, \quad u_0'(r) < 0,$ if $\ 0 < r \leq R, \ u_0''(r) < 0.$

Then the blow-up set of the solution consists of the only point $x = 0$.

Theorem1. 5 (Blow-up sets: general case.) Let us consider the problem (1. 1) in a general domain Ω with $\alpha > 1$. Given $x_0 \in \Omega$ and $\varepsilon > 0$, there exists an initial condition u_0 such that $B(u) \subset B_\varepsilon(x_0) = \{x \in \Omega; \|x - x_0\| < \varepsilon\}$.

2. Local existence and uniqueness.

This section is devoted to the proof of Theorem1. 1. Simultaneously, the comparison principle for the solutions of problem (1. 1) is also proved. The existence and uniqueness of the solution of problem (1. 1) will be obtained via *Banach*'s fixed point theorem.

Fix $t_0 > 0$ and consider the Banach space $X_{t_0} = C([0,t_0]; C(\overline{\Omega}))$ with the norm

$|||u||| = \max_{0 \leq t \leq t_0} \|u\|_{L^\infty(\Omega)}$.

We will obtain that solution for problem (1. 1) as fixed point of the operator $T : X_{t_0} \to X_{t_0}$ defined by

$$T_{u_0}(u)(x,t) = u_0(x) + \int_0^t \int_\Omega J\left(\frac{x-y}{u^\alpha(y,s)}\right) u^{1-N\alpha}(y,s)\,dyds - \int_0^t \int_\Omega J\left(\frac{x-y}{u^\alpha(x,s)}\right) u^{1-N\alpha}(x,s)\,dyds + \int_0^t e^{ku(x,t)}ds.$$

The following lemma is considerable important to our study and will be main ingredient of our proof.

Lemma 2. 1 Let u_0, v_0 nonnegative functions such that $u_0, v_0 \in C(\overline{\Omega})$, and $u, v \in X_{t_0}$, then there exists a positive constant $C = C(k, \|u\|_{X_{t_0}}, \|v\|_{X_{t_0}}, \Omega)$, such that

$$\|T_{u_0}(u) - T_{v_0}(v)\|_{X_{t_0}} \leq \|u_0 - v_0\|_{L^\infty(\overline{\Omega})} + Ct\|u - v\|_{X_{t_0}}. \tag{2. 1}$$

Proof: First we check that T_{u_0} maps X_{t_0} to X_{t_0}. For any $(x,t)\in\overline{\Omega}\times[0,t_0]$, we get

$$|T_{u_0}(u)(x,t)-u_0(x)|\le|\int_0^t\int_\Omega J(\frac{x-y}{u^\alpha(y,s)})u^{1-N\alpha}(y,s)dyds|+|\int_0^t\int_\Omega J(\frac{x-y}{u^\alpha(x,s)})u^{1-N\alpha}(x,s)dyds|+|\int_0^t e^{ku(x,s)}ds|$$

$$\le 2M|\Omega|t\|u\|_{X_{t_0}}+te^{k\|u\|_{X_{t_0}}}$$

$$\le Ct(\|u\|_{X_{t_0}}+e^{k\|u\|_{X_{t_0}}}).$$

Here $C=\max\{2M|\Omega|,1\}$ and $M=\|J\|_{L^\infty(R^N)}$.

And we obtain that $T_{u_0}(u)$ is continuous at $t=0$.

Now for any $(x,t_1),(x,t_2)\in(\overline{\Omega})\times[0,t_0],t_1<t_2$,

we have

$$|T_{u_0}(u)(x,t_2)-T_{u_0}(u)(x,t_1)|\le|\int_{t_1}^{t_2}\int_\Omega J(\frac{x-y}{u^\alpha(y,s)})u^{1-N\alpha}(y,s)dyds|+|\int_{t_1}^{t_2}\int_\Omega J(\frac{x-y}{u^\alpha(x,s)})u^{1-N\alpha}(x,s)dyds|$$

$$+|\int_{t_1}^{t_2}e^{ku(x,s)}ds|$$

$$\le 2M(t_2-t_1)|\Omega|\|u\|_{X_{t_0}}+(t_2-t_1)e^{k\|u\|_{X_{t_0}}}$$

$$\le C(t_2-t_1)(\|u\|_{X_{t_0}}+e^{k\|u\|_{X_{t_0}}}).$$

Therefore, mapping $T_{u_0}(u)$ is continuous in time for any a $t\in(0,t_0]$.

Since the convolution in space with the function J is also uniformly continuous, $T_{u_0}(u)$ is continuous as a function of x. Then for any $u_0\in C(\overline{\Omega})$ and $u\in X_{t_0}$, we can get $T_{u_0}(u)\in C([0,t_0];C(\overline{\Omega}))$, which shows that $T_{u_0}(u)$ maps X_{t_0} into X_{t_0}.

To get the estimate (2. 1) we argue as follows: for any $(x,t)\in\overline{\Omega}\times[0,t_0]$, we have that

$$|T_{u_0}(u)(x,t)-T_{v_0}(v)(x,t)|\le\|u_0-v_0\|_{L^\infty(\Omega)}+|\int_0^t(e^{ku(x,s)}-e^{kv(x,s)})ds|$$

$$+|\int_0^t\int_\Omega J(\frac{x-y}{u^\alpha(y,s)})u^{1-N\alpha}(y,s)-J(\frac{x-y}{v^\alpha(y,s)})v^{1-N\alpha}(y,s)dyds|$$

$$+|\int_0^t\int_\Omega J(\frac{x-y}{u^\alpha(x,s)})u^{1-N\alpha}(x,s)-J(\frac{x-y}{v^\alpha(x,s)})v^{1-N\alpha}(x,s)dyds|$$

$\leq \|u_0 - v_0\|_{L^\infty(\Omega)} + ke^{k\eta} t \|u - v\|_{X_{t_0}} + I_1 + I_2.$

where $\eta = \max\{\|u\|_{X_{t_0}}, \|v\|_{X_{t_0}}\}$.

To study I_1 and I_2 we proceed the way as stated in [1]. Let's consider the following sets

$A^+(s) = \{y \in \Omega \mid u(y,s) \geq v(y,s)\}$ and $A^-(s) = \{y \in \Omega \mid u(y,s) < v(y,s)\}$.

And we apply *Fubili*'s theorem to get

$I_1 \leq \int_0^t \int_\Omega |u(y,s) - v(y,s)| dyds$.

Furthermore we have that $I_1 \leq \int_0^t \int_\Omega |u(y,s) - v(y,s)| dyds \leq t|\Omega| \|u - v\|_{X_{t_0}}$.

Similarly, we can get $I_2 \leq \int_0^t \int_\Omega |u(x,s) - v(x,s)| dyds \leq t|\Omega| \|u - v\|_{X_{t_0}}$.

Hence we obtain

$$||| T_{u_0}(u)(x,t) - T_{v_0}(v)(x,t) ||| \leq \|u_0 - v_0\|_{L^\infty((\Omega))} + ke^{k\eta} t \|u - v\|_{X_{t_0}} + 2t|\Omega| \|u - v\|_{X_{t_0}}$$

$$\leq \|u_0 - v_0\|_{L^\infty(\Omega)} + (ke^{k\eta} + 2|\Omega|) t \|u - v\|_{X_{t_0}}.$$

Letting $u_0 \equiv v_0$ and choosing a proper t_0 such that $Ct_0 < 1$, (2. 1) ensures that T_{u_0} is a strict contraction

in the ball $B(u_0, 2\|u_0\|_{L^\infty(\bar{\Omega})})$ in X_{t_0}. Furthermore, for any u and v in such a ball, we obtain $|\eta| \leq C\|u_0\|_{L^\infty(\bar{\Omega})}$

In other words, there exists a constant C that depends on only J and u_0, such that

$\|T(u) - T(v)\| \leq Ct_0 \|u - v\|_{X_{t_0}}$.

It is not difficult for us to choose t_0 such that $Ct_0 < \frac{1}{2}$ to obtain a strict contraction in the ball $B(u_0, \ 2\|u_0\|_{L^\infty(\bar{\Omega})})$. The proof is completed.

Next, we will prove Theorem 1. 1.

Proof of Theorem 1. 1: As a consequence of *Banah*'s fixed point theorem and the lemma aforementioned, we can get the existence and uniqueness of solution for problem (1. 1).

Suppose there exists a positive S such that $\|u\|_{X_{t_0}} \le S$, and take it as initial datum $u(\cdot,t_0) \in C(\overline{\Omega})$, we will find it possible to extend the solution up to some interval $[0,t_1)$, of course $t_1 > t_0$.

From the equations $(1.\ 1)_1$, we can obtain u verifies the identity as follow

$$u(x,t)-u_0(x)=\int_0^t\int_\Omega J\Big(\frac{x-y}{u^\alpha(y,s)}\Big)u^{1-N\alpha}(y,s)dyds-\int_0^t\int_\Omega J\Big(\frac{x-y}{u^\alpha(x,s)}\Big)u^{1-N\alpha}(x,s)dyds+\int_0^t e^{ku(x,s)}ds.$$

Integrating in the variable x and applying *Fubini*'s theorem, we can get

$$\int_\Omega u(x,t)dx-\int_\Omega u_0(x)dx=\int_0^t\int_\Omega e^{ku(x,s)}dxds.$$

Remark 2. 1 u is a solution of (1. 1) if and only if u satisfies

$$u(x,t)=u_0(x)+\int_0^t\int_\Omega J\Big(\frac{x-y}{u^\alpha(y,s)}\Big)u^{1-N\alpha}(y,s)dyds-\int_0^t\int_\Omega J\Big(\frac{x-y}{u^\alpha(x,s)}\Big)u^{1-N\alpha}(x,s)dyds+\int_0^t e^{ku(x,t)}ds.$$

3. Blow-up and blow-up rates.

Proof of Theorem 1. 2

Since $u_t(x,t)=\int_\Omega J\Big(\frac{x-y}{u^\alpha(y,t)}\Big)u^{1-N\alpha}(y,t)dy-\int_\Omega J\Big(\frac{x-y}{u^\alpha(x,t)}\Big)u^{1-N\alpha}(x,t)dy+e^{ku(x,t)}$, we integrate in $x\in\Omega$ and apply *Fubini*'s theorem and *Jensen*'s inequality to get

$$\frac{\partial}{\partial t}\int_\Omega u(x,t)dx=\int_\Omega e^{ku(x,t)}dx \ge |\Omega| e^{\frac{k\int_\Omega u(x,t)dx}{|\Omega|}}.$$

Noting that $\int_\Omega u(x,t)dx$ is not global; thus $u(x,t)$ cannot be global either. By Theorem1. 1, we have that the solution of the problem (1. 1) blows up in $L^\infty(\Omega)-norm$. Moreover, integrating the above inequality we obtain the following estimate for the life span

$$T \le \frac{e^{-\frac{k\int_\Omega u(x,t)dx}{|\Omega|}}}{k}.$$

Proof of Theorem 1. 3: Let $T<\infty$ is the maximal time of existence of a blowing up solution and suppose $x_0 \in \overline{\Omega}$ be such that $\max_{0\le t\le t_0} u(x,t) = u(x_0,t)$. From equation $(1.\ 1)_1$ with this point, we can get

$$u_t(x,t) = \int_\Omega J(\frac{x-y}{u^\alpha(y,t)})u^{1-N\alpha}(y,t)dy - \int_\Omega J(\frac{x-y}{u^\alpha(x_0,t)})u^{1-N\alpha}(x_0,t)dy + e^{ku(x_0,t)} \le e^{ku(x_0,t)}.$$

So integrating in (t,T) for the inequality, we obtain

$$\max_{x\in\overline{\Omega}} u(x,t) \ge -\frac{1}{k}\ln[k(T-t)]. \tag{3. 1}$$

Noticing that for any $(x,t)\in(\overline{\Omega},t)$ it holds that

$$u_t(x,t) \ge -u(x,t) + e^{ku(x,t)}.$$

Since $u(x,t)$ blows up, we can get for all $\varepsilon > 0$, there exists a time t_0 such that for $t\in(t_0,T)$

$$u_t(x_0,t) \ge (1-\varepsilon)e^{ku(x_0,t)}.$$

Similarly integrating in (t,T), it holds

$$\max_{x\in\overline{\Omega}} u(x,t) \le -\frac{1}{k}\ln[(1-\varepsilon)k(T-t)]. \tag{3. 2}$$

Now let $\varepsilon \to 0$, combining (3. 1) and (3. 2), we obtain

$$\lim_{t\to T^-} \frac{\|u(.,t)\|_{L^\infty(\overline{\Omega})}}{\ln[k(T-t)]} = -\frac{1}{k}.$$

4. Blow-up sets.

In this section we give some results concerning the blow-up sets for the solution to the problem (1. 1). We suppose that u is a solution to (1. 1) blowing up at time T and investigate the symmetric case. To simplify, we only consider the case in one-dimension, that is $\Omega = (-L, L)$, the radial case is analogous.

First, we prove a lemma that says if the initial condition has a unique maximum at the origin, then the solution has a unique maximum at this point for every $t \in (0,t)$.

Lemma 4. 1 For any k, under the hypothesis on the initial condition mentioned in Theorem1. 4, we have that the solution is symmetric and such that $u_x < 0$ in $(0, L] \times (0, T)$.

Proof: Notice that $u(-x,t)$ is also a solution of the problem (1. 1), the symmetry follows from the uniqueness.

If we donate $w(x,t) = u_x(x,t)$, then $w(x,t)$ verifies the equation

$$w_t(x,t) = \int_{-L}^{L} J'(\frac{x-y}{u^\alpha(y,t)})u^{1-(N+1)\alpha}(y,t) - J'(\frac{x-y}{u^\alpha(x,t)})u^{1-(N+1)\alpha}(x,t)dy$$

$$+\int_\Omega J'(\frac{x-y}{u^\alpha(x,t)})\alpha(x-y)w(y,t)u^{-(N+1)\alpha}(y,t)dy$$

$$+\int_\Omega J(\frac{x-y}{u^\alpha(x,t)})(1-N\alpha)u^{-N\alpha}(x,t)w(x,t)dy + \alpha e^{\alpha u(x,t)}w(x,t).$$

We will find it easy to obtain a contradiction from the equation. If we assume that there exists a point $(x_0, t_0) \in (0, L) \times (0, T)$ at which $w(x_0, t_0) = 0$. We use here that J' is odd and the symmetry of u to obtain $w(x,t) = u_x(x,t) < 0$ in $(0, L) \times (0, T)$.

Now, time is right for us to prove Theorem 1. 4.

Proof of Theorem 1. 4. The proof consists of several steps, following the ideas of [12] blow-up for a nonlocal diffusion problem with Neumann boundary conditions and a reaction term.

Step 1 First, we prove that the only blow-up point that verifies the blow-up estimate is $x=0$. For a fixed $x_0>0$, let $\psi(t)=u(0,t)-u(x_0,t)$ and employing the mean value theorem and the way we processed before, then the function $\varphi(t)$ holds

$$\psi'(t)=\int_{-L}^{L}[J(\frac{-y}{u^{\alpha}(y,t)})u^{1-N\alpha}(y,t)-J(\frac{-y}{u^{\alpha}(0,t)})u^{1-N\alpha}(0,t)]dy$$

$$-\int_{\Omega}[J(\frac{x_0-y}{u^{\alpha}(y,t)})u^{1-N\alpha}(y,t)-J(\frac{x_0-y}{u^{\alpha}(x_0,t)})u^{1-N\alpha}(x_0,t)]dy+\alpha\psi(t)e^{\alpha\gamma(t)}$$

$$\geq\int_{-L}^{L}[J(\frac{-y}{u^{\alpha}(y,t)})-J(\frac{x_0-y}{u^{\alpha}(y,t)})]u^{1-N\alpha}(y,t)dy-[u(0,t)-u(x_0,t)]+\alpha\psi(t)e^{\alpha\gamma(t)}$$

$$\geq-\psi(t)+\alpha\psi(t)e^{\alpha\gamma(t)}.$$

where $\gamma(t)\in[u(x_0,t),u(0,t)]$. Integrating the above inequality we have

$$\ln(\psi)(t)-\ln(\psi)(t_0)\geq\int_{t_0}^{t}-1+\alpha e^{\alpha\gamma(s)}ds. \tag{4. 1}$$

Now we argue by contradiction. Assume that $\lim_{t\to T^-}e^{u(x_0,t)}(T-t)^{\frac{1}{\alpha}}=C_\alpha$,

since $u(x_0,t)\leq\gamma(t)\leq u(0,t)$, we get $\lim_{t\to T^-}e^{\gamma(t)}(T-t)^{\frac{1}{\alpha}}=C_\alpha$.

Therefore we have $\gamma(t)\geq(C_\alpha^\alpha-\varepsilon)(T-t)^{-1}$, as $\varepsilon\to0^+$.

Moreover, if $u(0,t)$ is bounded, we can get

$$1+\alpha e^{\alpha\gamma(s)}ds\geq\alpha\int_{t_0}^{t}\frac{C_\alpha^\alpha-\varepsilon}{T-s}ds-C$$

$$=-\alpha(C_\alpha^\alpha-\varepsilon)\ln(T-t)-C. \tag{4. 2}$$

From (4. 1) and (4. 2), we can get

$$\psi(t)\geq Ce^{-\alpha(C_\alpha^\alpha-\varepsilon)\ln(T-t)}=C(T-t)^{-1+\alpha\varepsilon}.$$

Using this fact, we have

$$0=\lim_{t\to T^-}\frac{\psi(t)}{\ln[\alpha(T-t)]}\leq\lim_{t\to T^-}\frac{C(T-t)^{-1+\alpha\varepsilon}}{\ln[\alpha(T-t)]}=-\infty,$$

a contradiction that proves our claim.

Step 2. We conclude by showing that the only possible blowing up point is the origin. To this end, let us perform the following change of variables

$$Z(x,s)=e^{u(x,t)}(T-t)^{-\frac{1}{\alpha}},(T-t)=e^{-s}. \tag{4. 3}$$

$$Z_s(x,s)=e^{u(x,t)-\frac{s}{\alpha}}(T-t)^{-\frac{1}{\alpha}}\cdot(u_t e^{-s}-\frac{1}{\alpha})$$

$$=Z(x,s)e^{-s}(\int_{-L}^{L}[J(\frac{x-y}{u(y,t)})-J(\frac{x-y}{u(x,t)})]dy+e^{\alpha u(x,t)})-\frac{1}{\alpha}Z(x,s)$$

$$=Z(x,s)e^{-s}\int_{-L}^{L}[J(\frac{x-y}{u(y,t)})-J(\frac{x-y}{u(x,t)})]dy-\frac{1}{\alpha}Z(x,s)+Z^{\alpha+1}(x,s).$$

Note that the blow-up rate of u implies that $Z(x,s)\le C$ for every $(x,s)\in[-L,L]\times(-\ln T,+\infty)$.

Besides we have $|\int_{-L}^{L}[J(\frac{x-y}{u(y,t)})-J(\frac{x-y}{u(x,t)})]dy|\le\int_{-L}^{L}|J(\frac{x-y}{u(y,t)})-J(\frac{x-y}{u(x,t)})|dy$

$$\le\int_{-L}^{L}|u(y,t)-u(x,t)|dy\le 2\int_{-L}^{L}u(x,t)dx.$$

Therefore, we can obtain that from $\lim_{t\to T^-}u(x,t)=-\frac{1}{\alpha}\ln[\alpha(T-t)]$,

$$Z_s(x,s)\le Z(x,s)e^{-s}2(-\frac{1}{\alpha}\ln(\alpha e^{-s})-\frac{1}{\alpha}Z(x,s)+Z^{\alpha+1}(x,s)$$

$$\le C_1e^{-s}+C_2se^{-s}-\frac{1}{\alpha}Z(x,s)+Z^{\alpha+1}(x,s). \tag{4. 4}$$

At the same time, we notice that for any $x\neq 0$, $Z(x,s)$ is bounded and doesn't converge to $\alpha^{-\frac{1}{\alpha}}$, so we can conclude that $Z(x,s)\to 0$, as $s\to+\infty$.

Thus, for a given $\varepsilon>0$, using $Z(x,s)\to 0$ in (4. 4), we have

$$Z_s(x,s)\le C_1e^{-s}+C_2se^{-s}-(\frac{1}{\alpha}-\varepsilon)Z(x,s).$$

By a comparison argument, it follows that

$$Z(x,s)\le C_1e^{-s}+C_2(1+s)e^{-s}+C_3e^{-(\frac{1}{\alpha}-\varepsilon)s}. \tag{4. 5}$$

Going back to the equation verified by $Z(x,s)$, it holds

$$(e^{\frac{s}{\alpha}}Z(x,s))_s=e^{\frac{s}{\alpha}}\cdot\frac{1}{\alpha}Z(x,s)+e^{\frac{s}{\alpha}}Z_s(x,s)$$

$$=e^{\frac{s}{\alpha}}[Z(x,s)\cdot e^{-s}\int_{-L}^{L}J(\frac{x-y}{u(y,t)})-J(\frac{x-y}{u(x,t)})dy+Z^{k+1}(x,s)]$$

$$\le e^{\frac{s}{\alpha}}(C_1e^{-s}+C_2se^{-s}+Z^{\alpha+1}(x,s)).$$

Integrating on (s_0,s), we have

$$Z(x,s)\le e^{-\frac{s}{\alpha}}[C_1+\int_{s_0}^{s}e^{-(1-\frac{1}{\alpha})\xi}(C_2+C_3\xi)+e^{\frac{\xi}{\alpha}}Z^{\alpha+1}(x,\xi)d\xi].$$

From (4. 5), we obtain $e^{\frac{s}{\alpha}}Z^{\alpha+1}(x,s)\to 0$, as $\alpha\to+\infty$. While $Z(x,s)$ is bounded, so we get that

$$Z(x,s)\le e^{-\frac{s}{\alpha}}(C_1+\int_{s_0}^{s}(1+\xi)e^{-(1-\frac{1}{\alpha})\xi}C_2d\xi).$$

Due to $\alpha>1$, we have

$$Z(x,s)\le C_1e^{-\frac{s}{\alpha}}.$$

This implies that $e^{u(x,t)}$ verifies

$$e^{u(x,t)}=Z(x,s)e^{\frac{s}{\alpha}}\le C_1.$$

Consequently we can get $u(x,t)\le C$.

In other words, $u(x,t)$ is bounded, so the theorem 1. 4 is proved.

Next, we are ready to seek blow-up set in a general domain Ω with initial condition for the problem (1. 1). We will find that blow-up set is localized around a given point in $\overline{\Omega}$.

Proof of Theorem 1. 5 Given $x_0 \in \overline{\Omega}$ and $\varepsilon > 0$, what we want to do is to construct an initial condition u_0 such that

$$B(u) \subset B_d(x_0) = \{x \in \overline{\Omega} : \|x - x_0\| < \varepsilon\}. \qquad (4.\ 6)$$

To this end we will consider u_0 concentrated near x_0 and small away from x_0.

Let ϕ be a nonnegative smooth function such that $\mathrm{supp}(\phi) \subset B_{\varepsilon/2}(x_0)$ and $\phi(x) > 0$ for all $x \in B_{\varepsilon/2}(x_0)$.

Now, let

$$u_0(x) = N\phi(x) + \delta .$$

We will choose N large and δ small in such a way that (4. 6) holds.
Taking N large enough, from the estimate in Theorem 1. 2, we have

$$T \le \frac{e^{-\frac{\alpha \int_\Omega u_0(x,t)dx}{|\Omega|}}}{\alpha} \le \frac{1}{\alpha} \cdot e^{-C(\Omega,\alpha,\phi)N} .$$

Besides we can get some T as small as we need.

Now let $\zeta(x,t) = e^{u(x,t)}$, using the upper bound for the blow-up rate

$$\max_{x\in\Omega} u(x,t) \le -\frac{1}{\alpha} \ln[(1-\varepsilon)\alpha(T-t)] \le -C \ln[\alpha(T-t)] \ ,$$

and $\max_{x\in\overline{\Omega}} e^{u(x,t)} \le C(T-t)^{-\frac{1}{\alpha}}$, we can get the following conclusion for any $\overline{x} \in \Omega$,

$$\zeta_t(\overline{x},t) = \zeta(x,t)[\int_{-L}^{L} [J(\frac{\overline{x}-y}{u(y,t)}) - J(\frac{\overline{x}-y}{u(\overline{x},t)})]dy + e^{\alpha u(\overline{x},t)}]$$

$$\le (T-t)^{-\frac{1}{\alpha}} C_{(J,k,\Omega)}\{-\ln[k(T-t)]\} + \zeta^{\alpha+1}(\overline{x},t). \qquad (4.\ 7)$$

To this consideration, $\zeta(\overline{x},t)$ is a subsolution to the following equation,

$$\theta(\overline{x},t) = (T-t)^{-\frac{1}{\alpha}} C_{(J,k,\Omega)}\{-\ln[\alpha(T-t)]\} + \theta^{\alpha+1}(\overline{x},t). \qquad (4.\ 8)$$

Hence, if we assume $\zeta(\overline{x},0) \le \theta(0)$, we have

$$\zeta(\bar{x},t)\le\theta(t). \tag{4. 9}$$

Now we just have to prove that a solution θ to (4. 8) beginning with $t=0$ (or $\theta(0)=e^{\delta}$) remains bounded up to $t=T$, provided that δ and T are small enough. To get this we also use some ideas from [10].

Let $z(s)=(T-t)^{\frac{1}{\alpha}}\theta(t)$ and $(T-t)=e^{-s}$.

We will find it easy to see that $z(s)$ verifies

$$z'(s)=-\frac{1}{\alpha}\cdot e^{-\frac{s}{\alpha}}\theta(t)+e^{-\frac{s}{\alpha}}\cdot\theta_t(t)\cdot e^{-s}=Ce^{-s}(-\ln[\alpha(T-t)])-\frac{z(s)}{\alpha}+z^{\alpha+1}(s),$$

and $z(-\ln T)=T^{\frac{1}{\alpha}}e^{\delta}$. When $t=0$, δ and T are small enough, the following formula is true $z'(-\ln T)=T^{\frac{\alpha+1}{\alpha}}e^{(\alpha+1)\delta}-\frac{1}{\alpha}T^{\frac{1}{\alpha}}e^{\delta}-CT\ln(\alpha T)<0$.

From this fact, we can easily prove that $z'(s)<0$, for all $s>-\ln T$, and $z(s)\to-\infty$, as $s\to\infty$. Going back to the equation verified by $z(s)$ and step 2 of Theorem 1. 5, we obtain that $z(s)\le Ce^{-\frac{s}{\alpha}}$, hence $\theta(t)<C$, for all $t\in[0,T)$. It will not be difficult to obtain that $w(\bar{x},t)\le\theta(t)<C$ from the fact that $\theta(t)$ is bounded and (4.8), so $u(\bar{x},t)<C$ for any $\bar{x}\in\bar{\Omega}\setminus B_{\varepsilon}(x_0)$. To this end, our proof is completed.

Acknowledgements

This work was partially supported by NNSF of China ((11301419), partially supported by Projects Supported by Scientific Research Funds of SiChuan Provincial Education Department (13ZA0010, 14ZB0143) and Technology Innovation Projects of China West Normal University (427120).

References

[1] M. Bogoya, *Blowing up boundary conditions for a nonlocal nonlinear diffusion equation in several space dimensions.* Nonlinear Anal, 72(2010), 143-150.

[2] M. Bogoya, *A nonlocal nonlinear diffusion equation in higher space dimensions*, J. Math. Anal. Appl. 344(2008), 601-615.

[3] E. Chasseigne, M. Chaves, J. D. Rossi, *Asymptotic behavior for nonlocal diffusion equations*, *Journal de Math´emati -ques Pures et Appliqu´ees. Neuviemes´erie*, 86(3), (2006), 271–291.

[4] C. Cortazar, M. Elgueta, J. D. Rossi, N. Wolanski. *How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems,* Arch. Ration. Mech. Anal. 187(2008), 137-156.

[5] K. Deng, H. A. Levine, *The role of critical exponents in blow-up theorems: the sequel,* J. Math. Anal. Appl. 243 (2000), 85-126.

[6] P. Fife, *Some nonclassical trends in parabolic-like evolution,* in: Trends in Nonlinear Analysis, Springer, Berlin, 2003.

[7] A. Friedman, B. McLeod, *Blow-up of positive solutions of semilinear heat equations*. Indiana Univ. Math. J. 34 (1985), 425-447.

[8] J. Garcia-Melian, F. Quiros, *Fujita exponents for evolution problems with nonlocal diffusion*, Journal of Evolution Equations, 10(1), (2010), 147–161.

[9] L. I. Ignat, J. D. Rossi, A. S. Antolin, *Lower and upper bounds for the first eigenvalue of nonlocal diffusion problems in the whole space*, J. Differential Equations. 252 (2012), 6429-6447.

[10] Z. P. Li, C. L. Mu. *Global existence and blow-up analysis for a nonlinear diffusion equation with inner absorportion and boundary flux,* Dynamical Systems. 26(2), (2011), 147-159.

[11] Z. P. Li, C. L. Mu, *Critical exponents and blow-up rate for a nonlinear diffusion equation with logarithmic boundary flux*, Nonlinear Anal. 73(2010), 933-939.

[12] M. Pérez-Llanos, J. D. Rossi. *Blow-up for a non-local diffusion problem with Neumann boundary conditions and a reaction term,* Nonlinear Anal. 70 (2009), 1629-1640.

[13] M. Pérez-Llanos, J. D. Rossi, *Numerical Approximations for a Nonlocal Evolution Equation, SIAM J. Numer. Anal.* 49 (2011): 2103-2123.

[14] Y. L. Wang, Z. Y. Xiang, J. S. Hu, *Blow-up Analysis for a Nonlocal Diffusion Equation with Reaction and Absorp -tion,* Journal of Applied Mathematics, doi:10. 1155/2012/648067, 2012.