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Global Dynamic of Einstein-Maxwell system for a perfect charged relativistic fluid in a Bianchi type I space-time

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Abstract

We discuss the global existence and uniqueness of solution to the Einstein-Maxwell system with the cosmological constant. The space-time considered being a Bianchi of type I, in the case of a perfect charged relativistic fluid.We obtain a global existence theorem in the single case where the derivatives of initial data of potentials of gravitation a, b and the cosmological constant Λ are positive. The problem of initial constraints is also highly studied. In the end, we remark that if $a \neq b$, the space-time never becomes empty to future infinity as it is the case when $a = b$.

AMS subject classification: 83Cxx.
Keywords: Bianchi type, Differential system, problem of constraints, charged particles, local existence, global existence.

1. Introduction

In this paper, we study the evolution of a charged perfect fluid of pure radiation type, the background space-time being the time-oriented Bianchi type 1 space-time, which is an immediate generalization of the flat Friedman-Lemaitre-Robertson-Walker space-time, also known to be the basic space-time of Cosmology. In Cosmology, homogeneous phenomena such as the one we consider here are relevant. The whole universe is modeled and particles in the kinetic theory may be particles of ionized gas as nebular galaxies or even cluster of galaxies, burning reactors, solar wind, for which only the evolution in time is really significant. In the case we consider, the evolution is governed by the coupled Einstein-Maxwell system, the Einstein equations for gravitational field inquiring about gravitational effects, whereas the Maxwell equations for the electromagnetic field inform about electromagnetic effects.

The Einstein theory stipulates that the gravitational field, which in the case we consider, depends on the two real valued functions a and b, called potentials of gravitation, is determined through the Einstein equations, by the material and energetic content of space-time. The space-time content in this case is represented by a stress-matter tensor defined by both the matter density ρ and the electromagnetic field F.

The Maxwell equations are the basic equations of Electromagnetism and determine the electromagnetic field F created by the charged particles. We consider the case where the electromagnetic field F is generated through the Maxwell equations by a charged density e, and a future pointing unit vector u, tangent at any point to the temporal axis. The system is coupled in the sense that a and b which are subjected to the Einstein equations also determine the electromagnetic field , whereas the electromagnetic field F, which is subjected to the Maxwell equations also appears in the Einstein equations through the Maxwell tensor $\tau_{\alpha\beta}$.

The Einstein-Maxwell system coupled to the conservation laws, turns out to be a non-linear differential system to determine $a,\ b,\ \rho$ and F.

The main objective of this paper is to extend to the case where the gravitational field depends on two real valued functions a and b, the result obtained by N. Noutchegueme and C. Nangne[1] , where the gravitational field was depending only on a single real valued function a.

We prove using a change of variables, that if the cosmological constant $\Lambda > 0$ and if the initial datum $\dot{b}_0$ of the derivative with respect to t of b is positive, then there exists a global solution to the coupled system. And we also prove that if this derivative is negative, or if the cosmological constant $\Lambda < 0$ even if $\dot{b}_0 > 0$, then there cannot exist global solutions. The fact that we prove global existence with cosmological constant Λ, is with a great interest, in the sense that some recent observations show that the whole universe is in an accelerated expansion, and it is the presence of the cosmological constant in the Einstein equations which mathematically shapes this phenomenon.

The paper organizes as follows:

In section 2, we introduce equations and give some preliminaries. In section 3, we study the global existence.

2. Preliminaries and Equations

2.1. The Spacetime, the Unknowns and the reduced system

We consider the Bianchi type I space-time $\left(\mathbb{R}^4, g\right)$ and the usual coordinates in $\mathbb{R}^4$; where $g = \left(g_{\alpha\beta}\right)$ stands for the metric tensor of hyperbolic signature $(-, +, +, +)$ that can be written:

$$g = -(dt)^2 + a^2(t)\left(dx^1\right)^2 + b^2(t)\left[\left(dx^2\right)^2 + \left(dx^3\right)^2\right] \tag{1}$$

in which $a > 0,\ b > 0$ are two continuously differentiable unknown functions of the single variable t.

Following Lichnerowicz[2] , the Einstein-Maxwell system with the cosmological constant Λ reads:

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + g_{\alpha\beta}\Lambda = 8\pi\left(T_{\alpha\beta} + \tau_{\alpha\beta}\right) \tag{2}$$

$$\nabla_\alpha F^{\alpha\beta} = eu^\beta \tag{3}$$

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0, \tag{4}$$

where:

- (2) are the Einstein equations for the metric tensor g that represents the gravitational field, and:

  $R_{\alpha\beta}$ is the Ricci tensor, contracted of the curvature tensor;

  $R = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar curvature, contracted of the Ricci tensor;

  $T_{\alpha\beta}$ and $\tau_{\alpha\beta}$ are respectively the matter tensor and the Maxwell tensor we specify below;

- (3) and (4) are respectively the first and the second groups of the Maxwell equations for the electromagnetic field $F = \left(F^{0i}, F_{ij}\right)$ which is a closed unknown antisymmetric $2-$form depending on the single variable t, F^{0i} and F_{ij} stand for the electric and magnetic parts respectively.

  In (3) , $e \geq 0$ is an unknown real-valued function of t,representing the charge density, and $u = \left(u^\alpha\right)$ is a time-like future pointing unit vector **tangent at any point to the time axis;** eu^β is the Maxwell current. ∇_α is the usual covariant derivative in g.

- (4) only expresses the fact that $dF = 0$, because F is closed.

The general expression of the matter tensor of a relativistic perfect fluid is, in the chosen signature of g :

$$T_{\alpha\beta} = (\rho + p)\,u_\alpha u_\beta + pg_{\alpha\beta}$$

in which $\rho \geq 0$ and $p \geq 0$ are unknown functions of t, representing respectively the matter density and the pressure.

We consider here a perfect fluid of pure radiation type, which means that $p = \frac{\rho}{3}$. In that situation, one obtains:

$$T_{\alpha\beta} = \frac{4}{3}\rho u_\alpha u_\beta + \frac{1}{3}\rho g_{\alpha\beta}. \tag{5}$$

In order to simplify, we consider a **co-moving fluid**. This implies that:

$$u^i = u_i = 0, \; u^0 = 1. \tag{6}$$

The particles are then supposed to be spatially at rest.

In what follows, using (3) , the well known identities given below will also be very useful:

$$\nabla_\alpha \nabla_\beta F^{\alpha\beta} = 0. \tag{7}$$

Next, the Maxwell tensor $\tau_{\alpha\beta}$ is defined by:

$$\tau_{\alpha\beta} = -\frac{g_{\alpha\beta}}{4} F^{\lambda\mu} F_{\lambda\mu} + F_{\beta\lambda} F^{\lambda}_{\alpha}. \tag{8}$$

Using (7) , the Einstein-Maxwell System (2) − (3) − (4) then yields:

$$\begin{gathered} R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + g_{\alpha\beta}\Lambda = 8\pi \left(T_{\alpha\beta} + \tau_{\alpha\beta}\right) \\ \nabla_\alpha \left(e u^\alpha\right) = 0 \\ \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0. \end{gathered} \tag{9}$$

The following property of $\tau_{\alpha\beta}$ will be very useful:

Proposition 1.

$$\nabla_\alpha \tau^\alpha_\beta = F_{\beta\lambda} \nabla_\alpha F^{\alpha\lambda}. \tag{10}$$

Proof. See Noutchegueme & Nangne [1]. ∎

Proposition 2. $e, \; \rho, \; F^{0i}, \; F_{ij}$ satisfy the following equations:

$$\begin{gathered} \dot{e} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) e = 0 \\ 3\dot{\rho} + 4\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)\rho = 0 \\ \dot{F}^{0i} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) F^{0i} = 0 \\ \dot{F}_{ij} = 0 \end{gathered} \tag{11}$$

and are given by the relations:

$$
\begin{aligned}
e &= \left(\frac{a_0}{a}\right)\left(\frac{b_0}{b}\right)^2 e_0 \\
\rho &= \rho_0 \left(\frac{a_0}{a}\right)^{\frac{4}{3}}\left(\frac{b_0}{b}\right)^2 \\
F^{0i} &= \left(\frac{a_0}{a}\right)\left(\frac{b_0}{b}\right)^2 E^i \\
F_{ij} &= \phi_{ij},
\end{aligned}
\tag{12}
$$

where $e_0 = e\,(0)\,,\ \rho_0 = \rho\,(0)\,,\ a_0 = a\,(0)\,,\ b_0 = b\,(0)\,,\ F^{0i}\,(0) = E^i\,,\ F_{ij}\,(0) = \phi_{ij}$.

Proof. See Noutchegueme & Nangne [1]. ■

Proposition 3. We have for the Einstein tensor:

$$
\begin{aligned}
S_{00} &= 2\frac{\dot{a}\dot{b}}{ab} + \left(\frac{\dot{b}}{b}\right)^2,\ S_{11} = -a^2\left[2\frac{\ddot{b}}{b} + \left(\frac{\dot{b}}{b}\right)^2\right] \\
S_{22} &= S_{33} = -b^2\left[\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab}\right] \\
S_{\alpha\beta} &= 0, \quad \text{if } \alpha \neq \beta.
\end{aligned}
\tag{13}
$$

Proof. See Noutchegueme & Nangne [1] . ■

Proposition 4. $T_{\alpha\beta}$ and $\tau_{\alpha\beta}$ are given by the following relations

$$
T_{00} = \rho,\ T_{11} = a^2\frac{\rho}{3},\ T_{22} = T_{33} = b^2\frac{\rho}{3},\ T_{\alpha\beta} = 0 \text{ if } \alpha \neq \beta \tag{14}
$$

$$
\begin{cases}
\tau_{00} = \frac{1}{2}\left[\left(a_0E^1\right)^2\left(\frac{b_0}{b}\right)^4 + \left(a_0E^2\right)^2\left(\frac{b_0^2}{ab}\right)^2 + \left(a_0E^3\right)^2\left(\frac{b_0^2}{ab}\right)^2\right] \\
\qquad + \frac{1}{2}\left[\left(\frac{\phi_{12}}{ab}\right)^2 + \left(\frac{\phi_{13}}{ab}\right)^2 + \left(\frac{\phi_{23}}{b^2}\right)^2\right] \\
\tau_{11} = \frac{1}{2}\left[-\left(aa_0E^1\right)^2\left(\frac{b_0}{b}\right)^4 + \left(a_0^2b_0^2E^2\right)^2\left(\frac{b_0}{b}\right)^2 + \left(b_0^2E^3\right)^2\left(\frac{b_0}{b}\right)^2\right] \\
\qquad + \frac{1}{2}\left[\left(\frac{\phi_{12}}{b}\right)^2 + \left(\frac{\phi_{13}}{b}\right)^2 - \left(\frac{a\phi_{23}}{b^2}\right)^2\right] \\
\tau_{22} = \frac{1}{2}\left[\left(a_0b_0E^1\right)^2\left(\frac{b_0}{b}\right)^2 - \left(b_0^2E^2\right)^2\left(\frac{a_0}{a}\right)^2 + \left(b_0^2E^3\right)^2\left(\frac{a_0}{a}\right)^2\right] \\
\qquad + \frac{1}{2}\left[\left(\frac{\phi_{12}}{a}\right)^2 - \left(\frac{\phi_{13}}{a}\right)^2 + \left(\frac{\phi_{23}}{b}\right)^2\right] \\
\tau_{33} = \frac{1}{2}\left[\left(a_0b_0E^1\right)^2\left(\frac{b_0}{b}\right)^2 + \left(b_0^2E^2\right)^2\left(\frac{a_0}{a}\right)^2 - \left(b_0^2E^3\right)^2\left(\frac{a_0}{a}\right)^2\right] \\
\qquad + \frac{1}{2}\left[-\left(\frac{\phi_{12}}{a}\right)^2 + \left(\frac{\phi_{13}}{a}\right)^2 + \left(\frac{\phi_{23}}{b}\right)^2\right] \\
\tau_{0i} = -\left(\frac{a_0}{a}\right)\left(\frac{b_0}{b}\right)^2 E^j\phi_{ij},\ \tau_{12} = \frac{1}{b^2}\left(-a_0^2b_0^4E^1E^2 + \phi_{13}\phi_{23}\right) \\
\tau_{13} = -\frac{1}{b^2}\left(a_0^2b_0^4E^1E^3 + \phi_{12}\phi_{23}\right),\ \tau_{23} = \frac{1}{a^2}\left(-a_0^2b_0^4E^2E^3 + \phi_{12}\phi_{13}\right).
\end{cases}
\tag{15}
$$

Proof.

i) For (14) , we use the expression of $T_{\alpha\beta}$ given by (5) invoking the fact that $u_0 = 1$, $u_i = 0$ and $g_{00} = -1$.

ii) For (15) , we use the expression of $\tau_{\alpha\beta}$ given by (8) which writes:

$$\tau_{\alpha\beta} = -\frac{g_{\alpha\beta}}{4}F^{\lambda\mu}F_{\lambda\mu} + F_{\beta\lambda}F_{\alpha}^{\lambda}.$$

- We then have:

$$\tau_{00} = -\frac{g_{00}}{4}F^{\lambda\mu}F_{\lambda\mu} + F_{0\lambda}F_0^{\lambda}.$$

But $F^{\lambda\mu}F_{\lambda\mu} = -2g_{ii}\left(F^{0i}\right)^2 + g^{ii}g^{jj}\left(F_{ij}\right)^2$ and $F_{0\lambda}F_0^{\lambda} = g_{ii}\left(F^{0i}\right)^2$, so

$$\tau_{00} = \frac{1}{2}g_{ii}\left(F^{0i}\right)^2 + \frac{1}{4}g^{ii}g^{jj}\left(F_{ij}\right)^2.$$

Using the relation (1) , and the relation (12) , one obtains:

$$\tau_{00} = \frac{1}{2}\left[\left(a_0E^1\right)^2\left(\frac{b_0}{b}\right)^4 + \left(a_0E^2\right)^2\left(\frac{b_0^2}{ab}\right)^2 + \left(a_0E^3\right)^2\left(\frac{b_0^2}{ab}\right)^2\right] + \frac{1}{2}\left[\left(\frac{\phi_{12}}{ab}\right)^2 + \left(\frac{\phi_{13}}{ab}\right)^2 + \left(\frac{\phi_{23}}{b^2}\right)^2\right].$$

- We also have $\tau_{0i} = F_{0\lambda}F_i^\lambda$, since $F_{0\lambda}F_i^\lambda = -F_{ij}F^{0j}$, consequently by (1) and (12) , we obtain:

$$\tau_{0i} = \tau_{0i} = -\left(\frac{a_0}{a}\right)\left(\frac{b_0}{b}\right)^2 E^j\phi_{ij}.$$

$$\tau_{ij} = -g_{ii}g_{jj}F^{0i}F^{0j} + g^{kk}F_{ik}F_{jk},\ i \neq j$$

and

$$\tau_{kk} = -\frac{g_{kk}}{4}\left[-2g_{ii}\left(F^{0i}\right)^2 + g^{ii}g^{jj}\left(F_{ij}\right)^2\right] - \left(g_{kk}\right)^2\left(F^{0k}\right)^2 + g^{jj}\left(F_{kj}\right)^2.$$

Invoking once more (1) and (12) , we find:

$$\tau_{ij} = -\frac{g_{ii}g_{jj}a_0^2b_0^4}{a^2b^4}E^iE^j + g^{kk}\phi_{ik}\phi_{jk},\ i \neq j.$$

Finally, we obtain the waiting results for τ_{11}, τ_{22}, τ_{33}, τ_{12}, τ_{13} and τ_{23}. ■

According to the relations (9) and (12) , we claim:

Proposition 5. The Einstein-Maxwell system (2) − (3) − (4) reduces to the following system in a and b:

$$2\frac{\dot{a}\dot{b}}{ab} + \left(\frac{\dot{b}}{b}\right)^2 - \Lambda = 8\pi\left[T_{00} + \tau_{00}\right] \tag{16}$$

$$-a^2\left[2\frac{\ddot{b}}{b} + \left(\frac{\dot{b}}{b}\right)^2 - \Lambda\right] = 8\pi\left[T_{11} + \tau_{11}\right] \tag{17}$$

$$-b^2\left[\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} - \Lambda\right] = 8\pi\left[T_{22} + \tau_{22}\right], \tag{18}$$

in the condition that $\tau_{\alpha\beta} = 0$ if $\alpha \neq \beta$ and $\tau_{22} = \tau_{33}$.

Proof. The Einstein equations (2) with the cosmological constant Λ write for:

- $\alpha = \beta = 0 :\ S_{00} + g_{00}\Lambda = 8\pi\ (T_{00} + \tau_{00})$;
- $\alpha = \beta = i :\ S_{ii} + g_{ii}\Lambda = 8\pi\ (T_{ii} + \tau_{ii})$;

- $\alpha \neq \beta : \tau_{\alpha\beta} = 0$.

Using propositions 3 and 5, we immediately obtain equations (16) , (17) and (18) , to which we add the problem of constraints $\tau_{\alpha\beta} = 0$ if $\alpha \neq \beta$ and $\tau_{22} = \tau_{33}$,that we have to solve. ■

2.2. The Cauchy problem and the problem of constraints

The above system is a system of second order non-linear differential equations in a and b. We also suppose that $a_0 > 0$, $b_0 > 0$, $\rho_0 > 0$, $\dot{a}_0$, $\dot{b}_0$ and E^i,$\phi_{ij}, i, j = 1, 2, 3$ are given real numbers and we look for solutions a and b of the system $(16) - (17) - (18)$ and of course the system $(2) - (3) - (4)$ satisfying the initial conditions:

$$\begin{cases} \rho(0) = \rho_0 \\ F^{0i}(0) = E^i, F_{ij}(0) = \phi_{ij}, i, j = 1, 2, 3 \\ a(0) = a_0, b(0) = b_0, \dot{a}(0) = \dot{a}_0, \dot{b}(0) = \dot{b}_0. \end{cases} \tag{19}$$

The aim of the present work is then to prove the global existence of solutions on $[0, +\infty[$ to the Cauchy problem $(16) - (17) - (18) - (19)$. The values prescribed at $t = 0$ will be called initial data, and it is obvious that the signs of $\dot{a}_0$ and $\dot{b}_0$ play an important role.

Proposition 6.
1• The Einstein equation (16) , called the *Hamiltonian constraint*, is satisfied all over the domain of the solutions a and b , if and only if, the initial data $\rho_0, a_0, b_0, \dot{a}_0, \dot{b}_0, E^i$, $\phi_{ij}, i, j = 1, 2, 3$ satisfy the initial condition:

$$2\frac{\dot{a}_0\dot{b}_0}{a_0 b_0} + \left(\frac{\dot{b}_0}{b_0}\right)^2 = \Lambda + 8\pi\left[\rho_0 + \frac{1}{2}\left[\left(a_0 E^1\right)^2 + \left(b_0 E^2\right)^2 + \left(b_0 E^3\right)^2\right]\right]$$
$$+8\pi\left[\frac{1}{2}\left[\left(\frac{\phi_{12}}{a_0 b_0}\right)^2 + \left(\frac{\phi_{13}}{a_0 b_0}\right)^2 + \left(\frac{\phi_{23}}{b_0^2}\right)^2\right]\right] \tag{20}$$

2• The remaining Einstein equations:

$$S_{0i} + g_{0i}\Lambda = 8\pi\left(T_{0i} + \tau_{0i}\right); S_{ij} + g_{ij}\Lambda = 8\pi\left(T_{ij} + \tau_{ij}\right) \tag{21}$$

are identically satisfied by any solutions a and b of $(16) - (17) - (18)$ if the initial data $a, b_0, E^i, \phi_{ij}, i, j = 1, 2, 3$ verify:

$$E^i \phi_{ij} = 0 \tag{22}$$

$$\sum_k \phi_{ik}\phi_{jk} - a_0^2 b_0^4 E^i E^j = 0, \ i \neq j \tag{23}$$

$$\phi_{12}^2 - \phi_{13}^2 - a_0^2 b_0^4\left[\left(E^2\right)^2 - \left(E^3\right)^2\right] = 0. \tag{24}$$

Proof. The problem of constraints $\tau_{\alpha\beta} = 0$ if $\alpha \neq \beta$ and $\tau_{22} = \tau_{33}$, appearing in proposition 3 is easily computed, using proposition 4, to give the system $(22) - (23) - (24)$. ■

Remark 7.

1) The system of constraints $(22) - (23) - (24)$ has non trivial solutions. Notice in fact that if $i \neq j$, and because $a_0 > 0$, $b_0 > 0$, the choice:

$$\begin{cases} E^1 \neq 0; \ E^2 = E^3 = 0 \\ \phi_{23} \neq 0; \ \phi_{12} = \phi_{13} = 0, \end{cases} \tag{25}$$

gives a non trivial solution $a_0, b_0, E^i, \phi_{ij}, \ i, j = 1, 2, 3, \ i \neq j$ of $(22) - (23) - (24)$.

2) In what follows, one supposes that the initial data $\rho_0, a_0, b_0, \dot{a_0}, \dot{b_0}, E^i, \phi_{ij}, i, j = 1, 2, 3$ satisfy the constraints (20), (22), (23) and (24). One must also remark that if the cosmological constant Λ is non negative and if $\rho_0, a_0, b_0, , E^i, \phi_{ij}, i, j = 1, 2, 3$ are given, (20) requires that:

$$\dot{a_0} > 0 \text{ and } \dot{b_0} > 0.$$

In the next section, we look for global existence of solutions a and b of equations (17) and (18), called evolution equations. The Hamiltonian constraint (16) will be used as a property of the solutions.

3. Global Existence of Solutions

3.1. Change of variables

For the study of the Einstein system $(17) - (18)$, we write:

$$\begin{aligned} \widetilde{\rho} = 8\pi \left(T_{00} + \tau_{00}\right), \ P_1 = \frac{8\pi \left(T_{11} + \tau_{11}\right)}{a^2}, \\ P_2 = \frac{8\pi \left(T_{22} + \tau_{22}\right)}{b^2}, \ R_+ = \frac{P_2 - P_1}{\widetilde{\rho}}. \end{aligned} \tag{26}$$

$\widetilde{\rho}$ is called the *energy density*.

Now, following Rendall and Uggla(2000), we make the change of variables as indicated below:

$$H = \frac{1}{3}\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right), \ z = \frac{1}{a^{-2} + 2b^{-2} + 1}, \ s = \frac{b^2}{b^2 + 2a^2}, \ \Sigma_+ = \frac{1}{H}\frac{\dot{b}}{b} - 1. \tag{27}$$

H is called *Hubble variable*.

We also set:

$$\Omega = \frac{\widetilde{\rho}}{3H^2}, \quad q = 2\Sigma_+^2 + \Omega. \tag{28}$$

Ω is the *normalized energy density* and q is the *parameter of deceleration*.

Lemma 1. We have:

$$0 < z < 1, \;\; 0 < s < 1, \;\; a^2 = \frac{z}{s(1-z)}, \;\; b^2 = \frac{2z}{(1-s)(1-z)} \tag{29}$$

$$\begin{cases} \Omega = 1 - \Sigma_+^2 - \dfrac{\Lambda}{3H^2}, \;\; \Omega \geq 0 \\ P_1 = \dfrac{\pi(1-s)^2(1-z)^2}{z^2}\left(-\left(a_0 b_0^2 E^1\right)^2 - \phi_{23}^2\right) + \\ \dfrac{2\pi s(1-s)(1-z)^2}{z^2}\left(\left(a_0 b_0^2 E^2\right)^2 + \left(a_0 b_0^2 E^3\right)^2 + \phi_{12}^2 + \phi_{13}^2\right) + \dfrac{4\pi\rho_0 a_0^{\frac{4}{3}} b_0^2 s^{\frac{2}{3}}(1-s)(1-z)^{\frac{5}{3}}}{z^{\frac{5}{3}}} \\ P_2 = \dfrac{\pi(1-s)^2(1-z)^2}{z^2}\left(\left(a_0 b_0^2 E^1\right)^2 + \phi_{23}^2\right) + \\ \dfrac{2\pi s(1-s)(1-z)^2}{z^2}\left(-\left(a_0 b_0^2 E^2\right)^2 + \left(a_0 b_0^2 E^3\right)^2 + \phi_{12}^2 - \phi_{13}^2\right) + \dfrac{4\pi\rho_0 a_0^{\frac{4}{3}} b_0^2 s^{\frac{2}{3}}(1-s)(1-z)^{\frac{5}{3}}}{z^{\frac{5}{3}}}. \end{cases} \tag{30}$$

Proof.

(a) It is clear, in view of formulas (27) which give

$$z = \frac{1}{a^{-2} + 2b^{-2} + 1}, \; s = \frac{b^2}{b^2 + 2a^2},$$

solved in a^2 and b^2, that:

$$0 < z < 1, \;\; 0 < s < 1, \;\; a^2 = \frac{z}{s(1-z)}, \;\; b^2 = \frac{2z}{(1-s)(1-z)}.$$

(b) Now we want to show that $\Omega = 1 - \Sigma_+^2 - \dfrac{\Lambda}{3H^2}$.

The equation (16) writes $2\dfrac{\dot{a}\dot{b}}{ab} + \left(\dfrac{\dot{b}}{b}\right)^2 - \Lambda = \widetilde{\rho}$.

So by (28) one has:

$$2\frac{\dot{a}\dot{b}}{ab} + \left(\frac{\dot{b}}{b}\right)^2 - \Lambda = 3H^2\Omega. \tag{31}$$

But invoking (27) one obtains:

$$\frac{\dot{b}}{b} = H(1 + \Sigma_+). \tag{32}$$

And using (27) and (32) , we get:

$$\frac{\dot{a}}{a} = H\left(1 - 2\Sigma_+\right). \tag{33}$$

Then reporting the expressions of $\frac{\dot{a}}{a}$ and $\frac{\dot{b}}{b}$ given by (33) and (32) in (31) , we obtain:

$$2\left(H\right)\left(1 - 2\Sigma_+\right)\left(H\right)\left(1 + \Sigma_+\right) + H^2\left(1 + \Sigma_+\right)^2 - \Lambda = 3H^2\Omega.$$

Consequently:

$$\Omega = 1 - \Sigma_+^2 - \frac{\Lambda}{3H^2}.$$

(c) $\Omega \geq 0$ because $\widetilde{\rho} \geq 0$.

(d) Using (29) and the values of T_{ii}, τ_{ii}, $i = 1, 2$ given by (14) and (15) , we find the given results for P_1 and P_2.

One should also remark using (14) , (15) that

$$\widetilde{\rho} = P_1 + 2P_2. \tag{34}$$

■

Remark 2.

1) By (30) we have $\Omega \geq 0$. So the first equation in (30) gives the inequality:

$$3H^2\left(1 - \Sigma_+^2\right) \geq \Lambda. \tag{35}$$

Since the cosmological constant Λ takes both negative and non negative values, (35) leads to the fact that if $\Lambda \geq 0$, then:

$$\Sigma_+ \in [-1, 1].$$

2) From now on, the variables H and Σ_+ should verify the inequality (35) .

3) We also have:

$$q \geq 0, \quad \forall \Lambda \in \mathbb{R}, \quad \text{and} \quad 0 \leq q \leq 2 \quad \text{if} \quad \Lambda \geq 0. \tag{36}$$

Theorem 3. **(No global existence in the case $\Lambda \geq$ and $\dot{b}_0 < 0$)** If $\Lambda \geq 0$ and $\dot{b}_0 < 0$, then the Einstein-Maxwell system $(16) - (17) - (18)$ has no global solution on $[0, +\infty[$.

Proof. Let us assume that $\Lambda \geq 0$ and $\dot{b}_0 < 0$. We set:

$$U = \frac{\dot{a}}{a}, V = \frac{\dot{b}}{b}, W = \frac{1}{ab^2},$$

then

$$\dot{W} = -(U + 2V)\, W. \tag{i}$$

First of all, we observe that since $\Lambda \geq 0$, $\dot{b}$ is never vanishes.

So, using equation (16) and $\Lambda \geq 0$, we have:

$$\begin{aligned} &\frac{\dot{b}}{b}\left(\frac{\dot{b}}{b} + 2\frac{\dot{a}}{a}\right) \\ &= \Lambda + 8\pi\,(T_{00} + \tau_{00}) \geq 8\pi\rho \\ &= \frac{8\pi\rho_0 a_0 b_0^2}{ab^2} := \frac{C_0^2}{ab^2} \end{aligned} \tag{ii}$$

where $C_0^2 = 8\pi\rho_0 a_0 b_0^2 > 0$.

Thus $\frac{\dot{b}}{b}(t) \neq 0,\ \ \forall t$ and then $\dot{b}(t) \neq 0,\ \forall t$.

If there exists a global solution, then by (ii), $V\,(V + 2U) \geq C_0^2 W$.

Since $\dot{b_0} < 0$ and $\dot{b}$ never vanishes , then by the Weierstrass intermediate value theorem:

$$\dot{b} < 0.$$

So

$$V < 0, \quad V + 2U < 0.$$

It then follows that:

$$V\,(V + 2U) = V^2 + 2UV = (U + V)^2 - U^2 \geq C_0^2 W,$$

and that

$$(U + V)^2 \geq C_0^2 W > 0.$$

Consequently $|U + V| \geq C_0\sqrt{W}$. But

$$V + U = \frac{1}{2}(2V + 2U) = \frac{1}{2}[V + (V + 2U)] < 0.$$

So $|U + V| = -U - V \geq C_0\sqrt{W}$ implies that:

$$U + V \leq -C_0\sqrt{W}. \tag{iii}$$

Invoking now (i) and (iii), one has:

$$\begin{aligned} \frac{d}{dt}\left[\frac{1}{\sqrt{W}}\right] &= \frac{-\dot{W}}{2W\sqrt{W}} \\ &= \frac{(U + 2V)\,W}{2W\sqrt{W}} \\ &= \frac{U + 2V}{2\sqrt{W}} = \frac{(U + V) + V}{2\sqrt{W}} \end{aligned}$$

$$\leq \frac{U+V}{2\sqrt{W}} \leq \frac{-C_0\sqrt{W}}{2\sqrt{W}} = -\frac{C_0}{2}.$$

Thus:

$$\frac{d}{dt}\left[\frac{1}{\sqrt{W}}\right] \leq -\frac{C_0}{2}. \tag{iv}$$

Integrating (iv) over $[0, t]$, one gets:

$$\frac{1}{\sqrt{W}} \leq \frac{1}{\sqrt{W_0}} - \frac{C_0}{2}t, \tag{v}$$

where $t > 0$ is an arbitrary real number , since the solution is global.

The $r.h.s$ of (v) vanishes after a finite time

$$t^* = \frac{2}{C_0\sqrt{W_0}}$$

and this implies that $\frac{1}{\sqrt{W}}$ also vanishes. This is a contradiction because following (i)

$$W^* = \frac{1}{a_0 b_0^2} exp\left(\int_0^t -(U+2V)(s)\,ds\right) > 0.$$

This completes the proof of theorem 1. ■

Using the values of $\widetilde{\rho}$, P_1 and P_2,the system $(16)-(17)-(18)$, becomes:

$$2\frac{\dot{a}\dot{b}}{ab} + \left(\frac{\dot{b}}{b}\right)^2 - \Lambda = \widetilde{\rho} \tag{37}$$

$$-\left[2\frac{\ddot{b}}{b} + \left(\frac{\dot{b}}{b}\right)^2 - \Lambda\right] = P_1 \tag{38}$$

$$-\left[\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} - \Lambda\right] = P_2. \tag{39}$$

3.2. The new system

In view of obtaining a differential system of first order, we will next show that, combining conveniently the equations of the system $(37)-(38)-(39)$, we get to the following system:

$$\frac{\ddot{a}}{a} = \frac{2}{3}\left(\left(\frac{\dot{b}}{b}\right)^2 - \frac{\dot{a}\dot{b}}{ab}\right) - \frac{\widetilde{\rho}}{6} + \frac{P_1 - 2P_2}{2} + \frac{\Lambda}{3} \tag{40}$$

$$\frac{\ddot{b}}{b}=\frac{1}{3}\left(\frac{\dot{a}\dot{b}}{ab}-\left(\frac{\dot{b}}{b}\right)^2\right)-\frac{\widetilde{\rho}}{6}-\frac{P_1}{2}+\frac{\Lambda}{3}. \tag{41}$$

Proposition 4. The Einstein system of equations $(40)-(41)$ in a and b ensures to express all derivatives of second order as functions of derivatives of first order and leads to the non linear first order differential system in $H,\ s,\ z,\ \Sigma_+$ given below:

$$\frac{dH}{dt}=-H^2(1+q)+\frac{\Lambda}{3} \tag{42}$$

$$\frac{ds}{dt}=6s(1-s)\Sigma_+H \tag{43}$$

$$\frac{dz}{dt}=2z(1-z)(1+\Sigma_+-3s\Sigma_+)H \tag{44}$$

$$\frac{d\Sigma_+}{dt}=-(2-q)\Sigma_+H+\Omega R_+H-\frac{\Lambda\Sigma_+}{3H}. \tag{45}$$

Proof. It shall be done in two steps.

i) First of all, let us prove that the system $(40)-(41)$ implies the system $(42)-(43)-(44)-(45)$.

It will be sufficient to establish the relation (42) ,the demonstration of the other relations being identical.

Suppose that $(40)-(41)$ is realized. Since by (27) $H=\frac{1}{3}\left(\frac{\dot{a}}{a}+2\frac{\dot{b}}{b}\right)$, then:

$$\frac{dH}{dt}=\frac{1}{3}\left(\frac{\ddot{a}}{a}+2\frac{\ddot{b}}{b}-\left(\frac{\dot{a}}{a}\right)^2-2\left(\frac{\dot{b}}{b}\right)^2\right).$$

Summing (40) and the double of (41) , one has:

$$\frac{\ddot{a}}{a}+2\frac{\ddot{b}}{b}-\left(\frac{\dot{a}}{a}\right)^2-2\left(\frac{\dot{b}}{b}\right)^2=\frac{1}{2}(\widetilde{\rho}+P_1+2P_2)-\left(\frac{\dot{a}}{a}\right)^2-2\left(\frac{\dot{b}}{b}\right)^2+\Lambda.$$

Thus:

$$\frac{dH}{dt}=\frac{1}{3}\left(-\frac{1}{2}(\widetilde{\rho}+P_1+2P_2)-\left(\frac{\dot{a}}{a}\right)^2+2\left(\frac{\dot{b}}{b}\right)^2\right)+\frac{\Lambda}{3}.$$

But $\frac{\dot{a}}{a}=\left(\frac{\dot{a}}{a}+2\frac{\dot{b}}{b}\right)-2\frac{\dot{b}}{b}$, so:

$$\frac{dH}{dt}=\frac{1}{3}\left(-\frac{1}{2}(\widetilde{\rho}+P_1+2P_2)-\left(\left(\frac{\dot{a}}{a}+2\frac{\dot{b}}{b}\right)-2\frac{\dot{b}}{b}\right)^2\right)$$
$$-\frac{1}{3}\left(4\frac{\dot{b}}{b}\left(\left(\frac{\dot{a}}{a}+2\frac{\dot{b}}{b}\right)-2\frac{\dot{b}}{b}\right)-6\left(\frac{\dot{b}}{b}\right)^2\right)+\frac{\Lambda}{3}.$$

Now $P_1 + 2P_2 = \widetilde{\rho}$, according to (34) . Since $\Omega = \frac{\widetilde{\rho}}{3H^2}$(given by (28)), one gets:

$$\frac{dH}{dt} = \frac{1}{3}\left(\frac{6\Omega H^2}{2} - 9H^2 + 6H^2\left(\Sigma_+^2 - 1\right)\right) + \frac{\Lambda}{3} = -H^2\left[\Omega + 2\Sigma_+^2 + 1\right] + \frac{\Lambda}{3}.$$

Using (28) which also implies $q = 2\Sigma_+^2 + \Omega$, one obtains:

$$\frac{dH}{dt} = -H^2(1+q) + \frac{\Lambda}{3}.$$

We have proved that the system $(40) - (41)$ implies the system $(42) - (43) - (44) - (45)$. There is still to prove the converse.

ii) Proof of the converse.

Let (H, s, z, Σ_+) be a solution of the system $(42) - (43) - (44) - (45)$. We are going to prove that if we set:

$$\begin{gathered} a^2 = \frac{z}{s(1-z)},\ b^2 = \frac{2z}{(1-s)(1-z)} \\ \Sigma_+ = \frac{3\dot{b}}{\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)b} - 1,\ H = \frac{\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)}{3} \end{gathered} \tag{46}$$

then a and b verify the system $(40) - (41)$.

From the second line of relation (46) one has:

$$\frac{\dot{b}}{b} = H(1+\Sigma_+) \tag{47}$$

$$\frac{\dot{a}}{a} = H(1-2\Sigma_+). \tag{48}$$

On the one hand we have, taking the derivative with respect to t:

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2, \tag{49}$$

and on the other hand, using (48) :

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \dot{H}(1-2\Sigma_+) - 2H\dot{\Sigma}_+. \tag{50}$$

From equations (49) and (50) , one deduces that:

$$\frac{\ddot{a}}{a} = \dot{H}(1-2\Sigma_+) - 2H\dot{\Sigma}_+ + \left(\frac{\dot{a}}{a}\right)^2.$$

Since (H, s, z, Σ_+) is a solution of the system $(42)-(43)-(44)-(45)$, one obtains:

$$\frac{\ddot{a}}{a} = \left(-H^2(1+q) + \frac{\Lambda}{3}\right)(1-2\Sigma_+) -$$

$$2H\left(-(2-q)\Sigma_+ H + \Omega R_+ H - \frac{\Lambda\Sigma_+}{3H}\right) + \left(\frac{\dot{a}}{a}\right)^2. \tag{51}$$

Now, using (28) , and from $P_1 + 2P_2 = \widetilde{\rho}$ given by the relation (34) one gets:

$$q = 2\Sigma_+^2 + \Omega = 2\Sigma_+^2 \frac{\widetilde{\rho}}{3H^2} = 2\Sigma_+^2 \frac{\widetilde{\rho}}{6H^2}\left(1 + \frac{P_1 + 2P_2}{\widetilde{\rho}}\right). \tag{52}$$

Invoking the second line of the relations (46) ,the equation (52) yields:

$$q = 2\left(\frac{1}{H}\frac{\dot{b}}{b} - 1\right)^2 + \frac{1}{6H^2}(\widetilde{\rho} + P_1 + 2P_2)$$

$$= \frac{2}{H^2}\left(\frac{\dot{b}}{b}\right)^2 - \frac{4\dot{b}}{bH} + 2 + \frac{1}{6H^2}(\widetilde{\rho} + P_1 + 2P_2). \tag{53}$$

Consequently:

$$\begin{aligned} 1 + q &= \frac{2}{H^2}\left(\frac{\dot{b}}{b}\right)^2 - \frac{4\dot{b}}{bH} + 3 + \frac{1}{6H^2}(\widetilde{\rho} + P_1 + 2P_2) \\ -q &= \frac{-2}{H^2}\left(\frac{\dot{b}}{b}\right)^2 + \frac{4\dot{b}}{bH} - \frac{1}{6H^2}(\widetilde{\rho} + P_1 + 2P_2). \end{aligned} \tag{54}$$

Carrying forward the relations (54) inside (51) , and taking in account the equalities $\Omega = \frac{\widetilde{\rho}}{3H^2}$, $R_+ = \frac{P_2 - P_1}{\widetilde{\rho}}$, one obtains the equalities:

$$\begin{aligned} \frac{\ddot{a}}{a} &= -2\left(\frac{\dot{b}}{b}\right)^2 + \frac{4\dot{b}}{bH} - 3H^2 - \frac{\widetilde{\rho}}{6} - \frac{1}{6}(P_1 + 2P_2) + 6H^2\Sigma_+ \\ &+ \left(\frac{\dot{a}}{a}\right)^2 - \frac{2}{3}(P_2 - P_1) + \frac{\Lambda}{3} - 2\left(\frac{\dot{b}}{b}\right)^2 + \frac{4\dot{b}}{bH} - \frac{\widetilde{\rho}}{6} - \frac{1}{6}(P_1 + 2P_2) \\ &+3H^2(2\Sigma_+ - 1) + \left(\frac{\dot{a}}{a}\right)^2 - \frac{2}{3}(P_2 - P_1) + \frac{\Lambda}{3}. \end{aligned} \tag{55}$$

So using (48) to express $2\Sigma_+ - 1$ and then $H = \frac{\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)}{3}$ to express H, equalities (55) imply that:

$$\frac{\ddot{a}}{a} = \frac{2}{3}\left(\left(\frac{\dot{b}}{b}\right)^2 - \frac{\dot{a}\dot{b}}{ab}\right) - \frac{\widetilde{\rho}}{6} + \frac{P_1 - 2P_2}{2} + \frac{\Lambda}{3}.$$

This proves (40) . Similarly, one establishes the relation (41) . In conclusion, the proof of proposition 7 is complete. ■

Remark 5. We are going to use this new system of the Einstein equations to prove that there is no global existence of solutions in the case $\Lambda < 0$. Since the aim of this work is the proof of global existence of solutions, based on theorem 1, we will in the next take an interest to the single case $\Lambda \geq 0$ and $\dot{b}_0 > 0$. This because $\dot{b}$ never vanishes, and so $\dot{b}(0) = \dot{b}_0$ must be taken strictly non negative. In view of this, let us remind the following useful result:

Lemma 6. Let u and v be two continouosly differentiable real valued functions of the variable t, satisfying the following conditions in which $\alpha \neq 0$ is a constant and $t_0 \in \mathbb{R}$:

$$\begin{cases} \dot{u} \leq -\alpha^2 u^2 \\ \dot{v} = -\alpha^2 v^2 \\ u(t_0) = v(t_0). \end{cases}$$

Then:

$$u(t) \leq v(t), \quad \forall t \geq t_0.$$

Theorem 7. (No global existence in the case $\Lambda < 0$**)** The Einstein-Maxwell system with the cosmological constant $\Lambda < 0$, in the Bianchi type I space-time has no global solution on $[0, +\infty[$.

Proof. Suppose that $\Lambda < 0$. If there exists a global solution on $[0, +\infty[$, then by (35) and owing to $q \geq 0$, we have:

$$\frac{dH}{dt} \leq \frac{\Lambda}{3}. \tag{56}$$

Integrating (56) over $[0, t]$, $t > 0$, since H is defined on $[0, +\infty[$, we obtain:

$$H(t) \leq H(0) + \frac{\Lambda}{3} t. \tag{57}$$

Inequality (57) shows that:

$$\underset{t \to +\infty}{H(t)} \to -\infty \tag{58}$$

because $\Lambda < 0$.

There then exists according to (58) , $t_0 \in \mathbb{R}$ such that:

$$H(t) < 0, \quad \forall t \geq t_0. \tag{59}$$

Using once more (35) and the inequality $\Lambda < 0$, we obtain:

$$\frac{dH}{dt} \leq -H^2. \tag{60}$$

We consequently find on $[t_0, +\infty[$ the inequalities:

$$\begin{aligned} \dot{H} &\leq -H^2 \\ H(t_0) &< 0. \end{aligned} \tag{61}$$

According to (61) , we conclude using lemma 2 that:

$$H(t) \leq v(t), \quad \forall t \geq t_0 \tag{62}$$

$$\begin{aligned} \dot{v} &= -v^2 \\ v(t_0) &= H(t_0). \end{aligned} \tag{63}$$

But (63) shows that $\dot{v} < 0$, so v decreases. Consequently:

$$v(t) \leq v(t_0) = H(t_0) < 0, \tag{64}$$

and hence:

$$v(t) \neq 0, \ \forall t \geq t_0. \tag{65}$$

Separating and integrating (63) over $[t_0, t]$, we find:

$$v(t) = \frac{H(t_0)}{1 + H(t_0)(t - t_0)}. \tag{66}$$

The equality (66) shows that:

$$\lim_{t \to_< t^*} v(t) \to -\infty \tag{67}$$

where $t^* = t_0 - \dfrac{1}{H(t_0)} > t_0$.

Consequently:

$$\lim_{t \to_< t^*} H(t) \to -\infty. \tag{68}$$

This result is absurd since, H being a continuous function on $[0, +\infty[$, is bounded over the compact $\left[t_0, t^*\right]$. ■

In what follows, we assume that $\Lambda \geq 0$ and $\dot{b}_0 > 0$.

3.3. The global existence

We are going to solve the system (42) − (43) − (44) − (45) , whose unknown is (H, s, z, Σ_+), this will provide the solution of the equivalent system (40) − (41) whose unknown is (a, b).

We are looking for a solution (H, s, z, Σ_+) of the system (42)−(43)−(44)−(45) , on the interval $I = [0, T]$, $T > 0$ which satisfies at the initial instant $t = 0$, the condition:

$$(H, s, z, \Sigma_+)(0) = (H_0, s_0, z_0, \Sigma_{+0}) \tag{69}$$

where $H_0,\ s_0,\ z_0,\ \Sigma_{+0}$ are real numbers conveniently fixed. In fact we must return to definitions of $H,\ s,\ z,\ \Sigma_+$ and choose those data according to $a_0,\ b_0,\ \dot{a}_0,\ \dot{b}_0$ (and so to $\rho_0,\ E^i,\ \phi_{ij},\ i, j = 1, 2, 3$) and furthermore, subjected to the Hamiltonian constraint (20) .

Remark 8. Using the definition of q, the equalities $\widetilde{\rho} = P_1 + 2P_2$ and $\dfrac{\Omega H^2}{\widetilde{\rho}} = \dfrac{1}{3},\ \ \Omega = 1 - \Sigma_+^2 - \dfrac{\Lambda}{3H^2}$, to rewrite the equations (42) and (45) , the system (42)−(43)−(44)−(45) turns into a new system given by:

$$\begin{aligned} \frac{dH}{dt} &= -\frac{3}{2}H^2\left(1+\Sigma_+^2\right) - \frac{P_1+2P_2}{6} + \frac{\Lambda}{2} \\ \frac{ds}{dt} &= 6s\,(1-s)\,\Sigma_+ H \\ \frac{dz}{dt} &= 2z\,(1-z)\,(1+\Sigma_+ - 3s\Sigma_+)\,H \\ \frac{d\Sigma_+}{dt} &= -\frac{3}{2}H\Sigma_+\left(1-\Sigma_+^2\right) - \frac{P_1\,(\Sigma_+-2)}{6H} + \frac{P_2\,(\Sigma_++1)}{3H} - \frac{\Lambda\Sigma_+}{2H} \end{aligned} \tag{70}$$

and that we now solve.

Remark 9. Domain of the variables H, s, z, Σ_+. We will take the variables H, s, z, and Σ_+ on the set B defined below by:

$$B = \left\{(H, s, z, \Sigma_+) \in \mathbb{R}^4 / 0 < H \le H_0,\ 0 < s < 1,\ 0 < z < 1,\ -1 < \Sigma_+ < 1\right\}. \tag{71}$$

In fact, it is easily shown using (35) and the definitions of s, z that:

$$0 < s < 1,\ 0 < z < 1,\ -1 < \Sigma_+ < 1. \tag{72}$$

Now, using (35) , (42) and the fact that $0 < H_0$, one gets:

$$0 < H \le H_0 \tag{73}$$

where $H_0 = H\,(0)$.

Let us now give the following definition which shows helpful in the next:

Definition 10. If s is a real number such that $0 < s < 1$, we set:

$$\alpha\,(s) = inf\,(s, 1-s)\,.$$

Remark 11. Since $H_0 > 0$, one can assume that $\dot{a}_0 > 0$.

Under this assumption, initial data $a_0,\ b_0,\ \dot{a}_0,$ and $\dot{b}_0$ shall be such that:

$$a_0 > 0,\ b_0 > 0,\ \dot{a}_0 > 0,\ \dot{b}_0 > 0. \tag{74}$$

Inequalities (73) imply using $\frac{\dot{a}_0}{a_0} > 0$ and $\dot{b}_0 > 0$ that:

$$\begin{cases} -1 < \Sigma_{+0} < \frac{1}{2} \\ \Sigma_{+0} = \Sigma_+ (0) . \end{cases} \tag{75}$$

We now prove the global existence theorem of solutions to the Einstein system (70) with the initial data given by (69) and where

$$H_0 > 0,\ 0 < s_0 < 1,\ 0 < z_0 < 1,\ -1 < \Sigma_{+0} < \frac{1}{2}. \tag{76}$$

We will apply the standard theory on first order differential systems. With a view to succeed, we will study the function G defined using the *r.h.s* of the system (70) by:

$$G (t, H, s, z, \Sigma_+) = (G_1, G_2, G_3, G_4) (t, H, s, z, \Sigma_+) \tag{77}$$

where

$$\begin{cases} G_1 (t, H, s, z, \Sigma_+) = -\frac{3}{2} H^2 \left(1 + \Sigma_+^2\right) - \frac{P_1 + 2P_2}{6} + \frac{\Lambda}{2} \\ G_2 (t, H, s, z, \Sigma_+) = 6s (1 - s) \Sigma_+ H \\ G_3 (t, H, s, z, \Sigma_+) = 2z (1 - z) (1 + \Sigma_+ - 3s\Sigma_+) H \\ G_4 (t, H, s, z, \Sigma_+) = -\frac{3}{2} H \Sigma_+ \left(1 - \Sigma_+^2\right) - \frac{P_1 (\Sigma_+ - 2)}{6H} + \frac{P_2 (\Sigma_+ + 1)}{3H} - \frac{\Lambda \Sigma_+}{2H}. \end{cases} \tag{78}$$

Recall that G is defined on the set

$$B =]0, H_0] \times]0, 1[\times]0, 1[\times]-1, 1[.$$

We must prove that G is a continuous function of t, locally *Lipschitzian* in $X = (H, s, z, \Sigma_+) \in \mathbb{R}^4$ endowed with the norm:

$$\|X\|_{\mathbb{R}^4} = |H| + |s| + |z| + |\Sigma_+| . \tag{79}$$

G is obviously a continuous function of t, on one hand. On the other hand, G_2 and G_3 are polynomial functions in H, s, z and Σ_+, so are locally *Lipschitzian.*

Concerning now G_1 and G_4, we have:

Lemma 12. Let $\delta > 0,\ \ t_0 \geq 0$ to be given. If $s_1, s_2, z_1, z_2 \in]0, 1[,\ \ H_1, H_2 \in]0, H_0[$ then:

$$\begin{cases} |P_i (s_1, z_1) - P_i (s_2, z_2)| \leq \frac{C [|s_1 - s_2| + |z_1 - z_2|]}{\alpha^4 (s_i) \alpha^5 (z_1) \alpha^5 (z_2)} \\ |P_i (s, z)| \leq \frac{C}{\alpha^2 (z)} \\ \left| \frac{P_i (s_1, z_1)}{H_1} - \frac{P_i (s_2, z_2)}{H_2} \right| \leq \frac{C [|H_1 - H_2| + |s_1 - s_2| + |z_1 - z_2|]}{H_1 H_2 \alpha^4 (s_i) \alpha^5 (z_1) \alpha^5 (z_2)} \end{cases} \quad i = 1, 2. \tag{80}$$

Proof. We set:

$$\begin{cases} U_i = \frac{s_i (1-s_i)(1-z_i)^2}{z_i^2}, V_i = \frac{(1-s_i)^2 (1-z_i)^2}{z_i^2}, \ W_i = \frac{s_i^{\frac{2}{3}} (1-s_i)(1-z_i)^{\frac{5}{3}}}{z_i^{\frac{5}{3}}}, \ i = 1,2 \\ C_1 = -\pi \left(\left(a_0 b_0^2 E^1 \right)^2 + \phi_{23}^2 \right), \ C_2 = 2\pi \left(\left(a_0 b_0^2 E^2 \right)^2 + \left(a_0 b_0^2 E^3 \right)^2 + \phi_{12}^2 + \phi_{13}^2 \right) \\ C_3 = 2\pi \left(-\left(a_0 b_0^2 E^2 \right)^2 + \left(a_0 b_0^2 E^3 \right)^2 + \phi_{12}^2 - \phi_{13}^2 \right), \ C_4 = 4\pi \rho_0 a_0^{\frac{4}{3}} b_0^2. \end{cases} \tag{81}$$

We want to estimate

$$|P_i(s_1, z_1) - P_i(s_2, z_2)|, \ |P_i(s, z)|, \left| \frac{P_i(s_1, z_1)}{H_1} - \frac{P_i(s_2, z_2)}{H_2} \right|.$$

(**a**) Estimation of $|P_i(s_1, z_1) - P_i(s_2, z_2)|$. To handle the differences appearing in $P_i(s_1, z_1) - P_i(s_2, z_2)$, $i = 1, 2$, it will be sufficient to handle $U_1 - U_2$, $V_1 - V_2$ and $W_1 - W_2$ since $C_{1,} C_3$, C_3, and C_4 are absolute constants.

We have:

$$W_1 - W_2 = \frac{z_2^5 s_1^2 (1-s_1)^3 (1-z_1)^5 - z_1^5 s_2^2 (1-s_2)^3 (1-z_2)^5}{\left(a^2 + ab + b^2\right) z_1^{\frac{5}{3}} z_2^{\frac{5}{3}}} \tag{82}$$

where

$$a = z_2^{\frac{5}{3}} s_1^{\frac{2}{3}} (1-s_1)(1-z_1)^{\frac{5}{3}}, \ b = z_1^{\frac{5}{3}} s_2^{\frac{2}{3}} (1-s_2)(1-z_2)^{\frac{5}{3}}.$$

Since:

$$s_i, , z_i, 1-s_i, 1-z_i \in]0,1[, \ s_i, 1-s_i \geq \alpha(s_i), \ z_i, 1-z_i \geq \alpha(z_i), \ i = 1,2$$

it follows by usual factorizations that

$$\left| z_2^5 s_1^2 (1-s_1)^3 (1-z_1)^5 - z_1^5 s_2^2 (1-s_2)^3 (1-z_2)^5 \right| \leq C \left[|s_1 - s_2| + |z_1 - z_2| \right] \tag{83}$$

and

$$\left(a^2 + ab + b^2\right) z_1^{\frac{5}{3}} z_2^{\frac{5}{3}} \geq a^2 z_1^{\frac{5}{3}} z_2^{\frac{5}{3}} \geq \alpha^5(z_1) \alpha^5(z_2) \alpha^4(s_i), \ i = 1,2. \tag{84}$$

So, from (82), (83), (84), on concludes that

$$|W_1 - W_2| \leq \frac{C \left[|s_1 - s_2| + |z_1 - z_2| \right]}{\alpha^4(s_i) \alpha^5(z_1) \alpha^5(z_2)}, \ i = 1,2. \tag{85}$$

Similarly, on obtains:

$$\begin{aligned} |U_1 - U_2| &\leq \frac{C \left[|s_1 - s_2| + |z_1 - z_2| \right]}{\alpha^2(z_1) \alpha^2(z_2)} \\ |V_1 - V_2| &\leq \frac{C \left[|s_1 - s_2| + |z_1 - z_2| \right]}{\alpha^2(z_1) \alpha^2(z_2)}. \end{aligned} \tag{86}$$

We deduce owing all the preceding steps that:

$$|P_i(s_1, z_1) - P_i(s_2, z_2)| \leq \frac{C[|s_1 - s_2| + |z_1 - z_2|]}{\alpha^4(s_1)\alpha^5(z_1)\alpha^5(z_2)}. \tag{87}$$

(**b**) It is immediate that

$$|P_i(s, z)| \leq \frac{C}{\alpha^2(z)}, \ i = 1, 2. \tag{88}$$

(**c**) Handling of $\left|\frac{P_i(s_1, z_1)}{H_1} - \frac{P_i(s_2, z_2)}{H_2}\right|$.

We have:

$$\frac{P_i(s_1, z_1)}{H_1} - \frac{P_i(s_2, z_2)}{H_2} = \frac{H_1 - H_2}{H_1 H_2} P_i(s_1, z_1) + \frac{1}{H_2}(P_i(s_1, z_1) - P_i(s_2, z_2)).$$

It follows from (87) and (88) since $0 < H_i \leq H_0$ that: s

$$\left|\frac{P_i(s_1, z_1)}{H_1} - \frac{P_i(s_2, z_2)}{H_2}\right| \leq \frac{C[|H_1 - H_2| + |s_1 - s_2| + |z_1 - z_2|]}{H_1 H_2 \alpha^2(s_1)\alpha^9(z_1)\alpha^2(z_2)}. \tag{89}$$

This ends the proof of lemma 3. ■

We conclude using lemma 3 that:

$$|G_1(H_1, s_1, z_1, \Sigma_{+1}) - G_1(H_2, s_2, z_2, \Sigma_{+2})| \leq$$
$$\frac{C\left[|H_1 - H_2| + |s_1 - s_2| + |z_1 - z_2| + |\Sigma_{+1} - \Sigma_{+2}|\right]}{\alpha^4(s_i)\alpha^5(z_1)\alpha^5(z_2)}. \tag{90}$$

We similarly conclude that:

$$|G_4(H_1, s_1, z_1, \Sigma_{+1}) - G_4(H_2, s_2, z_2, \Sigma_{+2})| \leq$$
$$\frac{C\left[|H_1 - H_2| + |s_1 - s_2| + |z_1 - z_2| + |\Sigma_{+1} - \Sigma_{+2}|\right]}{H_1 H_2 \alpha^4(s_i)\alpha^5(z_1)\alpha^5(z_2)}. \tag{91}$$

Let $\left(H^0, s^0, z^0, \Sigma_+^0\right)$ be given in $B =]0, H_0] \times]0, 1[\times]0, 1[\times]-1, 1[$. Now consider the neighborhood V of $\left(H^0, s^0, z^0, \Sigma_+^0\right)$ defined by:

$$V = \left]\frac{H^0}{2}, H_0\right[\times \left]\frac{s^0}{2}, \frac{s^0 + 1}{2}\right[\times \left]\frac{z^0}{2}, \frac{z^0 + 1}{2}\right[\times]-1, 1[. \tag{92}$$

If we take

$$(H_1, s_1, z_1, \Sigma_{+1}), (H_2, s_2, z_2, \Sigma_{+2}) \in V,$$

then

$$\frac{H^0}{2} < H_i < H_0, \ \frac{s^0}{2} < s_i < \frac{s^0 + 1}{2}, \ \frac{z^0}{2} < z_i < \frac{z^0 + 1}{2}, \ i = 1, 2,$$

$$\frac{1}{H_i} < \frac{2}{H^0}, \quad \frac{1}{\alpha(s_i)} < \frac{2}{\alpha\left(s^0\right)}, \quad \frac{1}{\alpha(z_i)} < \frac{2}{\alpha\left(z^0\right)}, \quad i = 1, 2. \tag{93}$$

Using(90) , (91) , (93) , we obtain:

$$|G_1(H_1, s_1, z_1, \Sigma_{+1}) - G_1(H_2, s_2, z_2, \Sigma_{+2})| \leq$$
$$\frac{C\left[|H_1 - H_2| + |s_1 - s_2| + |z_1 - z_2| + |\Sigma_{+1} - \Sigma_{+2}|\right]}{\alpha^4\left(s^0\right)\alpha^{10}\left(z^0\right)}, \tag{94}$$

$$|G_4(H_1, s_1, z_1, \Sigma_{+1}) - G_4(H_2, s_2, z_2, \Sigma_{+2})| \leq$$
$$\frac{C\left[|H_1 - H_2| + |s_1 - s_2| + |z_1 - z_2| + |\Sigma_{+1} - \Sigma_{+2}|\right]}{\left(H^0\right)^2\alpha^4\left(s^0\right)\alpha^{10}\left(z^0\right)}. \tag{95}$$

We conclude that, like G_2 and G_3, the functions G_1 and G_4 are locally *Lipschitzia*n. So is for the function G.

We can now prove our main result:

Theorem 13. If $\Lambda \geq 0$ and $\dot{b}_0 > 0$, then the Cauchy problem for the Einstein-Maxwell system with the cosmological constant has a global solution on $[0, +\infty[$.

Proof. We have seen before that G given by (77) and defined by the *r.h.s* of the system (70) in (H, s, z, Σ_+) was *continuous and locally Lipschitzian*, so by the standard theory on first order differential systems, this system for the initial data $H_0,\ s_0,\ z_0,\ \Sigma_{+0}$ such that:

$$H_0 = \frac{1}{3}\left(\frac{\dot{a}_0}{a_0} + 2\frac{\dot{b}_0}{b_0}\right),\ s_0 = \frac{b_0^2}{b_0^2 + 2a_0^2},\ , z_0 = \frac{1}{a_0^{-2} + 2b_0^{-2} + 1},\ \Sigma_{+0} = \frac{1}{H_0}\frac{\dot{b}_0}{b_0} - 1, \tag{96}$$

where $a_0, b_0, \dot{a}_0, \dot{b}_0$ (and so $\rho_0,\ E^i,\ \phi_{ij},\ i, j = 1, 2$) satisfy the *Hamiltonian constraint (20),* has a unique **local solution** (H, s, z, Σ_+) . But moreover we have:

$$\begin{gathered} 0 < s < 1, 0 < z < 1, \\ -1 \leq \Sigma_+ \leq 1, 0 < H \leq H_0. \end{gathered} \tag{97}$$

Thus, the local solution (H, s, z, Σ_+) of the Cauchy problem is **uniformly bounded**, so is **global**.

From there, results the **global existence** of the Cauchy problem for the equivalent system $(40) - (41)$ in a and b on $[0, +\infty[$. And from the global solution (a, b), we deduce using proposition 2, the **global existence** of $\rho,\ F^{0i},\ F_{ij}$.

This completes the proof of theorem 3. ■

4. Conclusion

In this paper we have analyzed the Einstein-Maxwell system for perfect charged relativistic fluid in a Bianchi type I space-time with the cosmological constant. We have seen that there cannot exist a global solution when the cosmological constant $\Lambda < 0$. We have also proved that even if the cosmological constant $\Lambda \geq 0$, there cannot exist a global solution when $\dot{b}_0 < 0$. But we have obtained a unique global solution in the case $\Lambda \geq 0$ and $\dot{b}_0 > 0$. The result obtained here seems very important, because in the General Theory of Relativity, it is the cosmological constant which models the acceleration of the expansion of our universe. Several scientists are now working to better understanding such phenomena; which are essential to the future and the subsistence of humanity. One can also easily see by ρ given in (12) , that; conveniently choosing a in the case $a = b$ (Robertson-Walker space-time), the energy-momentum tensor $T_{\alpha\beta} + \tau_{\alpha\beta}$ given in (14) , (15) vanishes when $t \to +\infty$. *So the space-time in this case becomes empty to future infinity.* But in the case we have considered here where $a \neq b$, we see using (14) , (15) that $T_{\alpha\beta} + \tau_{\alpha\beta}$ never vanishes with $t \to +\infty$. *So the space-time would never be empty to future infinity.*

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