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A note on the Appell-type Daehee polynomials

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Abstract

In this paper, we present a systematic studies of some families of Appell-type Daehee polynomials. In particular, we derive some interesting identities for the Appell-type Daehee polynomials by using the basis properties of Appell-type Daehee polynomials of degree less than and equal to n.

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned} I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_0(x + p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x),, \end{aligned} \tag{1.1}$$

(see [5, 7, 8-10]). From (1.1), we have

$$I_0(f_1) = I_0(f) + f'(0), \tag{1.2}$$

where $f_1(x) = f(x+1)$.

The Daehee polynomials are defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \text{ (see [5, 7–9])}, \tag{1.3}$$

and there are many works related with Daehee numbers and polynomials (see [3, 6–9, 11, 14–17]).

As is well known, the *Bernoulli polynomials* are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \text{ (see [1, 2])}. \tag{1.4}$$

When $x = 0$, $B_n = B_n(0)$ are called the *Bernoulli numbers*, and *Bernoulli polynomials of the second kind* are defined by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \text{ (see [12, 13])}. \tag{1.5}$$

For $\lambda, t \in \mathbb{C}_p$ with $|\lambda t| < p^{-\frac{1}{p-1}}$, the *Daehee polynomials with parameter* λ are defined by the generating function to be

$$\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} D_n(x|\lambda)\frac{t^n}{n!}, \text{ (see [7, 16])}. \tag{1.6}$$

A polynomial sequence $\{p_n(x)\}$ is called *Appell sequence* if

$$\frac{d}{dx}p_n(x) = np_{n-1}(x)$$

for all positive integer n.

Let $V_n = \{p(x) \in \mathbb{Q}[x] | \deg p(x) \leq n\}$ be the $(n+1)$-dimensional vector space over $\mathbb{Q}$. Probably, $\left\{1, x, \ldots, x^n\right\}$ is the most natural basis for V_n.

In this paper, we prove that $\{\mathcal{D}_0(x|\lambda), \mathcal{D}_1(x|\lambda), \ldots, \mathcal{D}_n(x|\lambda)\}$ is also a good basis for the space V_n for our purpose of arithmetical and combinatorial applications of the Appell-type Daehee polynomials, and investigate some properties of those polynomials.

2. Degenerate Daehee polynomials

Let us assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$.

We consider the *Appell-type Daehee polynomials* which is defined by

$$\frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}} e^{xt} = \sum_{n=0}^{\infty} \mathcal{D}_n(x|\lambda)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $\mathcal{D}_n(\lambda) = \mathcal{D}_n(0|\lambda)$ are called the *Appell-type Daehee numbers*.

Note that

$$\begin{aligned}\frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}} e^{xt} &= \left(\sum_{n=0}^{\infty} \mathcal{D}_n(\lambda)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{x^m}{m!}t^m\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathcal{D}_{n-m}(\lambda)x^m\right)\frac{t^n}{n!},\end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we have the following theorem

Theorem 2.1. For $n \geq 0$, we have

$$\mathcal{D}_n(x|\lambda) = \sum_{m=0}^{n}\binom{n}{m}\mathcal{D}_{n-m}(\lambda)x^m.$$

From Theorem 2.1,

$$\begin{aligned}\frac{d}{dx}\mathcal{D}_n(x|\lambda) &= \sum_{m=1}^{n}\binom{n}{m}\mathcal{D}_{n-m}(\lambda)mx^{m-1} \\ &= n\sum_{m=1}^{n}\binom{n-1}{m-1}\mathcal{D}_{n-m}(\lambda)x^{m-1} \\ &= n\sum_{m=0}^{n-1}\binom{n}{m}D_{n-m}(\lambda)x^m \\ &= n\mathcal{D}_{n-1}(x|\lambda),\end{aligned} \tag{2.3}$$

and so we know that a sequence $\{\mathcal{D}_n(x|\lambda)\}$ is Appell sequence.

By (2.3), we have

$$\begin{aligned}\mathcal{D}_n(x|\lambda) &= \int \frac{d}{dx}\mathcal{D}_{n-1}(x|\lambda)dx = \frac{1}{n}\mathcal{D}_n(x|\lambda) \\ &= n\int \mathcal{D}_{n-1}(x|\lambda)dx.\end{aligned} \tag{2.4}$$

By (2.4), we obtain the following corollary.

Corollary 2.2. For $n \geq 0$, we have

$$\int \mathcal{D}_{n-1}(x|\lambda)dx = \frac{1}{n}\mathcal{D}_n(x|\lambda) + C$$

where C is a constant.

Since

$$\begin{aligned}e^{xt} &= \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{\log(1+t)}\sum_{n=0}^{\infty}\mathcal{D}_n(x|\lambda)\frac{t^n}{n!} \\ &= \frac{\log(1+\lambda t)}{\lambda t}\frac{t}{\log(1+t)}\sum_{m=0}^{\infty}\mathcal{D}_m(x|\lambda)\frac{t^m}{m!} \\ &= \left(\sum_{n=0}^{\infty}D_n\frac{\lambda^n t^n}{n!}\right)\left(\sum_{j=0}^{\infty}b_j\frac{t^j}{j!}\right)\left(\sum_{m=0}^{\infty}\mathcal{D}_m(x|\lambda)\frac{t^m}{m!}\right) \\ &= \left(\sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\binom{m}{l}\lambda^l D_l b_{n-l}\right)\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\mathcal{D}_k(x|\lambda)\frac{t^k}{k!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\binom{n}{m}\lambda^l D_l b_{n-l}\mathcal{D}_{n-m}(x|\lambda)\right)\frac{t^n}{n!},\end{aligned} \tag{2.5}$$

and

$$\sum_{n=0}^{\infty}x^n\frac{t^n}{n!} = e^{xt}, \tag{2.6}$$

by (2.5) and (2.6),

$$\begin{aligned}x^n &= \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\binom{n}{m}\lambda^l D_l b_{n-l}\mathcal{D}_{n-m}(x|\lambda) \\ &= \sum_{m=0}^{n}\sum_{l=0}^{n-m}\binom{n}{n-m}\binom{n-m}{l}\lambda^l D_l b_{n-l}\mathcal{D}_m(x|\lambda)\end{aligned} \tag{2.7}$$

for all nonnegative integers n. Hence, by (2.7), we obtain the following theorem.

Theorem 2.3. $\{\mathcal{D}_0(x|\lambda), \mathcal{D}_1(x|\lambda), \dots, \mathcal{D}_n(x|\lambda)\}$ is a basis of V_n.

As is well known, the *Bell polynomials* are defined by the generating function to be

$$e^{(e^t-1)x} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \text{ (see [4]).} \tag{2.8}$$

When $x = 0$, $Bel_n = Bel_n(0)$ are called the *Bell numbers*

By replacing t by $e^t - 1$ in (2.1), we have

$$\begin{aligned} e^{(e^t-1)x} &= \sum_{n=0}^{\infty} \mathcal{D}_n(x|1)\frac{1}{n!}\left(e^t-1\right)^m \\ &= \sum_{m=0}^{\infty} \mathcal{D}_m(x|1)\frac{1}{m!}m!\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \mathcal{D}_m(x|1)S_2(n,m)\right)\frac{t^n}{n!}, \end{aligned} \tag{2.9}$$

by (2.21) and (2.8), we obtain the following theorem.

Theorem 2.4. For $n \geq 0$, we have

$$Bel_n(x) = \sum_{m=0}^{n} \mathcal{D}_m(x|1)S_2(n,m).$$

Indeed,

$$\begin{aligned} e^{(e^t-1)x} &= \sum_{m=0}^{\infty} x^n\frac{1}{m!}\left(e^t-1\right)^m \\ &= \sum_{m=0}^{\infty} x^m\frac{1}{m!}m!\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} x^m S_2(n,m)\right)\frac{t^n}{n!}, \end{aligned} \tag{2.10}$$

and thus, by Theorem 2.4 and (2.22), we obtain the following corollary.

Corollary 2.5. For $n \geq 0$, we have

$$Bel_n(x) = \sum_{m=0}^{n} S_2(n,m)x^m,$$

and

$$\mathcal{D}_n(x|1) = x^n.$$

For $\lambda \neq 1$, since

$$\frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}} e^{xt} = \left(\sum_{l=0}^{\infty} \mathcal{D}_l(\lambda) \frac{t^l}{l!}\right) e^{xt}, \tag{2.11}$$

by replacing t by $e^t - 1$ in (2.11),

$$\begin{aligned}
\sum_{m=0}^{\infty} \mathcal{D}_m(x|\lambda) \frac{(e^t-1)^m}{m!} &= \left(\sum_{l=0}^{\infty} \mathcal{D}_l(\lambda) \frac{1}{l!}(e^t-1)^l\right)\left(\sum_{m=0}^{\infty} Bel_m(x) \frac{t^m}{m!}\right) \\
&= \left(\sum_{l=0}^{\infty} \mathcal{D}_l(\lambda) \sum_{k=l}^{\infty} S_2(k,l) \frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty} Bel_m(x) \frac{t^m}{m!}\right) \\
&= \left(\sum_{n=0}^{\infty} \sum_{l=0}^{n} \mathcal{D}_l(\lambda) S_2(n,l) \frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} Bel_m(x) \frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{l=0}^{k} \mathcal{D}_l(\lambda) S_2(k,l) Bel_{n-k}(x) \binom{n}{k}\right) \frac{t^n}{n!},
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
\sum_{m=0}^{\infty} \mathcal{D}_m(x|\lambda) \frac{(e^t-1)^m}{m!} &= \sum_{m=0}^{\infty} \mathcal{D}_m(x|\lambda) \frac{1}{m!} m! \sum_{l=m}^{\infty} S_2(l,m) \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \mathcal{D}_m(x|\lambda) S_2(n,m)\right) \frac{t^n}{n!}.
\end{aligned} \tag{2.13}$$

By (2.12) and (2.13), we obtain the following theorem.

Theorem 2.6. For $n \geq 0$, we have

$$\sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k} \mathcal{D}_l(\lambda) S_2(k,l) Bel_{n-k}(x) = \sum_{m=0}^{n} \mathcal{D}_m(x|\lambda) S_2(n,m).$$

Note that, by Theorem 2.1 and (2.3), we get

$$\begin{aligned}
\frac{d^k}{dx^k} \mathcal{D}_n(x|\lambda) &= \sum_{m=0}^{n} \binom{n}{m} \mathcal{D}_{n-m}(\lambda) \frac{d^k}{dx^k} x^m \\
&= \sum_{m=k}^{n} \binom{n}{m} \mathcal{D}_{n-m}(\lambda) (m)_k x^{m-k}
\end{aligned} \tag{2.14}$$

and so

$$\frac{d^k}{dx^k}\mathcal{D}_n(x|\lambda)\bigg|_{x=0} = \binom{n}{k}\mathcal{D}_{n-k}(\lambda)k!. \tag{2.15}$$

Let us assume that $p(x) \in V_n$. Then, by Theorem 2.3, $p(x)$ can be generated by $\mathcal{D}_0(x|\lambda), \mathcal{D}_1(x|\lambda), \dots, \mathcal{D}_n(x|\lambda)$ to be

$$p(x) = \sum_{m=0}^{n} a_m \mathcal{D}_m(x|\lambda). \tag{2.16}$$

Note that

$$\mathcal{D}_0(x|\lambda) = \lim_{t\to 0}\frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}e^{xt} = 1. \tag{2.17}$$

Then,

$$\frac{d^k}{dx^k}p(x) = \sum_{m=k}^{n} a_m \frac{d^k}{dx^k}\mathcal{D}_m(x|\lambda), \tag{2.18}$$

and so, by (2.15),

$$\frac{d^k}{dx^k}p(0) = \sum_{m=k}^{n} a_m \binom{m}{k}\mathcal{D}_{m-k}(\lambda)k!. \tag{2.19}$$

By (2.17) and (2.19),

$$a_m = \frac{1}{m!}\left(\frac{d^m}{dx^m}p(0) - m!\sum_{k=m+1}^{n} a_k\binom{k}{m}\mathcal{D}_{k-m}(\lambda)\right) \tag{2.20}$$

for each $m = 0, 1, \dots, n$. Thus we can determine a_m by inductively.

From now on, We consider the *higher-order Appell-type Daehee polynomials* which is defined by

$$\left(\frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^r e^{xt} = \sum_{n=0}^{\infty}\mathcal{D}_n^{(r)}(x|\lambda)\frac{t^n}{n!}. \tag{2.21}$$

When $x = 0$, $\mathcal{D}_n(\lambda) = \mathcal{D}_n(0|\lambda)$ are called the *higher-order Appell-type Daehee numbers*.

Since

$$\begin{aligned}
\left(\frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^r &= \left(\frac{\log(1+t)}{t}\right)^r\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^r e^{xt} \\
&= \left(\sum_{n=0}^{\infty} D_n^{(r)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} B_n^{(n-r+1)}(1)\frac{\lambda^n t^n}{n!}\right)\left(\sum_{n=0}^{\infty} x^n\frac{t^n}{n!}\right) \\
&= \left(\sum_{n=0}^{\infty}\sum_{m=0}^{n} B_m^{(m-r+1)}(1)D_{n-m}^{(r)}\binom{n}{m}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} x^n\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k} B_m^{(m-r+1)}(1)D_{k-m}^{(r)}\binom{k}{m}\binom{n}{k}x^{n-k}\right)\frac{t^n}{n!},
\end{aligned} \tag{2.22}$$

by (2.21) and (2.22), we obtain the following theorem.

Theorem 2.7. For given $n \geq 0$, we have

$$\mathcal{D}_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} \sum_{m=0}^{k} B_m^{(m-r+1)}(1) D_{k-m}^{(r)} \binom{k}{m} \binom{n}{k} x^{n-k}.$$

Competing interests

The authors declare that they have no competing interests.

Author contributions

The authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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