Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 5 (2015), pp. 2719-2734
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PARTIAL METRIC ON SPACE OF SUBSETS

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Abstract. This article introduced the concept of the Hausdroff partial metric. By using the Hausdorff partial metric we study the fixed point theory for set-valued mappings on partial metric spaces. Fixed point theorem that has been resulting by Nadler for set-valued mapping will be extended to partial metric spaces and the proof using Ekeland variational principle.
Key Words and Phrases: fixed point, partial metric Hausdroff, set-valued mappings,
2010 Mathematics Subject Classification: 47H10, 54H25, 26E25,

1. Introduction

In [2] has been introduced partial Hausdroff metric, but it is less than perfect that is only meets three from four of partial metric axioms. Therefore in this paper will be refined. Before we start we write return Hausdorff metric induced by the metric d in [3].

Let (X, d) be a metric space and $cb(X)$ denotes the family of all non-empty closed bounded subsets of X. For every set $A, B \in cb(X)$, define

$$h(A, B) = \max\{d(A, B), d(B, A)\}$$

where $d(A, B) = \sup\{d(a, B) \mid a \in A\}$ and $d(a, B) = \inf\{d(a, b) \mid b \in B\}$. It is know h is metric on $cb(X)$, called the Hausdorff metric induced by the metric d. Matthews [4] introduced the concept of a partial metric as a part of the study of denotational semantics of dataflow networks. The following notion of partial metric in [4].

Definition 1.1 Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions holds.

(P1): $p(x, x) \leq p(x, y)$

(P2): $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$
(P3): $p(x,y) = p(y,x)$
(P4): $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$

The pairs (X,p) is called a partial metric space.Note that the self-distance of any point need not be zero. Partial metric p will become a metric if $p(x,x) = 0$.

If p is a partial metric on X, then the function $d_p : X \times X \to [0,\infty)$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

for each $x, y \in X$, is a metric on X:

Lemma 1.2 *Let (X,p) be a partial metric space. If $p(x,y) = 0$ then $x = y$.*

Proof. From (P1) and (P2) imply that x = y. But the convers does not hold always. □

Let (X,p) partial metric spaces. Point $a \in X$ dan $\epsilon > 0$. The open ball for a partial metric p are sets of the form

$$B_\epsilon(a) = \{x \in X \mid p(a,x) < \epsilon\}.$$

Since $p(a,a) \geq 0$, open ball can also be presented as

$$B_{\epsilon+p(a,a)}(a) = \{x \in X \mid p(a,x) < \epsilon + p(a,a)\}.$$

Contrary to the metric space case, some open balls may be empty. If $\epsilon > p(a,a)$, then $B_\epsilon(a) = B_{\epsilon-p(a,a)}(a)$. If $0 < \epsilon \leq p(a,a)$, then

$$B_\epsilon(a) = \{x \in X \mid p(a,x) < \epsilon \leq p(a,a)\} = \emptyset.$$

This means that the open ball $B_{p(a,a)}(a)$ is an empty set. Therefore, the point $a \notin B_{p(a,a)}(a)$.

Definition 1.3 Let (X,p) be a partial metric space. A sequence $\langle x_n \rangle$ in X *converges* to the point $x \in X$ if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.

Definition 1.4 Let (X,p) be a partial metric space. A sequence $\langle x_n \rangle$ in X *properly converges* to the point $x \in X$ if $\langle x_n \rangle$ converges to x and

$$\lim_{n\to\infty} p(x_n,x_n) = p(x,x).$$

In other words, a sequence $\langle x_n \rangle$ properly converges to $x \in X$ if $\lim_{n\to\infty} p(x_n,x)$. and $\lim_{n\to\infty} p(x_n,x_n)$ exists and

$$\lim_{n\to\infty} p(x_n,x_n) = \lim_{n\to\infty} p(x_n,x) = p(x,x).$$

Definition 1.5 Let (X,p) be a partial metric space. A sequence $\langle x_n \rangle$ in X is said to be a *Cauchy sequence* if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and finite.

In other words, $\langle x_n \rangle$ is Cauchy sequence if the numbers sequence $p(x_n, x_m)$ converges to some $\lambda \in \mathbb{R}$ as n and m approach to infinity, that is, if $\lim_{n,m\to\infty} p(x_n, x_m) = \lambda < \infty$. This means for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $m, n \geq N, |p(X_n, x_m| < \epsilon$. If (X,p) is a metric space then $\lambda = 0$.

Theorem 1.6 *A sequence $\langle x_n \rangle$ in a partial metric space (X,p) is a Cauchy, if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq \mathbb{N}$ we have*

$$p(x_n, x_m) - p(x_m, x_m) < \epsilon.$$

Proof. Since $\langle x_n \rangle$ is Cauchy, there exists $\lambda \in \mathbb{R}$ such that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq \mathbb{N}$ we have

$$|p(x_n, x_m) - \lambda| < \frac{\epsilon}{2}.$$

Let $n = m \geq N$, then $|p(x_m, x_m) - \lambda| < \dfrac{\epsilon}{2}$. Therefore

$$|p(x_n, x_m) - p(x_m, x_m)| \leq |p(x_n, x_m) - \lambda| + |\lambda - p(x_m, x_m)| < \epsilon.$$

By (P1), we obtain $p(x_n, x_m) - p(x_m, x_m) < \epsilon$. Conversly it is obvious. ■

Definition 1.7 A partial metric space (X,p) is said a complete if every sequence Cauchy in X properly converges.

For $A, B \in cb(X)$ and point $x \in X$ is defined

$$p(x,A) = \inf_{a\in A} p(x,a), \quad p(A,B) = \sup_{a\in A} p(a,B), \quad p(B,A) = \sup_{b\in B} p(b,A)$$

In general $p(A,B) \neq p(B,A)$. For example $X = \mathbb{R}$ and function $p : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ we define

$$p(x,y) = \max\{x,y\} \quad x, y \in \mathbb{R}$$

Clear that p is a partial metric. If $A = [1,3]$ and $B = [2,4]$, then $p(A,B) = \sup_{a\in A} p(a,B) = p(1,B) = \inf_{b\in B} p(1,b) = p(1,2) = \max\{1,2\} = 2$ and $p(B,A) = \sup_{b\in B} p(b,A) = p(4,A) = \inf_{a\in A} p(4,a) = p(4,3) = \max\{4,3\} = 4$.

In [2] we obtain the following.

Proposition 1.8 *Let (X, p) be a partial metric space. For any $A, B, C \in cb(X)$ we have the following.*

(i): $p(A, A) = \sup_{a \in A} p(a, a)$;
(ii): $p(A, A) \leq p(A, B)$
(iii): $p(A, B) = 0$ implies that $A = B$
(iv): $p(A, B) \leq p(A, C) + p(C, B) - \inf_{c \in C} p(c, c)$.

2. Main results

2.1. **Hausdorff Partial Metric.** We start with the following Definitions and Lemmas needed to prove our main result.

Definition 2.1.1 Let (X, p) be a partial metric space and $A \neq \emptyset \subset X$.

(i): A is a *bounded* if there exists $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$ we have

$$p(x_0, a) \leq M + p(a, a).$$

(ii): The point $x \in X$ is said a *limit point* of A if all $\epsilon > 0$ there exists $a \neq x \in A$ such that

$$p(x, a) \leq \epsilon + p(x, x).$$

(iii): A is a closed set if for all its limit point belongs in A.

Lemma 2.1.2 *Let (X, p) be a partial metric space and $A \subset X$. The point $x \in X$ is a limit point of A if and only if $p(x, A) = p(x, x)$.*

Proof. By Definition 2.1.1 part (ii) for every $\epsilon > 0$ there exists $a \neq x \in A$ we have

$$p(x, a) \leq \epsilon + p(x, x).$$

Since $p(x, A) \leq p(x, a)$ for all $a \in A$, we have

$$p(x, A) \leq \epsilon + p(x, x). \tag{1}$$

On the other hand

$$p(x, a) - \epsilon \leq p(x, A). \tag{2}$$

From inequality (1) and (2)we obtained

$$p(x, a) - \epsilon \leq \epsilon + p(x, x).$$

Thus

$$p(x, x) - \epsilon \leq p(x, a) - \epsilon \leq p(x, A) \leq \epsilon + p(x, x).$$

The number $\epsilon > 0$ is arbitrary , $p(x, A) = p(x, x)$.
The converse, for all $\epsilon > 0$ there exists $a \in A$ and $p(x, a) < p(x, A) + \epsilon$. Since

$p(x, A) = p(x, x)$, then $p(x, a) < p(x, x) + \epsilon$. By Definition 2.1.1 part (ii), x is the limit point of A.

■

Corollary 2.1.3 *Let (X, p) be a partial metric space and A any nonempty set in (X, p)*

$$x \in \bar{A} \Longleftrightarrow p(x, x) = p(x, A), \tag{3}$$

where $\bar{A}$ is closure of A.

The following proposition is addition the above of Proposition 1.8.

Proposition 2.1.4 *Let (X, p) be a partial metric space. For any $A, B \subset X$ we have the following.*

(i): for any $a \in A$ $p(a, B) = p(a, a)$ if and only if $A \subset \bar{B}$;
(ii): for any $b \in B$ $p(b, A) = p(b, b)$ if and only if $B \subset \bar{A}$;
(iii): for any $a \in A$ $p(a, B) = 0$ then $a \in \bar{B}$

Proof (i) Let $a \in A$. Since $p(a, a) \leq p(a, b)$ for all $b \in B$ and $p(A, B) = p(a, a)$ for all $a \in A$, therefore we have $p(a, a) \leq \inf_{b \in B} p(a, b) = p(a, B) \leq p(A; B) = p(a; a)$ for all $a \in A$ and hence $p(a, B) = p(a, a)$. Using (3) we have $a \in \bar{B}$ whenever $a \in A$ so $A \subset \bar{B}$. The converse, If $A \subset \bar{B}$,then $p(x, B) = p(x, x)$ for all $x \in A$. We know that $p(A, B) = \sup_{a \in A} p(a, B)$ so that for every $\epsilon > 0$, there exists $a \in A$ such that

$$p(A, B) - \epsilon < p(a, B) \leq p(A, B) < p(A, B) + \epsilon.$$

This means

$$|p(a, B) - p(A, B)| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $p(A, B) = p(a, B)$. The other hand, for all $a \in A \subset \bar{B}, p(a, a) = p(a, B)$ so that $p(A, B) = p(a, a)$ for all $a \in A$. Similarly to prove part (ii). (iii) Suppose that $p(a, B) = 0$ for all $a \in A$. Consequently $p(A, B) = 0$. Putting $a \in A$. Since $p(a, a) \leq p(a, b)$ for all $b \in B$, we have $p(a, a) \leq p(a, B) \leq p(A, B) = 0$ for all $a \in A$, thus $p(a, a) = 0$, and hence $p(a, a) = p(A, B) = 0$. From (i), it follows that $a \in \bar{B}$.

■

Now, suppose (X, p) be a partial metric space. For any $A, B \in cb(X)$, the function $H : cb(X) \times cb(X) \longrightarrow \mathbb{R}^+$ we define

$$H(A, B) = \max\{p(A, B), p(B, A)\}. \tag{4}$$

Proposition 2.1.5 *Let $(X;p)$ be a partial metric space. For all $A, B, C \in cb(X)$ we have*

(h1): $H(A,A) \leq H(A,B)$;
(h2): $H(A,A) = H(A,B) = H(B,B)$ if and only if $A = B$;
(h3): $H(A,B) = H(B,A)$;
(h4): $H(A,B) \leq H(A,C) + H(C,B) - H(C,C)$.

Proof. From proposition 1.8 part (ii), we have $H(A,A) = p(A,A) \leq p(A,B) \leq H(A,B)$. For (h2), suppose $H(A,A) = H(B,B) = H(A,B)$. Since $A \subset A$ and $B \subset B$ and from Lemma 2.1.4, we have $H(A,A) = p(A,A) = p(a,a)$ for all $a \in A$ and $H(B,B) = p(B,B) = p(b,b)$ for all $b \in B$. Since $H(A,A) = H(B,B)$, we obtained $H(A,B) = p(a,a) = p(b,b)$. Therefore we have, $p(A,B) \leq H(A,B) = p(a,a)$ and the other hand, $p(a,a) \leq p(a,B) \leq p(A,B)$ thus

$$p(A,B) = p(a,a)$$

By Proposition 2.1.4 part (i) we have $A \subset B$. Whenever $p(B,A) \leq H(A,B) = p(b,b)$ and the other hand, $p(b,b) \leq p(b,A) \leq p(B,A)$ and hence

$$p(B,A) = p(b,b)$$

By Proposition 2.1.4 part (i), we have $B \subset A$ so $A = B$. The converse, $A = B$ this implies $H(A,B) = H(A,A) = p(A,A) = p(a,a)$ for all $a \in A$ and $H(A,B) = H(B,B) = p(B,B) = p(b,b)$ for all $b \in B$. Since $p(a,a) = p(A,A) \leq P(A,B) = P(B,B) = p(b,b)$ and $p(b,b) = p(B,B) \leq p(B,A) = p(A,A) = p(a,a)$, we obtain $p(a,a) = p(b,b)$ for all $a \in A$ and $b \in B$. We conclude that

$$H(A,A) = H(A,B) = H(B,B)$$

By Definition equation (4), (h3) holds obviously. Now for (h4), by using property (iv) of Proposition 1.8, we have

$$\begin{aligned} H(A,B) &= \max\{p(A,B), p(B,A)\} \\ &\leq \max\{p(A,C) + p(C,B) - p(C,C), p(B,C) + p(C,A) - p(C,C)\} \\ &= \max\{p(A,C) + p(C,B), p(B,C) + p(C,A)\} - P(C,C) \\ &\leq \max\{p(A,C), p(C,A)\} + \max\{p(C,B), p(B,C)\} - P(C,C) \\ &= H(A,C) + H(C,B) - P(C,C) \\ &= H(A,C) + H(C,B) - H(C,C) \end{aligned}$$

■

Corollary 2.1.6 *Let (X,p) be a partial metric space. For $A,B \in cb(X)$ the following holds*

$$H(A,B)=0 \quad \textit{implies} \quad \textit{that} \quad A=B.$$

Proof. Let $H(A,B) = 0$. By Definition of H, $p(A,B) = p(B,A) = 0$. This implies that $p(a,B) = 0$ for all $a \in A$ and $p(b,A) = 0$ for all $b \in B$. Using part (iii) of Proposition 2.1.4, we obtain $A \subset B$ and $B \subset A$. Thus $A = B$. ■

Remark 2.1.7 *The converse of Corollary 2.1.6 is false in general. (see [2])*
In view of Proposition 2.1.5, we call the mapping $H : cb(X) \times cb(X) \longrightarrow \mathbb{R}^+$ a Hausdorff partial metric induced by p. The pairs $(cb(X), H)$ is called partial metric space with $cb(X)$ is based of set and H is a partial metric on $cb(X)$.

Given a set $A \in cb(X)$ and a number $\epsilon > 0$, we define the set $(A+\epsilon)$ by

$$(A+\epsilon) = \{x \in X : p(x,A) < \epsilon + p(x,x)\}. \tag{5}$$

Clear that $A \subseteq (A+\epsilon)$. We need to show that such set is closed for all possible choices of A and $\epsilon > 0$. To do this, we will begin by choosing an arbitrary limit point of the set $(A+\epsilon)$, and then showing that it is contained in the set.

Proosition 2.1.8 *The set $(A+\epsilon)$ is closed for all of $A \in cb(X)$ and $\epsilon > 0$.*

Proof. Let x be a limit point of $(A+\epsilon)$. Then for all $\delta > 0$, there exists $y \neq x \in (A+\epsilon)$ such that $p(x,y) < \delta + p(x,x)$. Since $A \subseteq (A+\epsilon)$, choose $y \in A$. This is implies $p(x,A) \leq p(x,y) < \delta + p(x,x)$. If $\epsilon \geq \delta$, then $p(x,A) < \epsilon + p(x,x)$. In other words, $x \in (A+\epsilon)$.

■

Proosition 2.1.9 *Suppose that $A,B \in cb(X)$ and that $\epsilon > 0$. Then $H(A,B) - H(A,A) < \epsilon$ if and only if $A \subseteq (B+\epsilon)$ and $B \subseteq (A+\epsilon)$.*

Proof. By simmetry it is suffucient to prove $p(A,B) - p(A,A) < \epsilon$ if and only if $A \subseteq (B+\epsilon)$. Suppose $p(A,B) - p(A,A) < \epsilon$. Then for every $a \in A, p(a,B) - p(a,a) < \epsilon$. It follows by definition of $(B+\epsilon)$, that $A \subseteq (B+\epsilon)$. Now suppose $A \subseteq (B+\epsilon)$. By definition of the set $(B+\epsilon)$, for every $a \in A, p(a,B) < \epsilon + p(a,a)$. It follow that $p(A,B) < \epsilon + p(A,A)$.

■

Definition 2.1.10 Let $(cb(X), H)$ be a partial metric space. A sequence $\langle A_n \rangle$ in $cb(X)$ *converges* to set $A \in cb(X)$ if $\lim_{n\to\infty} H(A_n, A) = H(A,A)$.

Definition 2.1.11 Let $(cb(X), H)$ be a partial metric space. A sequence $\langle A_n \rangle$ in $cb(X)$ *properly converges* to set $A \in cb(X)$ if $\langle A_n \rangle$ converges to A and

$$\lim_{n\to\infty} H(A_n, A_n) = H(A, A).$$

In other words, a sequence $\langle A_n \rangle$ properly converges to $A \in cb(X)$ if $\lim_{n\to\infty} H(A_n, A)$. and $\lim_{n\to\infty} H(A_n, A_n)$ exists and

$$\lim_{n\to\infty} H(A_n, A_n) = \lim_{n\to\infty} H(A_n, A) = H(A, A).$$

Definition 2.1.12 Let $(cb(X), H)$ be a partial metric space. A sequence $\langle A_n \rangle$ in $cb(X)$ is said to be a Cauchy sequence if $\lim_{n,m\to\infty} H(A_n, A_m)$ exists and finite.

In other words,$\langle A_n \rangle$ is Cauchy sequence if the numbers sequence $H(A_n, A_m)$ converges to some $\alpha \in \mathbb{R}$ as n and m approach to infinity, that is, if $\lim_{n,m\to\infty} H(A_n, A_m) = \alpha < \infty$. This means for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $m, n \geq N$, we have

$$|H(A_n, A_m) - \alpha| < \epsilon. \tag{6}$$

Theorem 2.1.13 *A sequence $\langle A_n \rangle$ in a partial metric space $(cb(X), H)$ is a Cauchy, if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have*

$$H(A_n, A_m) - H(A_m, A_m) < \epsilon.$$

Proof. Since $\langle A_n \rangle$ is Cauchy, there exists $\alpha \in \mathbb{R}$ such that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq \mathbb{N}$ we have

$$|H(A_n, A_m) - \alpha| < \frac{\epsilon}{2}.$$

Let $n = m \geq N$, then $|H(A_m, A_m) - \alpha| < \dfrac{\epsilon}{2}$. Therefore

$$|H(A_n, A_m) - H(H_m, A_m)| \leq |H(A_n, A_m) - \alpha| + |\alpha - H(A_m, A_m)| < \epsilon.$$

By (P1), we obtain $H(A_n, A_m) - H(A_m, A_m) < \epsilon$. Conversly it is obvious.

■

Lemma 2.1.14 *Let $\langle A_n \rangle$ be a Cauchy sequence in $cb(X)$ and let $\langle n_k \rangle$ be an increasing sequence of positive integers. If $\langle y_{n_k} \rangle$ is a Cauchy in X for which $y_{n_k} \in A_{n_k}$ for all $k \in \mathbb{N}$, then there exists a Cauchy sequence $\langle x_n \rangle$ in X such that $x_n \in A_n$ for all n and $x_{n_k} = y_{n_k}$ for all k.*

Proof. Suppose $\langle y_{n_k}\rangle$ is a Cauchy in X for which $y_{n_k} \in A_{n_k}$ for all $k \in \mathbb{N}$. By definition of infmum, there exists $a_j \in A_n$ such that $p(y_{n_k}, a_j) < p(y_{n_k}, A_n) + \frac{1}{j}$. Suppose $\langle a_{n_j}\rangle$ in A_n that converges to $a \in A_n$. Then we find that

$$\begin{aligned} p(y_{n_k}, A_n) &\le p(y_{n_k}, a) \le p(y_{n_k}, a_{n_j}) + p(a_{n_j}, x) - p(a_{n_j}, a_{n_j}) \\ &< p(y_{n_k}, A_n) + \frac{1}{n_j} + p(a_{n_j}, x) - p(a_{n_j}, a_{n_j}). \end{aligned}$$

Since $\lim_{j\to\infty} \frac{1}{n_j} + [p(a_{n_j}, x) - p(a_{n_j}, a_{n_j})] = 0$, it follow that $p(y_{n_k}, A_n) = p(y_{n_k}, a)$ For each $n \le n_k$ to choose $x_n \in A_n$ such that $p(y_{n_k}, A_n) = p(y_{n_k}, x_n)$. Then we find $p(y_{n_k}, x_n) = p(y_{n_k}, A_n) \le p(A_{n_k}, A_n) \le H(A_{n_k}, A_n)$. Similarly $p(y_{n_k}, x_{n_k}) = p(y_{n_k}, A_{n_k}) \le p(A_{n_k}, A_{n_k}) \le H(A_{n_k}, A_n)$. Since, $\langle A_n\rangle$ is a Cauchy sequence, it follow that $p(y_{n_k}, x_{n_k}) = p(y_{n_k}, A_{n_k}) = 0$. This implies $y_{n_k} = x_{n_k}$. Let $\epsilon > 0$. Since $\langle y_{n_k}\rangle$ is a Cauchy sequence in X, there exists $K \in \mathbb{N}$ such that $p(y_{n_k}, y_{n_j}) - p(y_{n_j}, y_{n_j}) < \frac{\epsilon}{4}$ for all $k, j \ge K$. Since $\langle A_n\rangle$ is Cauchy sequence in $cb(X)$, there exists $N \ge n_K \ge K$ such that $H(A_n, A_m) - H(A_m, A_m) < \frac{\epsilon}{4}$ for all $n, m \ge N$. We choose $j, k \ge K$ such that $n_k \ge n$ and $n_j \ge m$. Then we find that

$$\begin{aligned} p(x_n, x_m) - p(x_m, x_m) &\le p(x_n, y_{n_k}) + p(y_{n_k}, x_m) - p(y_{n_k}, y_{n_k}) - p(x_m, x_m) \\ &\le p(x_n, y_{n_j}) + p(y_{n_j}, y_{n_k}) - p(y_{n_j}, y_{n_j}) + p(y_{n_k}, x_m) \\ &\quad - p(y_{n_k}, y_{n_k}) - p(x_m, x_m) \\ &= p(y_{n_k}, A_n) + p(y_{n_j}, y_{n_k}) - p(y_{n_j}, y_{n_j}) + p(y_{n_k}, A_m) \\ &\quad - p(y_{n_k}, y_{n_k}) - p(x_m, x_m) \\ &\le p(A_{n_k}, A_n) + p(y_{n_j}, y_{n_k}) - p(y_{n_j}, y_{n_j}) + p(A_{n_k}, A_m) \\ &\quad - p(A_{n_k}, A_{n_k}) - p(A_m, A_m) \\ &\le [p(A_{n_k}, A_n) - p(A_{n_k}, A_{n_k})] + \frac{\epsilon}{4} \\ &\quad + [p(A_{n_k}, A_m) - p(A_m, A_m)] \\ &\le [H(A_{n_k}, A_n) - H(A_{n_k}, A_{n_k})] + \frac{\epsilon}{4} \\ &\quad + [H(A_{n_k}, A_m) - H(A_m, A_m)] \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon. \end{aligned}$$

Therefore, by definition and from our earlier set up, $\langle x_n\rangle$ is a Cauchy sequence in X such that $x_n \in A_n$ for all n and $x_{n_k} = y_{n_k}$ for all k. This complete the proof. ■

Lemma 2.1.15 *Let (X,p) be a complete partial metric space and $\langle A_n \rangle$ be a sequence in $cb(X)$ and let $A = \{x \in X : \exists \langle x_n \rangle; x_n \in A_n; n \in \mathbb{N}; x_n \to x\}$. If $\langle A_n \rangle$ is a Cauchy sequence, then the set A is nonempty closed and bounded.*

Proof. Since $\langle A_n \rangle$ is a Cauchy sequence, by Theorem 2.1.13, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n, m \geq N$, we have

$$H(A_n, A_m) - H(A_m, A_m) < \epsilon. \tag{7}$$

Since $H(A_m, A_m) = p(A_m, A_m)$ and $p(A_n, A_m) \leq H(A_n, A_m)$, from (7) for every $n, m \geq N$ we obtain

$$p(A_n, A_m) - p(A_m, A_m) < \epsilon. \tag{8}$$

For each $x_n \in A_n$ and $x_m \in A_m$, $p(x_n, x_m) \leq p(A_n, A_m)$. Since $p(x_m, x_m) = p(A_m, A_m)$. From (8) for every $n, m \geq N$ we obtain

$$p(x_n, x_m) - p(x_m, x_m) < \epsilon. \tag{9}$$

By Theorem 1.6 show that,$\langle x_n \rangle$ is a Cauchy sequence. Since X is complete, the Cauchy sequence properly converges to a point $x \in X$. Since $x_n \in A_n$ for all $n \in \mathbb{N}$, then by definition of the set A, $x \in A$. Hence A is nonempty. Furthermore, we will prove that A is closed set. Suppose x is a point limit of A. Then there exists $a_k \neq x \in A$ such that

$$\lim_{k \to \infty} p(a_k, x) - p(x, x) = 0. \tag{10}$$

Since for each $a_k \in A$, there exists a sequence $\langle y_n \rangle$,with $y_n \in A_n$ for each $n \in \mathbb{N}$ and y_n converges to a_k. Consequently, there exists an integer n_1 such that $y_{n_1} \in A_{n_1}$ and $p(y_{n_1}, a_1) - p(a_1, a_1) < 1$. Similarly, there exists an integer $n_2 > n_1$ such that $y_{n_2} \in A_{n_2}$ and $p(y_{n_2}, a_2) - p(a_2, a_2) < \frac{1}{2}$. Continuing this process we can choose $n_k > n_{k+1}$ such that $y_{n_k} \in A_{n_k}$ and

$$p(y_{n_k}, a_k) - p(a_k, a_k) < \frac{1}{k}. \tag{11}$$

It follows that

$$p(y_{n_k}, x) - p(x, x) \leq p(y_{n_k}, a_k) + p(a_k, x) - p(a_k, a_k) - p(x, x).$$

From (10) and (11) we have

$$\lim_{k \to \infty} p(y_{n_k}, x) - p(x, x) = 0.$$

It follows that $\langle y_{n_k} \rangle$ converges to x. We show $\langle y_{n_k} \rangle$ is a Cauchy sequence. Let $\epsilon > 0$. Since $\langle A_n \rangle$ is Cauchy sequence in $cb(X)$, there exists $N \geq n_K \geq K$ such that

$H(A_n, A_m) - H(A_m, A_m) < \frac{\epsilon}{2}$ for all $n, m \geq N$. We choose $j, k \geq K$ such that $n_k \geq n$ and $n_j \geq m$. Then we find that

$$\begin{aligned}
p(y_{n_k}, y_{n_j}) - p(y_{n_j}, y_{n_j}) &\leq p(y_{n_k}, y_n) + p(y_n, y_{n_j}) - p(y_n, y_n) - p(y_{n_j}, y_{n_j}) \\
&= p(y_{n_k}, A_n) + p(y_{n_j}, A_n) - p(A_n, A_n) - p(A_{n_j}, A_{n_j}) \\
&\leq [p(A_{n_k}, A_n) + p(A_n, A_n)] + [p(A_{n_j}, A_n) - p(A_{n_j}, A_{n_j})] \\
&\leq [H(A_{n_k}, A_n) + H(A_n, A_n)] + [H(A_{n_j}, A_n) - p(A_{n_j}, A_{n_j})] \\
&< \epsilon
\end{aligned}$$

It follow that $\langle y_{n_k} \rangle$ is a Cauchy sequence for which $y_{n_k} \in A_{n_k}$ for all k. By Lemma 2.1.14 guarantees that there exists a Cauchy sequence $\langle x_n \rangle$ in X such that $x_n \in A_n$ for all n and $x_{n_k} = y_{n_k}$. Therefore $x \in A$, so A is closed.

Now, we show A is bounded set. For each $x \in A$ there exists $x_n \in A_n$ such that $p(x_n, x) < \epsilon + p(x, x)$, for all $\epsilon > 0$ and $n \leq N$. Suppose $x_0 \in X$ and $K = \max\{p(x_0, x_1), p(x_0, x_2), \cdots, p(x_0; x_{N-1}), \epsilon + p(x, x)\}$. For $n = 1, 2, \cdots (N-1)$, we have

$$p(x_0, x) \leq p(x_0, x_n) + p(x_n, x) - p(x_n, x_n) < K + \epsilon + p(x, x) = M + p(x, x).$$

■

Definition 2.1.16 A partial metric space $(cb(X), H)$ is said a complete if every Cauchy sequence properly converges in $cb(X)$.

Proposition 2.1.17 *If (X, p) is a complete partial metric space, then $(cb(X), H)$ is a complete partial metric space*

Proof. Let $\langle A_n \rangle$ be a Cauchy sequence in $cb(X)$ and $A = \{x \in X : \exists \langle x_n \rangle; x_n \in A_n; n \in \mathbb{N}; x_n \to x\}$. We must prove that $A \in cb(X)$ and $\langle A_n \rangle$ converges to A. By Lemma 2.1.15, the set A is nonempty, closed and bounded, so that $A \in cb(X)$. Now, we will prove that $\langle A_n \rangle$ converges to A. Let $\epsilon > 0$. We need to show that there exists $N \in \mathbb{N}$ such that $H(A_n, A) - H(A, A) < \epsilon$ for all $n \geq N$. To do this, Proposition 2.1.9 tells us that to show two conditions, that $An \subseteq (A + \epsilon)$ and $A \subseteq (A_n + \epsilon)$. Since $\langle A_n \rangle$ is Cauchy sequence, by Theorem 2.1.13 we have

$$H(A_n, A_m) - H(A_m, A_m) < \epsilon \tag{12}$$

for all $n, m \geq N$.
Since $p(A_n, A_m) \leq H(A_n, A_m)$ we obtain

$$p(A_n, A_m) - p(A_m, A_m) < \epsilon \tag{13}$$

for all $n, m \geq N$.
Let $x_n \in A_n$. Since $p(x_n, x_m) \leq p(A_n, A_m)$. From (13), we have

$$p(x_n, x_m) - p(x_m, x_m) < \epsilon \tag{14}$$

for all $n, m \geq N$.
It follows that, $\langle x_n \rangle$ is Cauchy sequence in X. By completeness of X, $\langle x_n \rangle$ properly converges to $x \in X$ and of course, sequence $\langle x_n \rangle$ converges to $x \in X$, that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$p(x_n, x) - p(x_n, x_n) < \epsilon \tag{15}$$

for every $n \geq N$.
If $x \in A$, then

$$p(x_n, A) - p(x_n, x_n) < \epsilon$$

By definition of the set $(A + \epsilon), x_n \in (A + \epsilon)$. Therefore

$$A_n \subseteq (A + \epsilon), \tag{16}$$

for every $n \geq N$.
Furthermore to prove $A \subseteq (An + \epsilon)$. Let $\epsilon > 0$. Since $\langle A_n \rangle$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that

$$H(A_n, A_m) - H(A_m, A_m) < \epsilon, \tag{17}$$

for all $n, m \geq N$.
By Proposition 2.1.9, $A_n \subseteq (A_m + \epsilon)$ for all $n, m \geq N$. Let $a \in A$. By definition of the set A, there exists a sequence $\langle x_n \rangle$ such that $x_n \in A_n$ for all $n \in \mathbb{N}$ and $\langle x_n \rangle$ converges to a, that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$p(a, A_n) - p(a, a) \leq p(a, x_n) - p(a, a) < \epsilon, \tag{18}$$

for every $n \geq N$.
From (18) and by definition of the set $(A_n + \epsilon), a \in (A_n + \epsilon)$. Therefore

$$A \subseteq (A_n + \epsilon), \tag{19}$$

for every $n \geq N$. From (16),(19) and by Proposistion 2.1.9, then that $H(A_n, A) - H(A, A) < \epsilon$ and thus $\langle A_n \rangle$ converges to A. Therefore if (X, p) is complete, then $(cb(X), H)$ is complete.

∎

2.2. **Fixed Point Theorem.** In this subsection, we focus our attention on the fixed point theory for set-valued maps.

Let X is a partial metric space and $\mathcal{P}(X)$ the collection of all non-empty subsets of X . The mapping $F : X \longrightarrow \mathcal{P}(X)$ is called set-valued maps. The points $x \in X$ is called a fixed point of F if $x \in F(x)$.

Definition 2.2.1 Let (X, p) be a partial metric space. The mapping $F : X \longrightarrow cb(X)$ is called contraction mapping if there exists real number $k \in [0, 1)$ such that

$$H(F(x), F(y))) - H(F(x), F(x)) < k[p(x, y) - p(x, x)], \tag{20}$$

for all $x, y \in X$.

In 1969, Nadler [6] established the following analogue of Banach contraction theorem for set-valued maps.

Theorem 2.2.2 [6] *Let (X, d) be a complete metric space and let $F : X \longrightarrow cb(X)$ be a set-valued map. Assume that there exists $k \in [0, 1)$ such that*

$$h(F(x), F(y))) < kd(x, y), \tag{21}$$

for all $x, y \in X$. Then F has a fixed point.

Note that the above notation h is called Hausdorff metric induced by d.

The following, our similarly introduce fixed point theorem for set-valued maps based on partial metric space. Before, we present, the weak formulation of Ekeland's variational principle on partial metric space version.

Lemma 2.2.3 [5] *Let (X, p) be a complete partial metric space and $\varphi : X \longrightarrow [0, 1)$ be lower semicontinuous function. Then for any $\epsilon > 0$, there exists $x^* \in X$ such that*

$$\varphi(x^*) \leq \inf_{x \in X} \varphi(x) + \epsilon$$

and

$$\varphi(x^*) < \varphi(x) + \epsilon(p(x, x^*) - p(x^*, x^*))$$

for all $x \in X$ with $x \neq x^*$

The continuity of the self-maps in the partial metric spaces is, in fact, the sequential continuity.

Definition 2.2.4 Let (X, p) be a partial metric space. The mapping $f : X \longrightarrow X$ is called continuous at the point $x_0 \in X$ if, for any sequence $\langle x_n \rangle$ in X converges to x_0, then a sequence $\langle f(x_n \rangle$ converges to $f(x_0)$

Lemma 2.2.5 [7] *Let (X, p) be a partial metric space and the function $f : X \longrightarrow X$. Then for each $x \in X$, the function $\varphi : X \longrightarrow [0, 1)$ given by $\varphi_x(y) = p(x, f(y))$ is lower semicontinuous on (X, d_p).*

Proof. Assume that a sequence $\langle y_n \rangle$ converges to the point $y \in X$, then $\lim_{n\to\infty} d_p(f(y_n), f(y)) = 0$. Futhermore that

$$\begin{aligned}\varphi_x(y) &= p(x, f(y))\\ &\leq p(x, f(y_n)) + p(f(y_n), f(y)) - p(f(y_n), f(y_n))\\ &= p(x, f(y_n)) + d_p(f(y_n), f(y)) - (p(f(y_n), f(y)) - p(f(y), f(y))\\ &\leq p(x, f(y_n)) = \varphi_x(y_n)\end{aligned}$$

This yields $\liminf_{n\to\infty} \varphi_x(y_n) \geq \varphi_x(y)$ because $p(f(y_n), f(y)) \geq p(f(y), f(y))$. ■

Lemma 2.2.6 *Let (X, p) be a partial metric space and $F : X \longrightarrow cb(X)$ be a set-valued map. For each $x \in X$, the function $\varphi_x : X \longrightarrow [0, 1)$ given by $\varphi_x(y) = p(x, F(y))$. If F contraction, then the function φ_x is continuous on X.}*

Proof. *Assume that a sequence y_n converges to y in X, then $\lim_{n\to\infty} p(y_n, y) - p(y, y) = 0$. By (P4), for each $x \in X$, we have*

$$\begin{aligned}|\varphi_x(y_n) - \varphi_x(y)| &= |p(x, F(y_n)) - p(x, F(y))|\\ &\leq |p(F(y), F(y_n)) - p(F(y), F(y))|\\ &= p(F(y), F(y_n)) - p(F(y), F(y))\\ &\leq H(F(y), F(y_n)) - H(F(y), F(y))\\ &< k(p(y_n, y) - p(y, y))\end{aligned}$$

This yields $\liminf_{n\to\infty} \varphi_x(y_n) = \varphi_x(y)$ because $\lim_{n\to\infty} p(y_n, y) - p(y, y) = 0$. ■

Theorem 2.2.7 *Let (X, p) be a complete partial metric space. If set-valued map $F : X \longrightarrow cb(X)$ is contractive, then F has a fixed point.*

Proof. *Consider the function $\varphi : X \longrightarrow [0, 1)$ defined by*

$$\varphi(x) = p(x, F(x))$$

for all $x \in X$.

By Lemma 2.2.6, φ is a continuous on X. Choose $\epsilon > 0$ such that $0 < \epsilon < 1 - k$, *where* $k \in (0,1)$. *By Lemma 2.2.3, there exists* $x* \in X$ *such that*

$$\varphi(x^*) < \varphi(x) + \epsilon(p(x,x^*) - p(x^*,x^*))$$

for all $x \in X$ *with* $x \neq x^*$. *Putting* $x \in F(x^*)$ *and* $H((F(x^*),F(x*)) = 0$, *we have*

$$\begin{aligned}
p(x^*, F(x^*) &< p(x, F(x)) + \epsilon(p(x,x^*) - p(x^*),x^*)) \\
&\leq p(F(x^*), F(x)) + \epsilon(p(x,x^*) - p(x^*),x^*)) \\
&\leq H(F(x^*),F(x)) - H(F(x^*),F(x^*)) + \epsilon(p(x,x^*) - p(x^*),x^*)) \\
&\leq (k+\epsilon)(p(x,x^*) - p(x^*,x^*)) \\
&\leq \inf_{x \in F(x^*)} (k+\epsilon)p(x,x^*) \\
&= (k+\epsilon)p(x^*, F(x^*).
\end{aligned}$$

If $p(x^*, F(x^*)) \neq 0$, *then we obtain* $1 \leq (k+\epsilon)$, *which contradict to our assumption that* $1 > (k+\epsilon)$. *Therefore, we have* $p(x^*, F(x^*)) = 0$. *Since* $F(x^*)$ *is closed set, by Proposition 2.1.4 part (iii),* $x^* \in F(x^*)$. *We conclude that* F *has fixed point* $x^* \in X$.

■

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Received: June 2015 ; Accepted: Aug 2015