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Some Properties of Certain Analytic Functions

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Abstract

Defining the subclass $\mathcal{MD}(\rho, \mu, \alpha, \beta)$ of certain analytic functions f(z) in the open unit disk Δ, some properties for f(z) belonging to the class $\mathcal{MD}(\rho, \mu, \alpha, \beta)$ are discussed. In this present paper, some coefficient estimates and some interesting application of Jack's lemma for functions f(z) in the class $\mathcal{MD}(\rho, \mu, \alpha, \beta)$ are given.

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1. Introduction

Let $\mathcal{A}$ be the class of functions f(z) of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\Delta = \{z \in C: |z| < 1\}$.

Shams, Kulkarni and Jahangiri [4] have considered the subclass $\mathcal{SD}(\alpha,\beta)$ of $\mathcal{A}$ consisting of f(z) which satisfy

$$\text{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \quad (z \in \Delta).$$

for some α $(\alpha \geq 0)$ and β $(0 \leq \beta < 1)$.

The class $\mathcal{KD}(\alpha, \beta)$ is defined by the subclass of $\mathcal{A}$ consisting of f(z) such that $zf'(z) \in \mathcal{SD}(\alpha, \beta)$. In view of the classes $\mathcal{SD}(\alpha, \beta)$ and $\mathcal{KD}(\alpha, \beta)$, we introduce the subclass $\mathcal{MD}(\alpha, \beta)$ of $\mathcal{A}$ consisting of all functions f(z) which satisfy

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] < \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \quad (z \in \Delta)$$

for some α ($\alpha \leq 0$) and β ($\beta > 1$).

The class $\mathcal{ND}(\alpha, \beta)$ is also defined by $f(z) \in \mathcal{ND}(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{MD}(\alpha, \beta)$. The classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$ were introduced by Nishiwaki and Owa [3]. We discuss some properties of functions f(z) belonging to the classes $\mathcal{MD}(\alpha, \beta)$ and $\mathcal{ND}(\alpha, \beta)$.

We note if $f(z) \in \mathcal{MD}(\alpha, \beta)$, then $\frac{zf'(z)}{f(z)} = u + iv$ maps Δ onto the elliptic domain such that

$$\left(U - \frac{\alpha^2 - \beta}{\alpha^2 - 1}\right)^2 + \left(\frac{\alpha^2}{\alpha^2 - 1}\right)v^2 < \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2}$$

for $\alpha < -1$, the parabolic domain such that

$$u < -\frac{1}{2(\beta - 1)}v^2 + \frac{(\beta + 1)}{2} \quad \text{for} \quad \alpha = -1,$$

and the hyperbolic domain such that

$$\left(U - \frac{\alpha^2 - \beta}{\alpha^2 - 1}\right)^2 + \left(\frac{\alpha^2}{\alpha^2 - 1}\right)v^2 > \frac{\alpha^2(\beta - 1)^2}{(\alpha^2 - 1)^2}$$

for $-1 < \alpha < 0$. See also [2].

We now introduce a subclass $\mathcal{MD}(\rho, \mu, \alpha, \beta)$ of $\mathcal{A}$ as follows:

Definition 1.1.

Let $\mathcal{MD}(\rho, \mu, \alpha, \beta)$ denote the subclass of functions in $\mathcal{A}$ consisting of f(z) which satisfy

$$\mathrm{Re}\left[\frac{\rho\mu z^3 f'''(z) + (2\rho\mu + \rho - \mu)z^2 f''(z) + zf'(z)}{\rho\mu z^2 f''(z) + (\rho - \mu)zf'(z) + (1 - \rho + \mu)f(z)}\right]$$

$$< \alpha\left|\frac{\rho\mu z^3 f'''(z) + (2\rho\mu + \rho - \mu)z^2 f''(z) + zf'(z)}{\rho\mu z^2 f''(z) + (\rho - \mu)zf'(z) + (1 - \rho + \mu)f(z)} - 1\right| + \beta$$

for some α ($\alpha \leq 0$) and β ($\beta > 1$) and $0 \leq \lambda \leq 1$.

2. Coefficient Estimates for the Class $\mathcal{MD}(\rho, \mu, \alpha, \beta)$

By definition of $\mathcal{MD}(\rho, \mu, \alpha, \beta)$, we derive

Theorem 2.1.
If f(z) $\in \mathcal{MD}(\rho, \mu, \alpha, \beta)$, then

$$f(z) \in \mathcal{MD}\left(\rho, \mu, 0, \frac{\beta-\alpha}{1-\alpha}\right)$$

Proof.
If f(z) $\in \mathcal{MD}(\rho, \mu, \alpha, \beta)$ then

$$\begin{aligned}\mathrm{Re}\left[\frac{\rho\mu z^3 f'''(z)+(2\rho\mu+\rho-\mu)z^2 f''(z)+zf'(z)}{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}\right] &< \alpha\left|\frac{\rho\mu z^3 f'''(z)+(2\rho\mu+\rho-\mu)z^2 f''(z)+zf'(z)}{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}-1\right|+\beta \\ &\le \alpha\,\mathrm{Re}\left[\frac{\rho\mu z^3 f'''(z)+(2\rho\mu+\rho-\mu)z^2 f''(z)+zf'(z)}{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}-1\right]+\beta \quad (z\in\Delta)\end{aligned}$$

implies that

$$(1-\alpha)\mathrm{Re}\left[\frac{\rho\mu z^3 f'''(z)+(2\rho\mu+\rho-\mu)z^2 f''(z)+zf'(z)}{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}\right]<\beta-\alpha$$

That is,

$$\mathrm{Re}\left[\frac{\rho\mu z^3 f'''(z)+(2\rho\mu+\rho-\mu)z^2 f''(z)+zf'(z)}{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}\right]<\frac{\beta-\alpha}{1-\alpha} \quad (\alpha\le 0, \beta>1)$$

since $\frac{\beta-\alpha}{1-\alpha}>1,$ we prove the theorem. □

Theorem 2.2.
If f(z) $\in \mathcal{MD}(\rho, \mu, \alpha, \beta)$, then

$$|a_2| \le \frac{2(\beta-1)}{(1-\alpha)(2\rho\mu+\rho-\mu+1)} \quad \text{and}$$

$$|a_n| \le \frac{2(\beta-1)}{(1-\alpha)[n(n-1)^2\rho\mu+(n-1)^2\rho-(n-1)^2\mu+(n-1)]} \prod_{j=1}^{n-2}\left[1+\frac{2(\beta-1)(1+j\rho-j\mu+j(j+1)\rho\mu}{j(1+j\rho-j\mu+j(j+1)\rho\mu)(1-\alpha)}\right] \quad (n\ge 3)$$

Proof.
If f(z) $\in \mathcal{MD}(\rho, \mu, \alpha, \beta)$, then

$$(\beta-\alpha)+(\alpha-1)\mathrm{Re}\left[\frac{\rho\mu z^3 f'''(z)+(2\rho\mu+\rho-\mu)z^2 f''(z)+zf'(z)}{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}\right]>0$$

from Theorem 2.1. And let us define the function p(z) by

$$p(z)=\frac{(\beta-\alpha)+(\alpha-1)\operatorname{Re}\left[\dfrac{\rho\mu z^3 f'''(z)+(2\rho\mu+\rho-\mu)z^2 f''(z)+zf'(z)}{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}\right]}{(\beta-1)} \tag{1}$$

Then p(z) is analytic in Δ, p(0) = 1 and Re p(z) > 0 ($z \in \Delta$). Therefore, if we write

$$p(z)=1+\sum_{n=1}^{\infty} p_n z^n, \tag{2}$$

then $|p_n| \le 2$ ($n \ge 1$).

From (1) and (2), we obtain that

$$(\alpha-1)\sum_{n=2}^{\infty}[n^3\rho\mu+n^2(-2\rho\mu+\rho-\mu)+n(1-2\rho+2\mu+\rho\mu)-1+\rho-\mu]a_n z^n$$

$$(\beta-1)\sum_{n=1}^{\infty} p_n z^n\left[z+\sum_{n=2}^{\infty}[(n^2-n)\rho\mu+(\rho-\mu)n+(1-\rho+\mu)]a_n z^n\right]$$

Therefore, we have

$$a_n=\frac{\begin{array}{l}(\beta-1)[p_{n-1}+p_{n-2}a_2(2\rho\mu+\rho-\mu+1)+p_{n-3}a_3(6\rho\mu+2\rho-2\mu+1)\\ +\cdots+p_2a_{n-2}[(n-2)(n-3)\rho\mu+(n-3)\rho-(n-3)\mu+1]\\ +p_1a_{n-1}[(n-1)(n-2)\rho\mu+(n-2)\rho-(n-2)\mu+1]]\end{array}}{(\alpha-1)[n^3\rho\mu+n^2(-2\rho\mu+\rho-\mu)+n(1-2\rho+2\mu+\rho\mu)-1+\rho-\mu]} \quad \forall \quad n\ge 2$$

when n = 2,

$$|a_2| \le \frac{(\beta-1)}{(1-\alpha)(2\rho\mu+\rho-\mu+1)}|p_1|$$

$$\le \frac{2(\beta-1)}{(1-\alpha)(2\rho\mu+\rho-\mu+1)}$$

And when n = 3,

$$|a_3| \le \frac{(\beta-1)[|p_2|+|p_1|\,|a_2|(2\rho\mu+\rho-\mu+1)]}{2(1-\alpha)(6\rho\mu+2\rho-2\mu+1)}$$

$$\le \frac{2(\beta-1)}{2(1-\alpha)(6\rho\mu+2\rho-2\mu+1)}\left[1+\frac{2(\beta-1)}{(1-\alpha)}\right]$$

Let us suppose that,

$$|a_k| \le \frac{\begin{array}{l}2(\beta-1)[1+|a_2|(2\rho\mu+\rho-\mu+1)+|a_3|(6\rho\mu+2\rho-2\mu+1)+\cdots\\ +|a_{k-2}|[(k-2)(k-3)\rho\mu+(k-3)\rho-(k-3)\mu+1]\\ +|a_{k-1}|[(k-1)(k-2)\rho\mu+(k-2)\rho-(k-2)\mu+1]\end{array}}{(1-\alpha)[k(k-1)^2\rho\mu+(k-1)^2\rho-(k-1)^2\mu+(k-1)]}$$

$$\le \frac{2(\beta-1)}{(1-\alpha)[k(k-1)^2\rho\mu+(k-1)^2\rho-(k-1)^2\mu+(k-1)]} \tag{3}$$

$$\prod_{j=1}^{k-2}\left[1+\frac{2(\beta-1)(1+j\rho-j\mu+j(j+1)\rho\mu)}{(1-\alpha)j(1+j\rho-j\mu+j(j+1)\rho\mu)}\right] \qquad (k\ge 3)$$

Then we see,

$$\begin{aligned}
&1+|a_2|(2\rho\mu+\rho-\mu+1)+|a_3|(6\rho\mu+2\rho-2\mu+1)+\cdots\\
&\qquad+|a_{k-2}|[(k-2)(k-3)\rho\mu+(k-3)\rho-(k-3)\mu+1]\\
&\qquad+|a_{k-1}|[(k-1)(k-2)\rho\mu+(k-2)\rho-(k-2)\mu+1]\\
&\qquad\leq\prod_{j=1}^{k-2}\left[1+\frac{2(\beta-1)(1+j\rho-j\mu+j(j+1)\rho\mu)}{(1-\alpha)j(1+j\rho-j\mu+j(j+1)\rho\mu)}\right]
\end{aligned}\tag{4}$$

By using (3) and (4), we obtain that

$$\begin{aligned}
|a_{k+1}|&\leq\frac{2(\beta-1)}{(1-\alpha)[k^2(k+1)\rho\mu+k^2\rho-k^2\mu+k]}\\
&\quad[1+|a_2|(2\rho\mu+\rho-\mu+1)+\cdots\\
&\quad+|a_{k-1}|[(k-1)(k-2)\rho\mu+(k-2)\rho-(k-2)\mu+1]\\
&\quad+|a_k|[(k-1)k\rho\mu+(k-1)\rho-(k-1)\mu+1]\\
&\leq\frac{2(\beta-1)}{(1-\alpha)[k^2(k+1)\rho\mu+k^2\rho-k^2\mu+k]}\left(1+\frac{2(\beta-1)}{(k-1)(1-\alpha)}\right)\\
&\quad\prod_{j=1}^{k-2}\left[1+\frac{2(\beta-1)(1+j\rho-j\mu+j(j+1)\rho\mu)}{(1-\alpha)j(1+j\rho-j\mu+j(j+1)\rho\mu)}\right]\\
&\leq\frac{2(\beta-1)}{(1-\alpha)[k^2(k+1)\rho\mu+k^2\rho-k^2\mu+k]}\\
&\quad\prod_{j=1}^{k-1}\left[1+\frac{2(\beta-1)(1+j\rho-j\mu+j(j+1)\rho\mu)}{(1-\alpha)j(1+j\rho-j\mu+j(j+1)\rho\mu)}\right]
\end{aligned}$$

This completes the proof of the theorem. □

3. Application of Jack's Lemma for the Class $\mathcal{MD}(\rho, \mu, \alpha, \beta)$

In this section, application of Jack's lemma for f(z) belonging to the class $\mathcal{MD}(\rho, \mu, \alpha, \beta)$ is discussed. Next lemma was given by Jack [1].

Lemma 3.1.
Let the function w(z) be analytic in Δ with w(0) = 0. If

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|,$$

then

$$z_0w'(z_0)=kw(z_0),$$

where k is a real number and $k \geq 1$.
Applying the above lemma, we derive,

Theorem 3.1.
If $f(z) \in \mathcal{MD}(\rho, \mu, \alpha, \beta)$, then

$$\left|\left[\frac{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}{z}\right]^{\frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}}-1\right|<1+\delta \quad (\delta\geq 0)$$

for some α (α ≤ 0) and β (β > 1), or

$$\left|\left[\frac{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}{z}\right]^{\frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}}-1\right|<1+\delta \quad (\delta\geq 0)$$

for some α (α ≤ −1) and β (β > 1).

Proof.
Let us define

$$\gamma=\frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}>0$$

for α ≤ 0 and β > 1, and

$$\gamma=\frac{(1+\delta)(1+\alpha)}{(2+\delta)(\beta-1)}<0$$

for α ≤ −1 and β > 1.
Further, let the function w(z) be defined by

$$w(z)=\frac{\left[\frac{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}{z}\right]^{\gamma}-1}{(1+\delta)}$$

which is equivalent to

$$\frac{\rho\mu z^3 f'''(z)+(2\rho\mu+\rho-\mu)z^2 f''(z)+zf'(z)}{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}=\frac{zw'(z)(1+\delta)}{\gamma[w(z)(1+\delta)+1]}$$

Then we see that w(z) is analytic in Δ and w(0) = 0. On the other hand, if f(z) ∈ $\mathcal{MD}$(ρ, μ, α, β) (α ≤ 0, β > 1), then

$$\operatorname{Re}\left[\frac{\rho\mu z^3 f'''(z)+(2\rho\mu+\rho-\mu)z^2 f''(z)+zf'(z)}{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}\right]$$
$$-\alpha\left|\frac{\rho\mu z^3 f'''(z)+(2\rho\mu+\rho-\mu)z^2 f''(z)+zf'(z)}{\rho\mu z^2 f''(z)+(\rho-\mu)zf'(z)+(1-\rho+\mu)f(z)}-1\right|$$
$$=1+\frac{1}{\gamma}\operatorname{Re}\left[\frac{(1+\delta)zw'(z)}{(1+\delta)w(z)+1}\right]-\frac{\alpha}{|\gamma|}\left|\frac{(1+\delta)zw'(z)}{(1+\delta)w(z)+1}\right|<\beta$$

Furthermore, if there is a point z_0, ($z_0 \in \Delta$) which satisfies

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1,$$

then Lemma 3.1 gives us that

$$1+\frac{1}{\gamma}\mathrm{Re}\left[\frac{(1+\delta)z_0w'(z_0)}{(1+\delta)w(z_0)+1}\right]-\frac{\alpha}{|\gamma|}\left|\frac{(1+\delta)z_0w'(z_0)}{(1+\delta)w(z_0)+1}\right|$$

$$=1+\frac{k(1+\delta)}{\gamma}\mathrm{Re}\left[\frac{1}{(1+\delta)+e^{-i\theta}}\right]-\frac{\alpha k(1+\delta)}{|\gamma|}\left|\frac{1}{(1+\delta)+e^{-i\theta}}\right|$$

$$=1+\frac{k(1+\delta)}{\gamma}\left[\frac{(1+\delta)+\cos\theta-\alpha\sqrt{(1+\delta)^2+2(1+\delta)\cos\theta+1}}{(1+\delta)^2+2(1+\delta)\cos\theta+1}\right]$$

$$=F(\theta)$$

when $\gamma > 0$

$$F(\theta)\geq 1+\frac{k(1+\delta)(1-\alpha)}{\gamma(2+\delta)}$$

$$\geq 1+\frac{(1+\delta)(1-\alpha)}{\gamma(2+\delta)}=\beta,$$

because

$$\gamma=\frac{(1+\delta)(1-\alpha)}{(2+\delta)(\beta-1)}.$$

Further, when $\gamma < 0$,

$$F(\theta)\geq 1+\frac{k(1+\delta)(1+\alpha)}{\gamma(2+\delta)}$$

$$\geq 1+\frac{(1+\delta)(1+\alpha)}{\gamma(2+\delta)}=\beta.$$

This contradicts our condition of the theorem. Thus there is no $z_0 \in \Delta$ such that $|w(z_0)| = 1$. This completes the proof of the theorem. □

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