Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 5 (2015), pp. 2657-2669

http://www.ripublication.com

Analytical Solution Of Symmetrical, Multi Point Fermat-Weber Facility Location Problems By Resolving The Weights And Reducing The Number Of Demand Points

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Abstract

Logistics management is that part of supply chain management that deals with forward and reverse flow and storage of goods/ services between the origin point and the demand points. Fermat-Weber problem forms the basics of a specific area in this that optimizes the location of the supply point(s) in economic terms. This gained popularity last century and still attracts a large number of researchers. Exact solutions are available for three point problems. For more than three points, many iterative algorithms have been proposed over the years. This paper analyzes a few symmetrical multi point problems and solves analytically.

Keywords: Logistics; Facility Location; Fermat-Weber Problem; Optimal Point

1. Introduction

The facility location problem, also known as location analysis problem, is a branch of operations research and computational geometry concerned with the optimal placement of facilities to minimize the transportation costs while considering other constraints also. The Fermat problem only deals with equal attractive forces. It was first formulated, and solved geometrically in the triangle case, by Simpson in (1750). It was later popularized by Alfred Weber (1909) and nowadays popularly known as the Fermat-Weber problem.

A facility location problem in its simplest form is the Fermat-Weber problem, in which a single facility is to be located with an objective of minimizing the weighted sum of distances from a given set of demand points. The complex level increases with the increase in the number of demand points and imposition of constraints on the facilities.

For a three point problem geometrical solutions have been available since the introduction. The direct numerical solution was proposed by Tellier (1972) for a triangular case which can be applied for both the Fermat and Weber triangular problems involving only attractive forces (positive weights). For a triangular case the problem of repulsion and attraction was formulated and solved by the Canadian researcher Tellier in 1985.

The facility location problem is NP-hard to solve optimally for more than three demand points and hence, a number of approximation algorithms have been developed over the years.

The objective function can be written as: f(x, y) = $\sum_{i=1}^{n} wi.di(x,y)$; where,

n – number of demand points
wi – weight of i th demand point
di(x, y) – distance of i th point from the optimal point

Though a few approximate algorithms are effective, when using the iterative procedures, it is possible that it may take millions of iterations to arrive the solution point or may get stuck in a non-optimal point also. Katz (1969 and 1974) suggested to first check each demand point for optimality before proceeding with the iterations.

Barycentric coordinates solution generally used in surveying is known to researchers for many years. Gruulich (1999) in his paper analyzed the Fermat-Weber attraction only problem and presented two simple formulae to find the coordinates of the solution point. One simple geometrical solution was proposed by Nguyen Minh Ha and Bui Viet Loc (2011) for the attractions-repulsion problem.

Chen (2011) analyzed a few attraction only problems analytically and showed that for a few types of symmetrical problems, non-iterative solutions are very fast and give direct solutions.

This concept has many practical applications in various fields. Dhinesh Babu and Venkata Krishna (2013) applied the facility location techniques in operations management for evaluating the available potential locations and zero down to the best optimal facility location, which can be used to minimize the cost associated with setting up of data centre facilities.

A stochastic bi-objective model was developed by Arousha Haghighian Roudsari; Kuan Yew Wong (2014) for point and area destinations with the purpose of finding a single new location for a chain supermarket. They used a reduced gradient solution procedure as an algorithm for solving the model with the help of the MATLAB software.

Heuristics and metaheuristics also find their places in the solution literature. For example, Pradeepmon and Brijesh Paul (2012) proposed one simulated annealing procedure which is a hybrid algorithm for solving the uncapacitated facility location (UFL) problems. The results obtained were compared with some recent algorithms available in the literature.

The principle of basic mechanics has been used by many researchers for solving smaller size problems. In this paper, a few problems are analyzed and solved by resolving the weights (forces) and reducing the number of demand points.

2. Review of basics of resolution of forces applied to the Fermat-Weber problem

It may be recalled that the static equilibrium conditions include, $\sum$ Fv = 0 and $\sum$ Fh = 0. The three points Fermat problems can be conveniently solved by resolving the forces horizontally and vertically (eg. Chen, 2011). In this section, the concept is used to solve two, four demand point problems and the four point theorem is verified. The theorem states that, "the sum of the Euclidean distances to four fixed points in the plane is minimized at the point of intersection of the diagonals, when the fixed points form a convex quadrangle. Otherwise, the fixed point which belongs to the (closed) triangle formed by the three other fixed points". To start with, a three collinear Fermat problem is considered and P is the optimal point (Figure 1). All the three attractive forces are assumed to be of same intensity and equal to one.

The problem is to find a point P so that the sum of the distances from P to A, B and C is to be minimized. If the optimal point P lies in between B and C (Figure 1 (a)), the forces towards B and C get cancelled and the force acting towards the point A remains active affecting the equilibrium. Similarly, if P lies in between the demand points A and B; the force towards C will be still active (Figure 1 (b)). If P happens to coincide with either A or C; then the attractive forces towards other two points will be active without equilibrium. Hence, the P should coincide with B only, giving the solution.

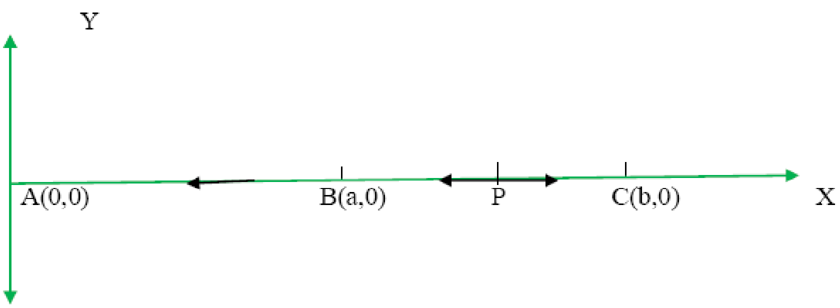

Figure 1(a), Three Point Collinear Fermat Problem – P assumed to be between B and C

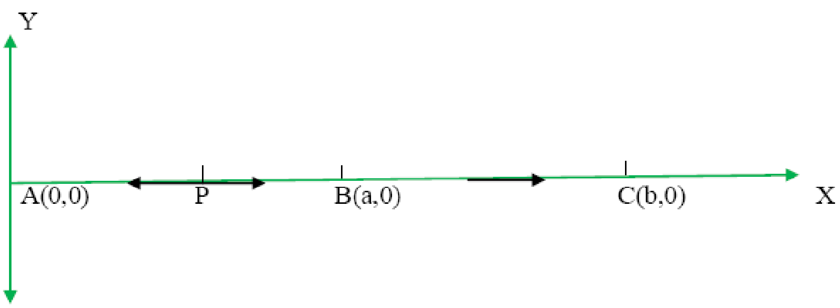

Figure 1(b), Three Point Collinear Fermat Problem – P assumed to be between A and B

Let us calculate the weights for the problem shown in Figure 2. It is assumed to be in equilibrium. Resolving the weights in vertical and horizontal directions we get,

$$w2.\text{Sin } \theta = w4.\text{Sin } \theta \qquad (1)$$

$$w3+w2.\text{Cos } \theta = w1+w4.\text{Cos } \theta \qquad (2)$$

It may be noted that, all the weights need not be equal for this Fermat-Weber problem. But, w2 = w4 and w1 = w3 will satisfy the equilibrium conditions. In a specific case; if w1=w2=w3=w4=1, this becomes a pure Fermat problem and the first part of the theorem is proved for the four point convex quadrangle Fermat problem.
For the convex quadrangle problem shown in Figure 3, let us assume that the optimal point P coincides with C. The weight w4 becomes inactive and the problem may be considered as a three point problem now.

$$w1 = w2.\text{Cos } \theta1+w3.\text{Cos } \theta2 \qquad (3)$$

$$w2.\text{Sin } \theta1 = w3.\text{Sin } \theta2 \qquad (4)$$

In this case also, all weights need not be equal, only the conditions are to be satisfied. If all the weights are equal and taken as 1 each,

equation (4) becomes: $\text{Sin } \theta1 = \text{Sin } \theta2 \Rightarrow \theta1 = \theta2 = \theta$ (5)

and equation (3) becomes: $2.\text{Cos } \theta = 1$ (6)

This gives the angle $\theta=60^o$. This is the solution for a three point Fermat problem. This also proves the second part of the theorem for the four point concave quadrangle Fermat problem.

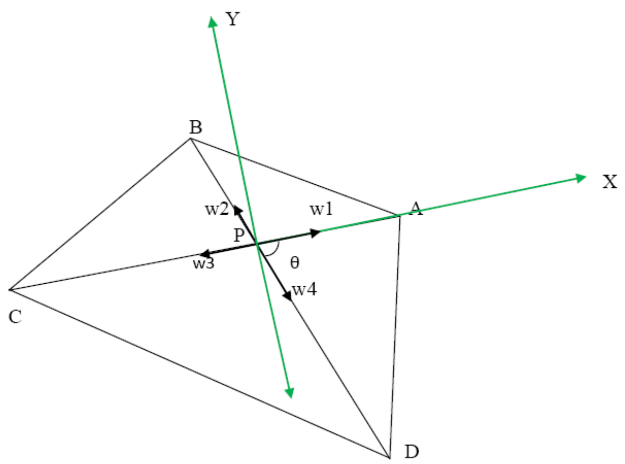

Figure 2, Four Point Fermat-Weber Problem - a Convex Quadrangle

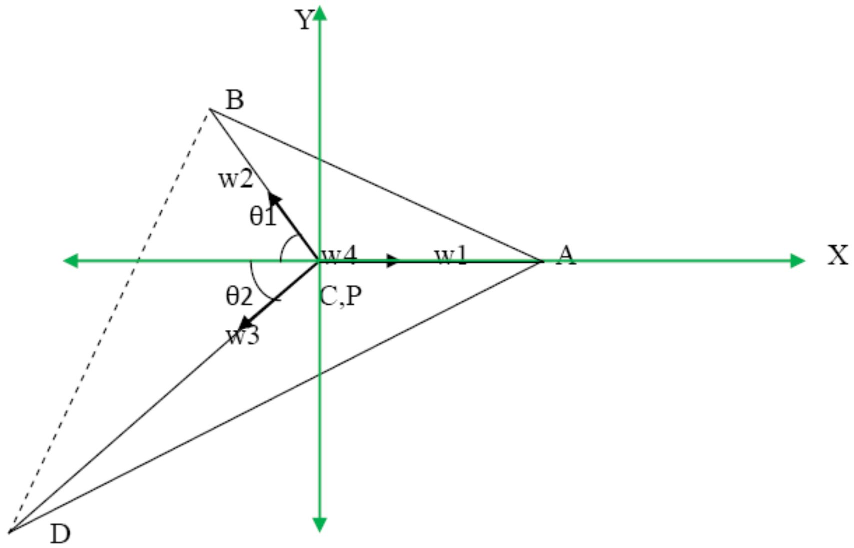

Figure 3, Four Point Fermat-Weber Problem - a Concave Quadrangle

3. Converting a multi point problem into a three point problem

In this section, two, four point problems and one five point problem proposed by earlier researchers have been studied and solved after converting them into equivalent three point problems.

Figure 4 (a) is the original problem given by Kuhn (1967) to demonstrate the possibility that the Weiszfeld method does not converge at all to the optimal solution. A1, A2, A3 and A4 are the demand points with the weights shown in the figure. It is symmetrical about the X-axis. Kuhn pointed out that the optimal solution is at the origin, (0, 0). The problem can be conveniently converted into a three demand point problem as given in Figure 4(b).

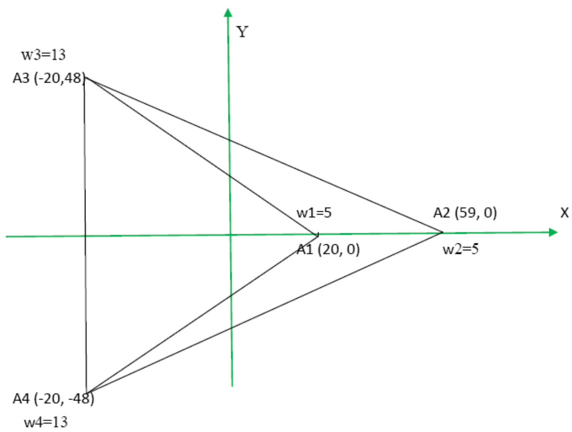

Figure 4 (a), Four Point problem proposed by Kuhn

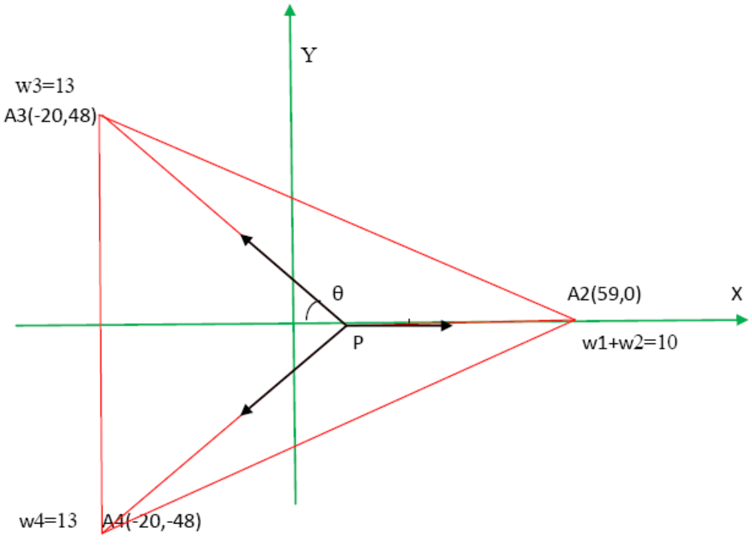

Figure 4(b), Kuhn's problem converted to a Three Demand Point problem

Since this is a symmetrical problem about the X axis, the solution point should lie on the X axis only. As all the angles of the covering triangle are acute and the forces are only attractive, the solution point shall be within the triangle. Also, the optimal point P cannot be in between A1 and A2 as w1 and w2 will balance each other and w3 and w4 will pull P towards left. Hence, the point P can only lie between the points (-20, 0) and (20, 0). Once it is known about P's approximate location, the forces w1 and w2 can be combined together and assumed to be acting at the outer demand point A2. Now, resolving the forces:

$$(2 \times 13).\cos\theta = 10 \text{ (or) } \cos\theta = (5/13) \quad (7)$$

Using trigonometric principles the triangle is completed (Figure 5(a)) and similar triangle principle the X coordinate of the point P (Figure 5(b)) can easily be found out. That is, the solution point P coincides with the origin (0, 0).

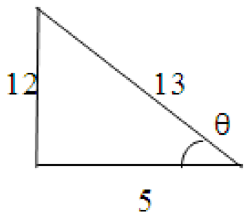

Figure 5(a), Cos θ

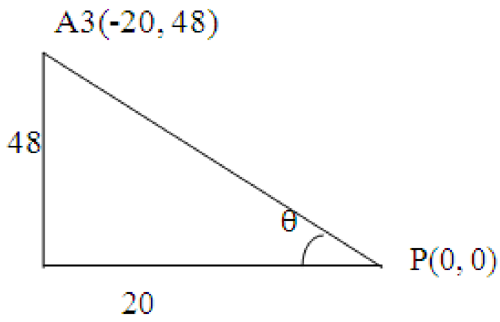

Figure 5(b), X coordinate of P

Another example discussed by Chen (2011) is analyzed here. There are four demand points and the coordinates and weights taken are shown in Figure 6(a). The coordinates are symmetrical and the forces at A1 and A2 are equal and hence the solution point P lies on the X axis. Without loss of generality, it is assumed as w4 > w3 and the P is between (0, 0) and (a, 0). To convert this into a three point problem, A4 is taken with a weight of (w4-w3) ignoring the demand point A3.
Resolving the forces, the vertical components will cancel each other and,

$$2w.\cos\theta = (w4-w3) \text{ (or) } \cos\theta = (w4-w3)/2w \quad (8)$$

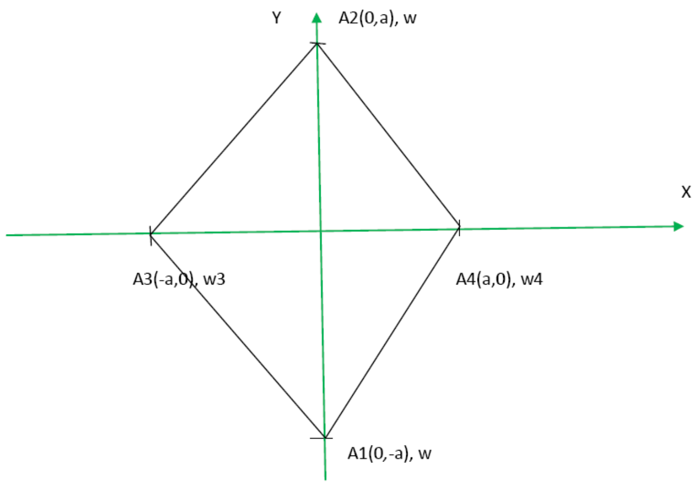

Figure 6(a), Chen's Four point problem

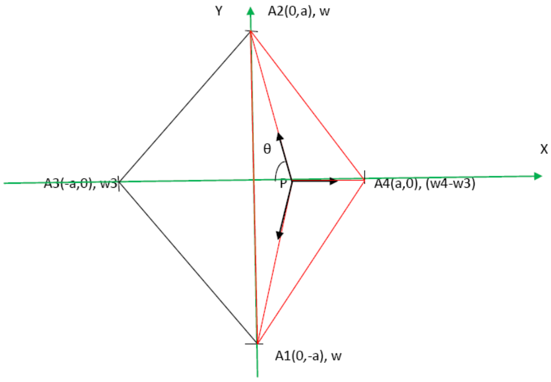

Figure 6(b), Chen's Four point problem converted to a Three Point problem

$y=\sqrt{[4w^2-(w4-w3)^2]}$

Using similar triangle principle:

$x/(w4-w3) = a/\sqrt{[4w^2-(w4-w3)^2]}$ (or) $x = a.(w4-w3)/\sqrt{[4w^2-(w4-w3)^2]}$ (9)

Using this simple equation, a few cases could be analyzed.
If w4=w3 then, x=0. That is, P coincides with the origin (0, 0).
If (w4-w3) is negative then, x becomes negative.

The condition for P to coincide with A4:
x=a and $(w4-w3) = \sqrt{[4w^2-(w4-w3)^2]}$. Squaring both the sides, we get,

$(w4-w3)^2 = [4w^2-(w4-w3)^2]$ (or) $4w^2 = 2(w4-w3)^2$ (or) $w = (w4-w3)/\sqrt{2}$ (10)

Using equation (9), if the weights and 'a' are known, the x coordinate of P can be computed.

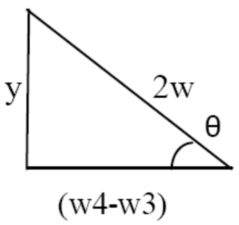

Figure 7(a), Cos θ

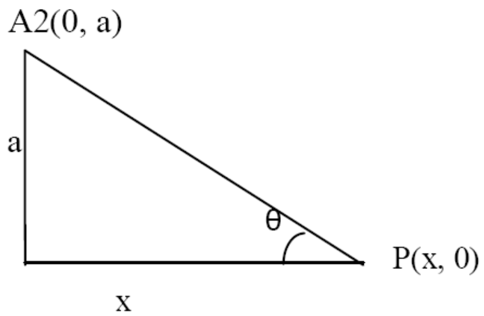

Figure 7(b), X coordinate of P

4. Five points Problem

Figure 8 represents a problem proposed and analyzed by Drenzer (1992, 1995) which is a difficult one to solve using Weiszfeld algorithm (1937). It is a symmetrical problem about the diagonal. It consists of five demand points, four located at the four corners of the unit square with coordinates (0, 0), (0, 1), (1, 0) and (1, 1) and a weight of 1 each. The fifth demand point is at (100, 100) with a weight of 4. It is assumed that the solution point P lies in between (1, 1) and (100, 100). This can be converted into a three point problem in two ways.

(I) The force towards the demand point (100, 100) is taken as it is (force = 4) and the unit forces towards the demand points (0, 0) and (1, 1) are resolved and combined with the unit forces towards the points (0, 1) and (1, 0).This results in (1+Sec θ) each (Figure 8(c))

(II) The unit forces towards the demand points (0, 0) and (1, 1) can be subtracted from the force acting towards the point (100, 100). This results in the forces of 2 and 1 each acting towards (100, 100), (0, 1) and (1, 0) respectively. This also clearly shows that (Figure 8(c), Right hand side) θ can only be zero for the given weights. If the solution point falls below (1, 1) this conversion requires a lot of care and needs modification.

For the purpose of analysis, only the first case is considered.
By resolving the forces we get, $2.(1+\sec\theta).\cos\theta = 4$ (or) $2.(1+\cos\theta) = 4$ (11)
This gives, $\theta = 0$, means that, the solution point will be pulled towards (100, 100) and settle there.
Also, from Figure 8(b):
$\tan\alpha = (\sqrt{2}/2)/[(100.\sqrt{2})-(\sqrt{2}/2)]$ (or) $\alpha = 0.287916066^{o}$ (12)
$\tan\theta = \sqrt{[(0-0.5)^2+(1-0.5)^2]} / \sqrt{[(a-0.5)^2+(a-0.5)^2]}$ (or) $\tan\theta = 1/(2.a-1)$ (13)
For computing the weight to have P at (100, 100); $\theta \Rightarrow \alpha = 0.287916066^{o}$
Equation (11) becomes, $2.(1+\cos\alpha) = W$ (or) $W = 3.999974749$ (14)
That is, a slight reduction in the weight changes the entire equilibrium condition.
To conclude: $\tan\theta = 1/(2a-1)$ and $2.(1+\cos\theta) = w$ are the required equations.
If $\theta = 0^{o}$; $a = \infty$ and $w=4$
If $\theta = \alpha^{o}$; $a = 100$ and $w=3.999974749$; are a few results obtained using equations (13) and (14).
If $\theta \Rightarrow 90^{o}$ then, $a \Rightarrow 0.5$. That is, P moves towards the point (0.5, 0.5).). Since the point is below (1, 1), this contradicts the assumption. If the solution point falls below (1, 1) in the line of the larger diagonal, the unit forces at points (0, 1) and (1, 1) will cancel each other and the weight towards (100, 100) also should be equal to zero.

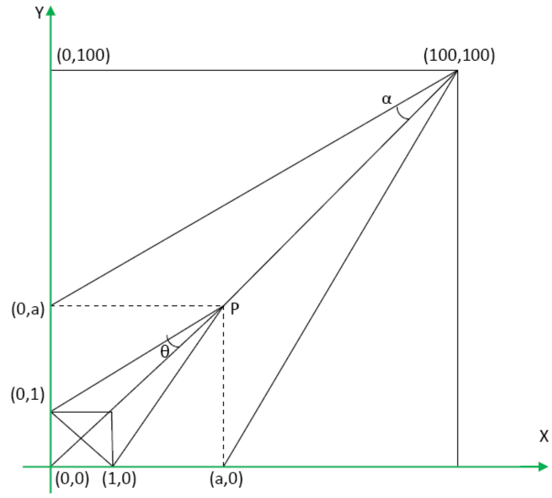

Figure 8(a), Drezner's Five Point problem

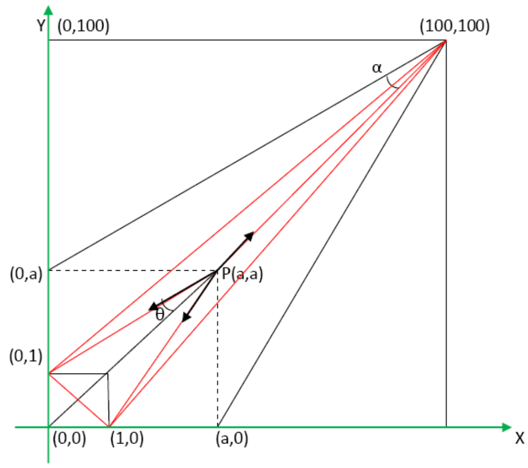

Figure 8(b), Drezner's Five Point problem converted to a Three Point Problem

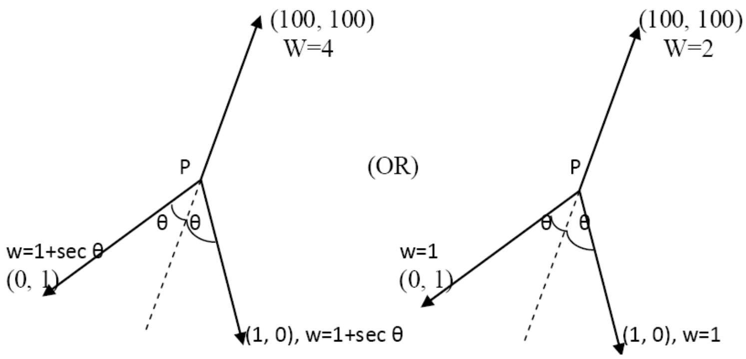

Figure 8(c), Two Possibilities of Forces in Drezner's Five Point problem

5. Solution for a Repulsion and Attraction Problem

Nguyen Minh Ha and Bui Viet Loc (2011) discussed an easy geometrical solution to the one repulsion force and two attraction forces triangular problem. The points A, B and C are the three demand points with known weights –w1, w2 and w3 respectively. Let a, b and c are the lengths of BC, CA and AB of a triangle ABC respectively. Without loss of generality, they assumed that the repulsion force acting away from A as 'a' and the objective function was modified as, f (P) = -a. PA + w2.PB + w3.PC. The constants w2 and w3 represent the attractive forces (weights) towards the demand points B and C respectively, and P being the optimal point. The triangle BOC was constructed inside the triangle ABC such that CO represents the weight w2 and BI the weight w3.

One circle was drawn through the points B, O and C as shown in figure 10. The point P in the circumference of the circle, obtained by extending the line joining the points A and O is the optimal point. This was geometrically proved by them. The repulsion force (negative weight) pertaining to point A pulls the solution point away from it as shown. It can be taken as the positive weight w1 away from P.

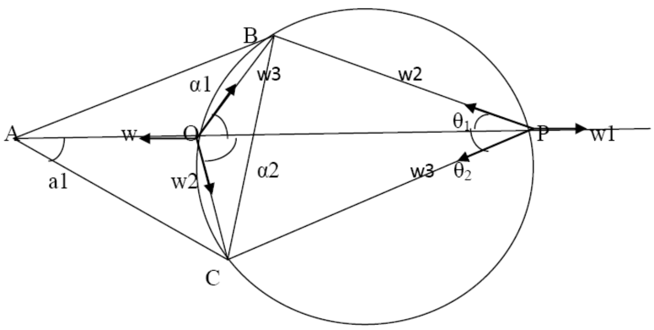

Figure 10 Geometric solution for the Repulsion problem

Resolving the forces (weights) at P horizontally along AP and vertically, perpendicular to AP we get:

$w2.\ \sin\theta1 = w3.\ \sin\theta2$ (15)

$w1 = w2.\ \cos\theta1 + w3.\ \cos\theta2$ (16)

Equations (15) and (16) can be solved to get θ1 and θ2.

$\sin\theta2 = (w2/w3).\ \sin\theta1$;

$\cos\theta1 = \sqrt{(1-\sin^2\theta1)}$; $\cos\theta2 = \sqrt{(1-\sin^2\theta2)}$ and hence,

$w1 = w2.\ \sqrt{(1-\sin^2\theta1)} + w3.\ \sqrt{(1-\sin^2\theta2)}$

$w1 = w2.\ \sqrt{(1-\sin^2\theta1)} + w3.\ \sqrt{[1-(w2/w3)^2.\ \sin^2\theta1)]}$;

If $\sin^2\theta1 = x$ then, $w1 = w2.\ \sqrt{(1-x)} + w3.\ \sqrt{[1-(w2/w3)^2.x)]}$ (17)

The only unknown is $\theta 1$ which can be computed by solving the expression using an ordinary scientific calculator. Subsequently the angle $\theta 2$ can be easily found using (15).
Since the points A, B and C are known, the apex angles at the corresponding vertices can be computed using any method. This follows:
$\tan \angle a1 = \frac{\operatorname{Sin}(C-\theta 1)}{(k3-\operatorname{Cos}(C-\theta 1))}$; where, $k3 = \frac{\operatorname{Sin} B . \operatorname{Sin}(\theta 1+\theta 2)}{\operatorname{Sin} A . \operatorname{Sin} \theta 2}$
Subsequently, any other angle can be found out.
As BOCP is a cyclic quadrangle, $\alpha 1 + \alpha 2 + \theta 1 + \theta 2 = 180^o$.
In Δ BCP: $(w2/w3) = (\sin \alpha 1/ \operatorname{Sin} \alpha 2)$
In Δ BOC: $\angle OCB = \theta 1$ and $OBC = \theta 2$; $(w2/w3) = (\sin \theta 2/ \operatorname{Sin} \theta 1)$

Condition for the point O to be inside the ΔABC: $\alpha 1 < 90^o$ and $\alpha 2 < 90^o$.
Then, $\theta 1, \theta 2 \Rightarrow 0^o$ and P will be at ∞. This happens when $w1 >> w2/ w3$. That is the repulsion effect is very high.
Suppose if $\alpha 1 = \alpha 2 = 30^o$ and $w2=w3=1$ then, $\theta 1 = \theta 2 = 60^o$

Conditions for only attraction forces are considered and the supply point to be at O:
$w2=w3=1$, $\alpha 1 = \alpha 2 = 30^o$ and $w2=w3=1$.
$w=2.\operatorname{Cos} 30$ (or) $w = 1.732$.
If $w = 0$, $\alpha 1 = \alpha 2 = 90^o$ and the solution point lies on the side BC.

6. Conclusion

For the Fermat-Weber facility location problems with three demand points, exact and direct solution methods are available. For more than three points, iterative procedures have been generally used which are time consuming. Sometimes, even after taking thousands of iterations and consuming considerable time, the solution point may get stuck in a non-optimal point. It has been shown with a few examples; it is not required to straightaway go for iterative procedures for obtaining the solution. Using the concept of resolution of forces, in some cases, it is possible to reduce the number of demand points. For symmetrical problems, this concept is more effective and exact solutions can be arrived in a relatively short time.
Chen (2011) used the concept of resolution of forces to solve a few symmetrical problems without reducing the number of demand points. In this paper, the concept of eliminating demand point(s) has been tried in conjunction with resolution of forces. Eliminating the demand points simplifies the solution further. This is evident from the Drenzer's problem analyzed in section 4. When two points are eliminated (Figure 8(c)), the solution is readily obtained. It has been shown that, if Weiszfeld algorithm is used, it is difficult to solve this problem.
Finally, the simplest problem with one repulsion and two attraction forces has also been analyzed using this concept. The same concept is being tried in a number of other problems also including non-symmetrical ones.

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