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Union, Intersection, Product and Direct Product of Prime Ideals

Bindu.P(1), Dr.Sarala.Y(2), Dr.Madhusudhana Rao.D(3), Dr.Anjaneyulu.A(4)
[1]Research Scholar, Department of Mathematics, K.L.University, A.P, India.
[2]Faculty of Mathematics, K.L.University, A.P, India.
[3]Faculty of Mathematics, VSR & NVR College, Tenali, A.P, India.
[4]Retired HOD, Department of Mathematics, VSR & NVR College, Tenali, A.P, India.
saralayella1970@gmail.com

Abstact

The main goal of this paper is to initiate the notion of product and direct product of prime ideals. We develop here some properties of prime ideals. It is shown that if a product of a finite set of prime right ideals of a ternary semigroup *T* is prime, then the product ideal contains at least one of the given ideals. It is also shown that the intersection of a set of prime right ideals of a ternary semigroup *T* is a prime right ideal of *T* if and only if it is a prime right ideal of the union of the given ideals.

Key words : Ideal, Prime ideal and Direct product.

Introduction

Prime ideals of ternary semigroups play a very important role in ternary semigroup. Several authors have worked on this important topic. These prime ideals was studied by Halen Bradely Grimble [3], Shabir [8] and Scwartz [7]. The notions of ternary semigroup is a natural generalization of ternary semigroup. The notion of ideal play very important role to study the algebraic structure. In Sioson [9] and Kar [5] studied ideal theory in ternary semigroups. In Lehmer [6] developed the theory of ternary semigroups. In this paper we study some interesting properties of prime ideals of ternary semigroups.

Definition 2.1

A left ideal P of a ternary semigroup T is said to be a prime if $a, b, c \in T$ and $abc \in P$ jointly imply that either $a \in P$ or $b \in P$ or $c \in P$. Clearly, a prime ideal might be defined equivalently as an ideal whose compliment is either empty (in case the ideal T itself, which is obviously prime) or a subternary semigroup of T.

It is also clear that a left zero element of T is a prime right ideal of T if and only if it has no proper divisors.

Theorem 2.2
The union of an arbitrary collection of prime right ideals of a ternary semigroup T is a prime right ideal of T.

Proof
Let R_i be a collection of prime right ideals of T, where i ranges over an index set M of arbitrary cardinality. Then if $a,b,c \in T$ and $abc \in \bigcup_{i \in M} P_i$ we must have $abc \in P_i$ for some $i \in M$. But P_i is prime, whence either $a \in P_i$ or $b \in P_i$ or $c \in P_i$. Therefore either $a \in \bigcup P_i$ or $b \in \bigcup P_i$ or $c \in \bigcup P_i$ whence $\bigcup P_i$ is prime.

Theorem 2.3
If a product of a finite set of prime right ideals of a ternary semigroup T is prime, then the product ideal contains at least one of the given ideals.

Proof
Let $R_1, R_2, ...R_n$ be a prime right ideals of T, and let $\prod_i^k R_i$ be an abbrevation for the product $R_k R_{k+1} .....R_n (1 \le k \le n)$ we wish to prove that if $\prod_{i=1}^n R_i$ is prime then $R_k \subseteq \prod_{i=1}^n R_i$ for some $k(1 \le k \le n)$. Suppose, to the contrary, that $R_k \not\subseteq \prod_{i=1}^n R_i$ for all k = 1,2,3....n. Then for each k there is an element $a_k \in R_k$ such that $a_k \notin \prod_{i=1}^n R_i$. Now $a_1 . \prod_{i=2}^n a_i = \prod_{i=1}^n a_i \in \prod_{i=1}^n R_i$, whence $\prod_{i=2}^n a_i \in \prod_{i=1}^n R_i$. Since $a_1 \notin \prod_{i=1}^n R_i$ and $\prod_{i=1}^n R_i$ is prime. Now if we make the inductive assumption that $a_k . \prod_{i=k+1}^n a_i = \prod_{i=k}^n a_i \in \prod_{i=1}^n R_i$, then since $a_k \notin \prod_{i=1}^n R_i$ we have $\prod_{i=k+1}^n a_i \in \prod_{i=1}^n R_i$. Hence by complete induction we conclude that $a_n \in \prod_{i=1}^n R_i$, contrary to our choice of a_n and the theorem is proved.

Theorem 2.4
If a product of prime right ideals of a ternary semigroup does not properly contain any of them, then the product is prime if and only if it is one of the given ideals.

Proof

The proof of theorem 2.3 holds mutatis mutandis for ideals, and for these we may drop the hypothesis concerning proper inclusion in theorem2.4. For, as we have already noted any product of ideals is contained in their intersection, so that the product cannot contain any one of them properly and the inclusion hypothesis in theorem 2.4 is satisfied automatically. Hence we have

Theorem 2.5

A product of prime ideals of a ternary semigroup is prime if and only if it is one of the given ideals.

Corollary 2.6

If a product of prime ideals of a ternary semigroup is prime, then the product of the ideals is just their intersection.

Proof

The converse of this corollary fails, however, because the intersection of the ideals may be a proper subset of each of them.

The intersection of prime right ideals need not be a prime. However, the question whether the nonempty intersection of a set of prime right ideals of a ternary semigroup *T* is prime in *T* may be reduced to the question whether the intersection is prime in the union of the given ideals.

Theorem 2.7

The intersection of a set of prime right ideals of a ternary semigroup *T* is a prime right ideal of *T* if and only if it is a prime right ideal of the union of the given ideals.

Proof

The necessity being obvious, we proceed to prove the sufficiency. Let $[R_i]$ be a set of prime right ideals of *T*; where *i* ranges over an arbitrary index set *M*. By hypothesis, $\bigcap_{i\in M} R_i$ is a prime right ideal of $\bigcup_{i\in M} R_i$ and hence is non-empty. Let $x, y, z \in T$; $xyz \in \bigcap_{i\in M} R_i$. Then if $x \notin \bigcap_{i\in M} R_i$ there is some $R_\lambda(\lambda \in M)$ such that $x \notin R_\lambda$; if $y \notin \bigcap_{L\in M} R_L$ then there is some $R_\mu(\mu \in M)$ such that $y \notin R_\mu$; and if $z \notin \bigcap_{j\in M} R_j$ then there is some $R_\xi(\xi \in M)$ such that $z \notin R_\xi$. Hence $x, y, z \notin R_\lambda \cap R_\mu \cap R_\xi$. But if $xyz \in \bigcap_{i\in M} R_i \subseteq R_\lambda \cap R_\mu \cap R_\xi$ and R_λ, R_μ, R_ξ are prime, whence either $x \in R_\lambda$ or $y \in R_\lambda$ or $z \in R_\lambda$ either $x \in R_\mu$ or $y \in R_\mu$ or $z \in R_\mu$ and either $x \in R_\xi$ or $y \in R_\xi$ or $z \in R_\xi$. Therefore we have either $x \in R_\lambda$, $y \in R_\mu$ and $z \in R_\xi$ or else $y \in R_\lambda$, $x \in R_\mu$, $z \in R_\xi$. In either case, $x, y, z \in R_\lambda \cup R_\mu \cup R_\xi \subseteq \bigcup_{L\in M} R_L$. But by hypothesis, $\bigcap_{L\in M} R_L$ is prime in

$\bigcup_{L \in M} R_L$, whence either $x \in \bigcap_{i \in M} R_i$ or $y \in \bigcap_{L \in M} R_L$ or $z \in \bigcap_{j \in M} R_j$. Hence $\bigcap_{L \in M} R_L$ is prime ideal of *T*.

Clearly the foregoing theorem is equally valid for ideals, but for finite sets of these we have the following more specific result.

Theorem 2.8
The intersection of a finite set of prime ideals of a ternary semigroup *T* is prime if and only if the intersection is one of the given ideals.

Proof
The sufficiency is trivial. To prove the necessity, let $A_1, A_2, \ldots, A_n$ be a prime ideals of T , and recall that their intersection must be non-empty. If $\bigcap_{i=1}^{n} A_i \neq A_k$ for all $k = 1, 2, \ldots, n$, then for each k there is an element $a_k \in A_k$ such that $a_k \notin \bigcap_{i=1}^{n} A_i$. Now, exactly as in the proof of theorem 2.3, we may prove inductively that if $\prod_{i=1}^{n} a_i \in \bigcap_{i=1}^{n} A_i$ then $a_n \in \bigcap_{i=1}^{n} A_i$, contrary to the choice of a_n , and hence conclude that $\prod_{i=1}^{n} a_i \notin \bigcap_{i=1}^{n} A_i$. But $\prod_{i=1}^{n} a_i \in \prod_{i=1}^{n} A_i \subseteq \bigcap_{i=1}^{n} A_i$. Hence the supposition that $\bigcap_{i=1}^{n} A_i \neq A_k$ for all $k = 1, 2, \ldots, n$ is contradicted, and our theorem is proved.

Definition 2.9
The direct product of three ternary semigroups *R, S* and *T* is defined to be a system whose elements are all the ordered pairs (r, s, t) with $r \in R$, $s \in S$ and $t \in T$ and with multiplication defined by $(r_1, s_1, t_1)\,(r_2, s_2, t_2)\,(r_3, s_3, t_3) = (r_1r_2r_3, s_1s_2s_3, t_1t_2t_3)$. The definition may be extended readily to the direct product of any finite set of ternary semigroups.

It is well known and easily proved that a direct product of ternary semigroups is a ternary semigroup.

Our purpose in introducing direct multiplication at this point is simply to point out that the direct product of prime ideals of three ternary semigroups need not be a prime in the direct product of the ternary semigroups.

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