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Generalization of Matkowski Contraction Principle in Fuzzy Metric Space

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Abstract

With a view to generalizing the Banach Contraction Principle, Matkowski extended the concept of Banach contraction to a system of equations on a finite product of metric spaces and obtain a fixed point theorem for such system of transformations. Many authors Singh, Gairola, Arora etc is used the results of Matkowski and proved some fixed point theorems. The concept of Matkowski type maps on product of fuzzy metric space and obtain a fixed point theorem for such system of transformation is introduced by Arora and Kumar. The main purpose of this paper is to obtain a common fixed point theorem for two systems of Matkowski type maps on fuzzy metric space.

Keyword: Fuzzy metric space, fixed point, Cauchy sequence

Introduction

Zadeh [5] introduced the concept of fuzzy sets in 1965 and in the next decade Kramosil and Michalek [2] introduced the concept of fuzzy metric space in 1975, which opened an avenue for further development of analysis in such spaces. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors viz [1], [6], [10] and others. The concepts of fuzzy metric spaces have been introduced in different ways by many authors. With a view to generalizing the Banach Contraction Principle, Matkowski extended the concept of Banach contraction to a system of equations on a finite product of metric spaces and obtain a fixed point theorem for such system of transformations [3], [4], [7], [11], [12] etc. Arora and Kumar [9] introduced the concept of Matkowski type maps on product of fuzzy metric space and obtain a fixed point theorem for such system of transformation. Matkowski type fixed point theorems are applicable in solving abstract equation on product spaces to convex solution of a system of functional equations and other abstract equation.

In this paper we obtained a common fixed point theorem for two systems of Matkowski type's transformation on product of fuzzy metric space.

Preliminaries

Definition 2.1
(Schweizer and Sklar [14]**)** A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $*$ satisfies the following conditions

[B.1] $*$ is commutative and associative

[B.2] $*$ is continuous

[B.3] $a*1=a \ \ \forall\ a\in[0,1]$

[B.4] $a*b\le c*d$ whenever $a\le c, b\le d$ and $a,b,c,d\in[0,1]$.

Definition 2.2
(A. George and P. Veeramani [1]**)** The $3-$tuple $(X,M,*)$ is called a fuzzy metric space if X is an arbitrary non –empty set, $*$ is a continuous t-norm and M is a fuzzy metric in $X^2\times[0,\infty]\to[0,1]$, satisfying the following conditions: for all $x,y,z\in X$, and $t,s>0$.

[FM.1] $M(x,y,0)=0$

[FM.2] $M(x,y,t)=1\ \forall\ \ t>0$ if and only if $x=y$.

[FM.3] $M(x,y.t)=M(y,x,t)$

[FM.4] $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$

[FM.5] $M(x,y,\cdot):[0,\infty]\to[0,1]$, is left continuous

[FM.6] $\lim_{n\to\infty} M(x,y,t)=1.$

Definition 2.3
(A. George and P. Veeramani [1]**)** Let $(X,M,*)$ be a fuzzy metric space and let a sequence $\{x_n\}$ in X is said to be converge to $x\in X$ if $\lim_{n\to\infty} M(x_n,x,t)=1$, for each $t>0$.

Definition 2.4
(A. George and P. Veeramani [1]**)** A sequence $\{x_n\}$ in X is called Cauchy sequence if $\lim_{n\to\infty} M(x_n,x_{n+p},t)=1$, for each $t>0$, and $p=1,2,3,\cdots$

Definition 2.5

(A. George and P. Veeramani [1]**)** A fuzzy metric space $(X,M,*)$ is said to be complete if every Cauchy sequence in X is convergent in X.

A fuzzy metric space in which every Cauchy sequence is convergent is called complete. It is called compact if every sequence contains a convergent subsequence.

Definition 2.6

(A. George and P. Veeramani [1]**)** A self mapping $T: X \to X$ is called fuzzy contractive mapping if $M(Tx,Ty,t) > M(x,y,t)$ for each $x \neq y \in X$ and $t > 0$.

In all, we generally follow the following notation and definition introduced in Matkowski and Singh et al. [3], [4], [7], [11].

Let a_{ik} be non- negative numbers $i,k = 1,\cdots,n,$ and $c_{ik}^{(l)}$ be square matrices defined in the following recursive manner

$$c_{ik}^{(0)} = \begin{cases} a_{ik}, & i \neq k \\ 1 - a_{ik}, & i = k \end{cases} \qquad i,k = 1,\cdots,n, \tag{1.1}$$

And $c_{ik}^{(l)}$ are defined recursively by

$$c_{ik}^{(l+1)} = \begin{cases} c_{11}^{(l)} c_{i+1,k+1}^{(l)} - c_{i+1,1}^{(l)} c_{1,k+1}^{(l)} & , i = k \\ c_{11}^{(l)} c_{i+1,k+1}^{(l)} + c_{i+1,1}^{(l)} c_{1,k+1}^{(l)} & , i \neq k \end{cases} \tag{1.2}$$

$i,k = 1,\cdots,n-l-1,\ l = 0,1,\cdots,n-2.$ If $n = 1,$ we define $c_{11}^{(0)} = a_{11},$ evidently, $c_{ik}^{(l)}$ is a $(n-l)\times(n-l)$ square matrix.

Lemma1

Let $c_{ik}^{(0)} \geq 0$ *for* $i,k = 1,\cdots,n,\ n \geq 2$ the system of inequalities

$$\sum_{\substack{k=1 \\ k \neq i}}^{n} c_{ik}^{(0)} r_k \geq c_{ii}^{(0)} r_i,\ i = 1,\cdots,n. \tag{1.3}$$

has a positive solution $r_1,\ldots,r_n,$ if and only if

$$c_{ii}^{(l)} > 0,\ i = 1,\cdots,n-l,\ l = 0,\cdots,n-1. \tag{1.4}$$

Proof

Suppose that $n = 2$. Since in the case $c_{12}^{(0)} = c_{21}^{(0)} = 0$, the proof of lemma is trivial, we assume that $c_{12}^{(0)}$ and $c_{21}^{(0)}$ are numbers which are different from 0 and $\frac{c_{12}^{(0)}}{c_{11}^{(0)}} > \frac{c_{22}^{(0)}}{c_{21}^{(0)}}$.

Note that the last inequality is fulfilled if and only if there exist positive numbers r_1 and r_2 such that $\frac{c_{12}^{(0)}}{c_{11}^{(0)}} > \frac{r_1}{r_2} > \frac{c_{22}^{(0)}}{c_{21}^{(0)}}$.

$$\Rightarrow \quad c_{12}^{(0)} r_2 \geq c_{11}^{(0)} r_1, \quad c_{21}^{(0)} r_1 \geq c_{22}^{(0)} r_2,$$

We get (1.3) for $n = 2$. Thus the lemma is true for $n = 2$.

Now suppose that the lemma is valid for $n-1, n \geq 3,$ and consider system (1.3) which is formed of n inequalities. The first of these inequalities can be written in the form

$$\frac{1}{c_{11}^{(0)}} \sum_{k=2}^{n} c_{1k}^{(0)} r_k \geq r_1 \tag{1.5}$$

If positive numbers $r_1, \ldots, r_n$, satisfy inequalities (1.3) then

$$\frac{c_{i1}^{(0)}}{c_{11}^{(0)}} \sum_{k=2}^{n} c_{1k}^{(0)} r_k + \sum_{\substack{k=2 \\ k \neq i}}^{n} c_{ik}^{(0)} r_k \geq c_{ii}^{(0)} r_i, \qquad i = 2, \cdots, n. \tag{1.6}$$

$$\Rightarrow \sum_{\substack{k=2 \\ k \neq i}}^{n} (c_{11}^{(0)} c_{ik}^{(0)} + c_{i1}^{(0)} c_{ik}^{(0)}) r_k \geq (c_{11}^{(0)} c_{ii}^{(0)} - c_{i1}^{(0)} c_{1i}^{(0)}) r_i, \; i = 2, \ldots, n.$$

Using (1.2) we can write this system of inequalities in the form

$$\sum_{\substack{k=1 \\ k \neq i}}^{n-1} c_{ik}^{(1)} r_{k+1} \geq c_{ii}^{(1)} r_{i+1}, \tag{1.7}$$

$$\Rightarrow \sum_{\substack{k=1 \\ k \neq i}}^{n-1} c_{ik}^{(l)} r_{k+1} \geq c_{ii}^{(l)} r_{i+1}, \quad i = 1, \ldots, n-l, l = 1, \cdots, n-1.$$

Main Result

Theorem

Let $F_i, G_i : X_i \to X_i$, $i = 1, \ldots, n,$ be two sets of transformations. If there exist non-negative numbers b and a_{ik}, $i, k = 1, \ldots, n,$ such that

$$M_i(T_i(x_1, \cdots, x_n), G_i(\bar{x}_1, \cdots, \bar{x}_n), t) \geq \sum_{k=1}^{n} a_{ik} M_k(x_k, \bar{x}_k, t) + b[M_i(x_i, F_i(x_1, \cdots, x_n), t) + M_i(\bar{x}_i, G_i(\bar{x}_1, \cdots, \bar{x}_n), t)]$$
$$\forall x_k, \bar{x}_k \in X_k, i = 1, \cdots, n. \tag{1}$$

and the numbers $c_{ik}^{(0)}$ and $c_{ik}^{(l)}$ defined by (1.1) and (1.2) fulfils the conditions

$c_{ii}^{(l)} > 0, \; i = 1, \cdots, n-l, \; l = 0, 1, \cdots, n-1.$

Where $0 \leq 2b \leq 1 - \nu$ and where $\nu = \max_i (r_i^{-1} \sum_{k=1}^{n} a_{ik} r_k)$ (2)

Then the system of equations

$$M_i(F_i x_i, G_i x_i, t) \geq M_i(x_i, x_i, t), \; i = 1, \ldots, n. \tag{3}$$

$$\Rightarrow \quad F_i(x_1,\cdots,x_n)=x_i=G_i(x_1,\cdots,x_n)$$

have a unique common solution $x_1,\cdots,x_n,$ such that $x_i \in X_i,\ i=1,\cdots,n.$

Proof

First of all we note that in view of the homogeneity of the system of inequalities (1.3), ν defined in (2) exist and $0<\nu<1.$ From Lemma (1) and equation (2) we may choose a system of positive numbers $r_1,\cdots,r_n,$ such that

$$\sum_{k=1}^{n} a_{ik} r_k \geq \nu r_i,\ i=1,\cdots,n.$$

Pick $x_i^0 \in X_i$ and choose a sequence $\{x_i^m\} \in X_i,\ i=1,\ldots,n,$ such that

$$M_i(x_i^{2m+1},x_i^{2m+2},t)=M_i(F_i x_i^{2m},F_i x_i^{2m+1},t)\geq M_i(x_i^{2m},x_i^{2m+1},t)$$

And

$$M_i(x_i^{2m+2},x_i^{2m+3},t)=M_i(G_i x_i^{2m+1},G_i x_i^{2m+2},t)\geq M_i(x_i^{2m+1},x_i^{2m+2},t),\ m=0,1,\cdots \tag{4}$$

without loss of generality, we assume that

$$M_i(x_i^1,x_i^0,t)\geq r_i,\quad r_i\leq 1,\ i=1,\ldots,n.$$

By (2) and (4)

$$M_i(x_i^1,x_i^2,t)=M_i(F_i(x_1^0,\cdots,x_n^0),G_i(x_1^1,\cdots,x_n^1),t)$$

$$\geq \sum_{k=1}^{n} a_{ik} M_k(x_k^0,x_k^1,t)+b[M_i(x_i^0,F_i(x_1^0,\cdots,x_n^0),t)+M_i(x_i^1,G_i(x_1^1,\cdots,x_n^1),t)]$$

$$\geq \sum_{k=1}^{n} a_{ik} r_k + b[M_i(x_i^1,x_i^0,t)+M_i(x_i^1,x_i^2,t)]$$

$$M_i(x_i^1,x_i^2,t)-bM_i(x_i^1,x_i^2,t)\geq \sum_{k=1}^{n} a_{ik} r_k + bM_i(x_i^1,x_i^0,t)$$

$$(1-b)M_i(x_i^1,x_i^2,t)\geq \nu r_i + br_i$$

$$M_i(x_i^1,x_i^2,t)\geq \frac{(\nu+b)}{(1-b)} r_i$$

$$M_i(x_i^1,x_i^2,t)\geq q r_i,\quad i=1,\cdots,n.$$

Where $0\leq q=\dfrac{(\nu+b)}{(1-b)}\leq 1$

Also from

$$M_i(x_i^2, x_i^3, t) = M_i(F_i(x_1^1, \cdots, x_n^1), G_i(x_1^2, \cdots, x_n^2), t)$$
$$\geq \sum_{k=1}^{n} a_{ik} M_k(x_k^1, x_k^2, t) + b[M_i(x_i^1, F_i(x_1^1, \cdots, x_n^1), t) + M_i(x_i^2, G_i(x_1^2, \cdots, x_n^2), t)]$$
$$M_i(x_i^2, x_i^3, t) \geq \sum_{k=1}^{n} a_{ik} r_i + b[M_i(x_i^1, x_i^2, t) + M_i(x_i^2, x_i^3, t)]$$

We obtain

$$M_i(x_i^2, x_i^3, t) \geq q^2 r_i, i = 1, \cdots, n.$$

Inductively

$$M_i(x_i^m, x_i^{m+1}, t) \geq q^m r_i \qquad (5)$$

It follows easily from (5) that $\{x_i^m\}$ be a sequence in X_i.

$$M_i(x_i^m, x_i^{m+1}, kt) \geq M_i(x_i^0, x_i^1, \frac{t}{k^{n-1}})$$

for all m and $t > 0$. Thus for any positive integer p we have

$$M_i(x_i^m, x_i^{m+p}, t) \geq M_i(x_i^m, x_i^{m+1}, \frac{t}{p}) * \cdots * M_i(x_i^{m+p-1}, x_i^{m+p}, \frac{t}{p})$$
$$\geq M_i(x_i^0, x_i^1, \frac{t}{pk^n}) * \cdots * M_i(x_i^0, x_i^1, \frac{t}{pk^n})$$
$$\lim_m M_i(x_i^m, x_i^{m+p}, t) \geq 1 * \cdots * 1 = 1.$$

Therefore $\{x_i^m\}$ is a Cauchy sequence for each $i = 1, \cdots, n$, hence convergent. Since $(X_i, M_i, *)$ is complete. Let $\{x_i^m\}$ converges to a point $u_i \in X_i$ such that

$$\lim_{m\to\infty} M_i(F_i x_i^m, F_i u_i, t) \geq \lim_{m\to\infty} M_i(x_i^m, u_i, t) = 1$$

Now we show that

$$u_i = F_i(u_1, \cdots, u_n) = G_i(u_1, \cdots, u_n), \ i = 1, \cdots, n.$$

For each $1 \leq i \leq n$

$$M_i(u_i, F_i(u_1, \cdots, u_n), t) \geq M_i(u_i, x_i^{2m+2}, t) + M_i(x_i^{2m+2}, F_i(u_1, \cdots, u_n), t)$$
$$\geq M_i(u_i, x_i^{2m+2}, t) + M_i(G_i(x_1^{2m+1}, \cdots, x_n^{2m+1}), F_i(u_1, \cdots, u_n), t)$$
$$\geq M_i(u_i, x_i^{2m+2}, t) + \sum_{k=1}^{n} a_{ik} M_k(u_k, x_k^{2m+1}, t) + b[M_i(u_i, F_i(u_1, \cdots, u_n), t) + M_i(x_i^{2m+1}, G_i(x_1^{2m+1}, \cdots, x_n^{2m+1}), t)]$$
$$\geq M_i(u_i, x_i^{2m+2}, t) + \sum_{k=1}^{n} a_{ik} M_k(u_k, x_k^{2m+1}, t) + b[M_i(u_i, F_i(u_1, \cdots, u_n), t) + M_i(x_i^{2m+1}, x_i^{2m+2}, t)]$$

Making $m \to \infty$, we obtain for each $0 \leq i \leq n$.

$(1-b)M_i(u_i, F_i(u_1,\cdots,u_n),t) \geq 0$

$\Rightarrow\ u_i = F_i(u_1,\cdots,u_n)$

And similarly

$u_i = G_i(u_1,\cdots,u_n)$

To prove the uniqueness of u_i. Let $u_i, \overline{u}_i$ be two distinct solutions. Then

$u_i = F_i(u_1,\cdots,u_n) = G_i(u_1,\cdots,u_n)$

And $\overline{u}_i = F_i(\overline{u}_1,\cdots,\overline{u}_n) = G_i(\overline{u}_1,\cdots,\overline{u}_n)$

We can assume that

$M_i(u_i,\overline{u}_i,t) \geq r_i, i = 1,\cdots,n.$

$$M_i(u_i,\overline{u}_i,t) = M_i(F_i(u_1,\cdots,u_n), G_i(\overline{u}_1,\cdots,\overline{u}_n),t)$$
$$\geq \sum_{k=1}^{n} a_{ik} M_k(u_k,\overline{u}_k,t) + b[M_i(u_i, F_i(u_1,\cdots,u_n),t) + M_i(\overline{u}_i, G_i(\overline{u}_1,\cdots,\overline{u}_n),t)]$$
$$\geq \sum_{k=1}^{n} a_{ik} M_k(u_k,\overline{u}_k,t) \geq \nu r_i$$

Inductively

$M_i(u_i,\overline{u}_i,t) \geq \nu^m r_i,\quad i = 1,\cdots,n.$

Therefore

$M_i(u_i,\overline{u}_i,t) = 1,\ \ i = 1,\cdots,n.$

This completes the proof.

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