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RC-Graphs and Their Fuzzifications

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Abstract

Around the year 2000, Gnanaseelan. T [2] in his Ph. D. thesis constructed a new finite, simple graph using complex irreducible characters of finite groups called Relative Character Graphs (RC-Graphs). This was later studied in depth by K. T. Nagalakshmi, N. Chitra and many others. (see [7] for details).
In this paper, we introduce a canonical fuzzification of these RC-graphs and study some of their basic properties.

Keywords: Relative Character Graphs; Chordal Graph; RC-Fuzzy Graph; Character degree ; Graph degree

AMS Subject Classification (2010) : 05C72

Introduction

Relative Character Graphs (RC-Graphs) have been the topic of study of K. T. Nagalakshmi, N. Chitra, Stella Maragatham and T. Gnanaseelan over the past few years. Several talks have been given by the speaker at various [7] conferences and seminars on RC-graphs. An international publication covering all the latest properties is in the pipe line.

Fuzzy graphs have recently attracted quite a few researchers. Even though fuzzy graphs exist independently, I would like to fuzzify (the new classical) RC-graphs and study some basic properties of these new graphs, which we call RC-Fuzzy Graphs.

RC-Graphs and Some Properties

But we shall first begin from the beginning. The Relative Character Graphs was first constructed by Gnanaseelan.T in his Madurai Kamaraj University Ph.D thesis [2] under the speaker's guidance.

S. Donkin, a well-known character-theorist observed that this construction is 'new and interesting'.

Definition

Let G be a finite group, fix a subgroup H of G; let IrrG and IrrH respectively denote the set of all distinct complex irreducible characters of G and H. (For details on Character theory of finite groups, we refer to J.P. Serre's lucid treatise [6]) and Isaac's book [4].

The Relative Character Graph $\Gamma(G,H)$, is constructed as follows : The vertex set V is the set IrrG and two elements φ,χ in IrrG are adjacent if and only if their restrictions φ_H and χ_H to H contain atleast one element θ of H.

This is equivalent to saying $(\varphi,\chi) > 0$

where $(\varphi,\chi) = \frac{1}{O(G)} \sum_s \varphi(s)\overline{\chi(s)}$

RC-graph is a finite, simple, undirected graph.

We skip some usual easy examples and straightaway take up some nice properties. (Of course, the proofs, which are the outcome of several months and years, are omitted.)

Theorem 1

RC-graph $\Gamma(G,H)$ is a null graph if and only if G = H.

Theorem 2

$\Gamma(G, (1))$ is always complete. But the converse is not true in general.

Problem 1.

Given arbitrary G, find all subgroups H such that $\Gamma(G,H)$ **is** complete.

(If G is simple non-abelian and H is generated by an element of order 2, then $\Gamma(G,H)$ is complete)

Theorem 3.

$\Gamma(G,H)$ is complete if and only if it is regular.

Theorem 4.

If $K \subseteq H \subseteq G$, then $\Gamma(G,H)$ is a subgraph of $\Gamma(G,K)$ (both having the same vertex set.)

Connected Components of Γ (G, H)

We first need a third equivalent definition for adjacency of vertices. Put $\chi = 1_H$, the character induced from the trivial character 1_H of H. Then, given $\phi, \psi \in \mathrm{Irr}G$, ϕ and ψ are adjacent in Γ(G, H) if and only if $\phi \subset \psi\chi (\Leftrightarrow \psi \subset \chi\phi)$

Theorem 5

Two vertices ϕ and ψ lie in the same connected component of Γ(G,H) if and only if $\phi \subset \psi\chi^s$ for some integer $s \geq 1$.

Corollary

Γ(G,H) is connected if and only if $\mathrm{Core}_G H = (1)$, where $\mathrm{Core}_G H$ = largest normal subgroup of G contained in H

$$= \bigcap_{x \in G} H^x \, (H^x = xHx^{-1})$$

For instance, if G is non-abelian simple, then Γ(G,H) is connected. Also for arbitrary G, if H is non-abelian simple, the Γ(G,H) is connected.

The Tree Problem

Definition

A finite group G is Frobenius if there exists a subgroup $H \neq (1)$ such that $H \cap H^x = (1)$ for all $x \notin G$. H is called a Frobenius complement(unique upto conjugacy) and the set $N = G - \bigcup_{x \in G} H^x \cup \{1\}$ is a normal subgroup called Frobenius Kernel (unique), and G is the semi direct product NH.

Examples :

S_3, A4, D_{10} etc.

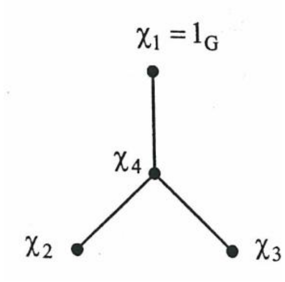
Note that Γ(G,N) is disconnected and Γ(G,H) is connected.

Theorem 6

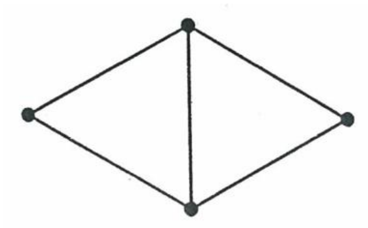
Γ(G,H) is a tree if and only if G = NH is Frobenius with N, elementary abelian of order p^m (for some prime p and $m \geq 1$) and $0(H) = p^m - 1$.

Example

$G = A_4 = V_4 \text{-} C_3$

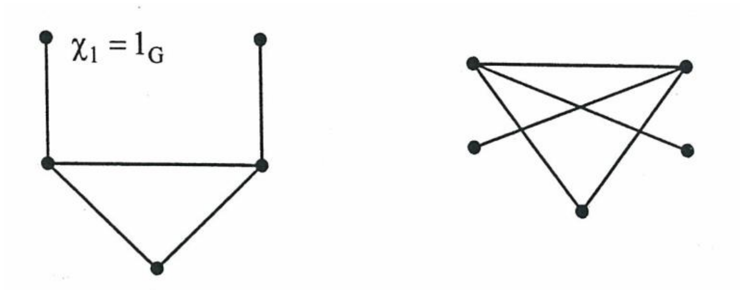


Though $D_{10} = C_5.C_2$ is Frobenius, $O(C_5) = 5$, $O(C_2) = 2 \neq 5 - 1 = 4$. The graph$\Gamma(D_{10},C_2)$ is the following (which of course is not a tree)



Complements

If Γ is disconnected (arbitrary) graph, then its complement $\bar{\Gamma}$ is connected. But if Γ is connected, then $\bar{\Gamma}$ may be disconnected or connected. In RC-graphs,$\Gamma(S_4,S_3)$ and $\bar{\Gamma}$ $(S_4,S_3)$are the following graphs:



$\Gamma(S_4,S_3)$ $\bar{\Gamma}(S_4,S_3)$

Both the graphs are connected, even though the second one is not $\Gamma(G,K)$ for any subgroup K. This raises the following question. ,

Problem 2

Find all finite groups having a pair of subgroups H,K such that $\bar{\Gamma}$ $(G,H) = \Gamma(G,K)$.

In this connection the Frobenius group discussed earlier again plays an important role.

Theorem 7
Let G = NH be a semi directed product, with N normal and H non-normal. Then Γ(G,H) = $\bar{\Gamma}$ (G,N) if and only if G is Frobenius with kernel N and complement H.

Triangulation

RC-graphs give natural room for concepts like domination, signed domination, gracefulness etc.

Also the Eigen Value Problem for RC-graphs is wide open.

We shall focus only the triangulation properly and the resulting passage to perfect graphs.

Definition
A connected graph is triangulated (or choral) if given any cycle C_n ($n \geq 4$), there exists a chord connecting two vertices of C_n.

Theorem 8
Any connected RC-graph is choral. (of course, any tree is trivially triangulated.)This leads to the very interesting result that any connected RC-graph is perfect.

Definitions
Let Γ be an arbitrary finite, simple, undirected graph.

The chromatic number $\chi(\Gamma)$ of Γ is the minimum number of colours needed to colour the vertices of Γ so that every two adjacent get different colours. The size of the biggest clique (= complete subgraph is denoted by $\omega(\Gamma)$)

$\Gamma$is called **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph H of Γ.(There are two more invariants θ, partition number and α, independent number and perfectness of Γ is equivalent to $\theta = \alpha$ for all induced subgraph).

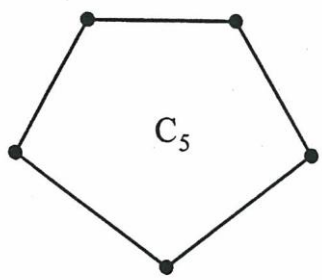
Theorem 9
Every choral graph is perfect.

In view of our earlier theorem that every RC-graph is choral, we get our final.

Theorem 10
Every RC-graph is perfect.

We conclude this section by giving a simple example of a graph Γ which is not perfect.

Here $\chi = 3$ and $\omega = 2$.



RC-Fuzzy graphs

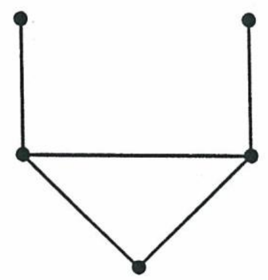
To define a fuzzy graph, one does not need a 'classical' (the so-called 'crisp') graph. What all we do here is to make any RC-graph into a fuzzy graph by introducing a membership function for the vertices and a relation function or weight function between a pair of vertices. In fact we introduce two entirely distinct fuzzifications.

Definition

A fuzzy graph $\Gamma = (V, \mu, \rho)$ is a triplet, where V is a nonempty set, a membership function $\boldsymbol{\mu : V \rightarrow [0,1]}$ and a relation function (or a weight function) $\rho : V \times V \rightarrow [0,1]$ satisfying $\boldsymbol{\rho(x,y) = \text{Min}\{\mu(x), \mu(y)\}}$ for all $x,y \in V$

In an RC-graph$\Gamma(G,H)$ each vertex v (= an irreducible character of G) has two degrees : the character degree D(v) and the graph degree d(v). There is no reason to expect these two degrees to coincide for all v, except in some special cases like $\Gamma(S_3,S_2)$ or $\Gamma(S_4,S_3)$.

$\Gamma(S_3,S_2)$



$\Gamma(S_4,S_3)$.

The ordered character degree sequence of the first graph is (1,1,2) and the corresponding graph degree sequence is also (1,1,2).

The ordered character degree sequence of the second graph is (1,1,2,3,3) and the corresponding graph degree sequence is also (1,1,2,3,3).

Definition

An RC Fuzzy Graph $\tilde{\Gamma}(G,H)$ is **Uniform** if D(v) = d(v) for all $v \in$ IrrG. Such graphs were (tentatively) called RC-uniform graphs.

However $\Gamma(C_3,(l))$, is the complete graph K_3 whose graph degree sequence is (2,2,2), but character degree sequence is (1,1,1).

Theorem 11

Every RC-graph $\Gamma(G,H)$ gives rise to an RC-fuzzy graph $\tilde{\Gamma}(G,H)$ Proof:

For the vertex set V, we take IrrG itself.

For any $v \in V$, define

$\mu : V \to [0,1]$ as

$\mu(v) = \dfrac{|Dv - dv|}{g}$ where g = 0(G).

Claim : That $0 \leq \mu(x) \leq 1$ for $v \in V$ (since both Dv and dv are positive) is obvious.

For the other inequality, first we observe that, since D(v) divides g for all v, D(v) < g (unless G is trivial).

Next we shall prove dv < g for all v.

Case(i)

If $\Gamma(G,H)$ is complete, then

$dv = |V| - 1 = |\, IrrG\, | - l \leq g - l < g$.

(since | IrrG | ≤ g and equality holds iff G is abelian).

Case(ii)

If $\Gamma(G,H)$ is not complete, then

$dv \leq |\, IrrG\, | - 1 \leq g - 1 < g$.

In any case dv < g which gives

$\mu(x) < 1$ for all v.

Next define $p:VxV \to [0,1]$ as

$$\rho(vw) = \frac{|Dv - dv|\,|Dw - dw|}{g^2}$$

Since | Dv - dv | < g for all v,

Clearly $\rho(vw) < 1$.

Finally | Dv - dv | < g, | Dw - dw | < g gives

$$\frac{|Dv - dv|\,|Dw - dw|}{g^2} \leq \text{Min}\left\{\frac{|Dv - dv|}{g^2}, \frac{|Dw - dw|}{g^2}\right\}$$

$\rho(vw) \leq$ Min { $\mu(v)$, $\mu(w)$} for all v,w$\in$ V.

Hence $(\Gamma(G,H),\mu,\rho)$ is an (RC-) fuzzy graph which we denote by $\tilde{\Gamma}$ **(G,H)**.

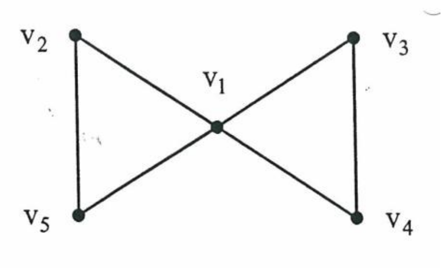
The graphs $\tilde{\Gamma}(S_3,S_2)$ and $\tilde{\Gamma}(S_4,S_3)$ get "trivial" fuzzification, that is, graph degrees and character degrees coincide for every vertex v.

Example

When $G = D_8 = \{ a,b \mid a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1} \}$

(The Dihedral graph) and H = (b)

$\Gamma(G,H)$ is the following graph.



The ordered character degree sequence is **1,1,1,1,2** and the corresponding degree sequence is **2,2,2,2,4** note that these two sequences are not identical .

$\mu(v_1) = \mu(v_2) = \mu(v_3) = \mu(v_4) = 1/8$, $\mu(v_5) = 1/4$,

$\rho(v_1v_2) = \rho(v_3v_4) = 1/64$,

$\rho(v_1v_5) = \rho(v_2v_5) = \rho(v_3v_5) = \rho(v_4v_5) = 1/32$

The situation where all the μ- values are equal and all the ρ-values are equal is shared by many classes of groups.

Abelian Group Case

Theorem 12.
If G = NH is a Frobenius group where both N and H are abelian then, μ and ρ are constant functions.

Proof:

The set IrrG = $B \cup A$, where

$B = \{ \chi \mid N \not\subset \text{Ker}\,\chi \}$ and

$A = \{ \chi \mid N \subset \text{Ker}\,\chi \}$

Also, | B | = t/h =b, where h = 0(H) and

IrrN | = t+l, where t=h-1

Also . |A| = | IrrH |

Let $B = \{v_1, v_2, ..., v_b\}$

$A = \{w_1, w_2, ..., w_h\}$

Then it can be proved that

$$\mu(v_i) = \frac{b-1}{(t+1)h} \quad \text{for all } v_i,$$

$$\mu(w_j) = \frac{b-1}{(t+1)h} \quad \text{for all } w_j,$$

Similarly we can prove that

$$\rho(v_iv_j) = \frac{(b-1)^2}{(t+1)^2h^2} \text{ for all } v_i, v_j,$$

$$\rho(v_iw_k) = \frac{(b-1)^2}{(t+1)^2h^2} \text{ for all } v_i, w_k,$$

Also, $\rho(w_k w_l) = \dfrac{(b-1)^2}{(t+1)^2 h^2}$ for all w_k, w_l.

This proves the theorem.

Definitions

An RC-graph is RC-lower uniform if $d(v) \leq D(v)$ for all $v \in V$ and is RC-upper uniform if $d(v) \geq D(v)$ for all v and as defined already ,equality gives RC-uniform graphs.

As examples, if $G = S_n$, the symmetric group on n letters and H **any** subgroup, then $d(v) \leq D(v)$ for all $v \in$ IrrG and hence is RC-lower uniform.

If G is abelian and H is of index atleast 3, then each of the 0(H) components is K_3 and hence $d(v) \geq D(v)$ for all v, showing that $\Gamma(G,H)$ is RC-upper uniform.

Conjecture:

We now based on evidences obtained. So far, We propose the following conjectures.

Classify all Pairs (G,H) for which

1. $\Gamma(G,H)$ is Lower Uniform
2. $\Gamma(G,H)$ is Upper Uniform
3. $\Gamma(G,H)$ is Uniform

Theorem 13

Let G be a RC-lower uniform. Define μ: V → [0,1] as ,

$\mu^*(v) = \dfrac{\sqrt{D(v_i)}}{\sqrt{g}}$ for all i and

$\mu^*(\text{vertex}) = \mu(\text{vertex})$

$$\rho^*(v_i, v_j) = \text{Min}\left\{\frac{\sqrt{D(v_i)d(v_j)}}{g}, \frac{\sqrt{D(v_j)d(v_i)}}{g}\right\}$$

Then $(\Gamma(G,H), \mu^*, \rho^*)$ is an RC-fuzzy graph denoted by $\tilde{\Gamma}^*(G,H)$.

Proof:

Since$D(v_i) < g$ for all i,

$0 \leq \mu^*(v) \leq 1$ is clear.

Let $\dfrac{\sqrt{D(v_i)d(v_j)}}{g} = \text{Min}\left\{\dfrac{\sqrt{D(v_i)d(v_j)}}{g}, \dfrac{\sqrt{D(v_j)d(v_i)}}{g}\right\}$

Since $D(v_j) \leq g$

$$\frac{\sqrt{D(v_i)}}{\sqrt{g}} \frac{\sqrt{d(v_j)}}{\sqrt{g}} \leq \frac{\sqrt{D(v_i)}}{\sqrt{g}}, \text{if } \sqrt{D(v_i)} = \text{Min}\{\sqrt{D(v_i)}\sqrt{D(v_j)}\}$$

If $\frac{\sqrt{D(v_j)}}{\sqrt{g}} = \text{Min}\left\{\frac{\sqrt{D(v_i)}}{g}, \frac{\sqrt{D(v_j)}}{g}\right\}$

Start with $\sqrt{d(v_i)} \leq \sqrt{D(v_j)}$ (assumption)

$\leq \sqrt{g} \Rightarrow \frac{\sqrt{d(v_j)}}{\sqrt{g}} \leq \frac{\sqrt{D(v_j)}}{\sqrt{g}}$

Hence $\frac{\sqrt{D(v_i)}}{\sqrt{g}} \frac{\sqrt{d(v_j)}}{\sqrt{g}} \leq 1. \frac{\sqrt{d(v_j)}}{\sqrt{g}} \leq \frac{\sqrt{D(v_j)}}{\sqrt{g}}$

This proves

$\rho(v_i,v_j) \leq \text{Min}\{\mu(v_i)\ \mu(v_j)\}$ in this case.

The other two cases can also be dealth with similarly proving (Γ(G,H), μ^*, p*) is an RC-fuzzy graph denoted by $\tilde{\Gamma}^*$ (G,H).

Recall:

An mxn matrix A = (a_{ij}) over the reals is a transition probability matrix if i) $0 \leq a_{ij} \leq l$ for all i,j ii) $\sum_j a_{ij} = 1$ for all i.

Theorem 14

If G is abelian and H, any subgroup of index 2 then $\tilde{\Gamma}^*$ (G,H) satisfies the t.p.m condition.

Proof:

We must check the two t.p.m conditions given above for $(\rho(v_{i,}v_j))$. The first condition is clear from the definitions.

Now (Γ(G,H) has O(H) connected components and each component is K_2.

Hence $d(v_i) = D(v_i)$ for all i.

Therefore, $\rho(v_i,v_j) = 1/g$ for all i,j.

Since G is abelian, g = | IrrG | = | V | and the resulting matrix is gxg with all entries = 1/g.

In particular each row sum = 1.

Hence $\tilde{\Gamma}^*$ (G,H) satisfies the t.p.m. condition.

The connection with transition probability matrix leads to many interesting problems in queuing theory and computer applications.

Hard research is being pursued at present by Harshal Baltar et al [3] on a technique called join-the-shortest-queue-technique which is used in web server farms in distributed systems. Also certain Network Processors problems are being tackled quite recently using the above probability techniques. In all these probability related works, analytical proofs are given involving state-diagrams, matrix – geometry, transition probability matrix , global and balance equations, Jackson model etc.

The above passage from character theory through RC-graphs to RC-fuzzy graphs could open a gateway hither to not widely known to a connection between character theory, graph theory (via pertinets too), probability and hard-core computer problems.

Conclusion

This is just a beginning of what promises to be an interesting study with myriad openings for applications .

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