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Edge Pair Sum Labeling of $WT(n:k)$ tree

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Abstract

An injective map $f : E(G) \to \{\pm 1, \pm 2, \cdots, \pm q\}$ is said to be an edge pair sum labeling of a graph $G(p,q)$ if the induced vertex function $f^* : V(G) \to Z - \{0\}$ defined by $f^*(v) = \sum_{e \in E_v} f(e)$ is one-one, where E_v denotes the set of edges in G that are incident with a vetex v and $f^*(V(G))$ is either of the form $\left\{\pm k_1, \pm k_2, \cdots, \pm k_{\frac{p}{2}}\right\}$ or $\left\{\pm k_1, \pm k_2, \cdots, \pm k_{\frac{p-1}{2}}\right\} \bigcup \left\{k_{\frac{p+1}{2}}\right\}$ according as p is even or odd. A graph which admits edge pair sum labeling is called an edge pair sum graph. In this paper we study the edge pair sum labeling of $WT(n:k)$ tree.

AMS subject classification: 05C78.
Keywords: Pair sum labeling, edge pair sum labeling, edge pair sum graph, $WT(n:k)$ tree.

1. Introduction

All graphs in this paper are finite, simple and undirected. For standard terminology and notations we follow Harary [1]. The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph. Ponraj et al. [7] introduced the concept of pair sum labeling. An injective map $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling of a graph $G(p,q)$ if the induced edge function $f_e : E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\left\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q}{2}}\right\}\}$ for even q or $\left\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q-1}{2}}\right\} \cup \left\{\pm k_{\frac{q+1}{2}}\right\}$ for odd q. A graph with a pair sum labeling is called a pair sum graph. Analogous to pair sum labeling, we introduced a new labeling called an edge pair sum labeling in [2].

We proved the edge pair sum behaviour of graphs like star graph, bistar, $K_{1,n} \cup K_{1,m}$, $c_n \cup c_n$, spider graph, ladder graph, one point union of cycles, friendship graph, total graph, shadow graph and some standard graphs in references [3]-[6]. In this paper we study the edge pair sum labeling of $WT(n:k)$ tree.

Definition 1.1. The tree $WT(n)$, $n \geq 1$, is obtained from a star $G_1 = K(1, n+2)$ with central vertex c_1 and end vertices $x_i : 1 \leq i \leq n+2$ and another star $G_2 = K(1, n+2)$ with central vertex c_2 and end vertices $y_j : 1 \leq j \leq n+2$ by identifying vertex x_{n+2} and y_{n+2} and call the resulting degree 2 vertex w. Hence

$$V(G) = \{(c_1, v_2, w, w') \cup (x_1, x_2, \ldots, x_n, x_{n+1}) \cup (y_1, y_2, \ldots, y_n)\} \text{ and}$$
$$E(G) = \{c_1x_1, c_1x_2, \ldots, c_1x_{n+1}\} \cup \{c_2y_1, c_2y_2, \ldots, c_2y_n\} \cup \{c_1w, c_2w\} \cup \{c_2w'\}.$$

Definition 1.2. A w-tree $WT(n:k)$ is obtained from k copies of $WT(n)$ by joining a new vertex a to vertex w'_n of each copy of $WT(n)$[1]. Let

$$\begin{aligned} V(WT(n:k)) = \{a\} &\cup \{x_i : 1 \leq i \leq k(n+1)\} \cup \{w_i : 1 \leq i \leq k\} \\ &\cup \{y_i : 1 \leq i \leq kn\} \cup \{c_{i,j} : 1 \leq i \leq k, j = 1,2\} \cup \{w'_i : 1 \leq i \leq k\} \end{aligned}$$

and

$$\begin{aligned} E(WT(n:k)) = \{u_i = c_{i,1}w_i : 1 \leq i \leq k\} &\cup \{u'_i = c_{i,2}w_i : 1 \leq i \leq k\} \\ \cup \{l_i = c_{i,2}w'_i : 1 \leq i \leq k\} &\cup \{l'_i = aw'_i : 1 \leq i \leq k\} \cup \{e_{(n+1)(i-1)+j} \\ = c_{i,1}x_{(n+1)(i-1)+j} : 1 \leq i \leq k \end{aligned}$$

and

$$1 \leq j \leq (n+1)\} \cup \{e'_{n(i-1)+j} = c_{i,2}y_{n(i-1)+j} : 1 \leq i \leq k$$

and $1 \leq j \leq n\}$ are the vertices and edges of the graph $WT(n:k)$.

The example for the ede pair sum graph labeling of $WT(3:4)$ is shown in Figure 1.

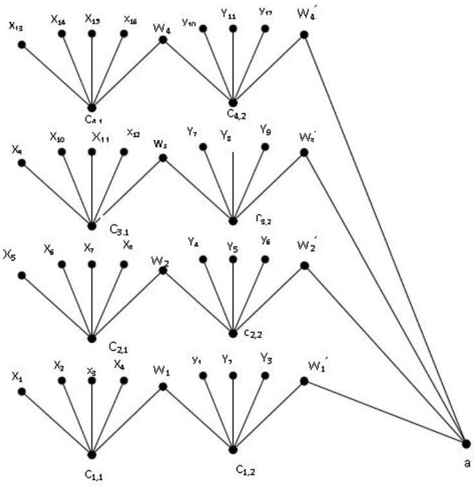

Figure 1:

2. Main Results

Theorem 2.1. Every w-tree $G = WT(n:k)$ admits an edge pair sum labeling.

Proof. If $v = |V(G)|$, $e = |E(G)|$, then $v = (2n+5)k+1$ and $e = v-1$. We consider the following four cases.

Case (i). k and n are odd.

Subcase (a). If $n = 1$ and $k \geq 1$.
Define

$$\begin{aligned}
f(e_1) &= 1,\ f(e_2) = -5 = -f(e_1'),\ f(u_1) = 3,\ f(u_1') = -6,\\
f(l_1) &= 4,\ f(l_1') = -2, \ \text{ for } 1 \leq i \leq \frac{k-1}{2},\\
f(e_{2i+1}) &= (2k+2+2i) = -f(e_{k+2i}),\\
f(l_{1+i}) &= (5k-1+2i) = -f\left(l_{\frac{k+1}{2}+i}\right),\\
f(u_{1+i}) &= (3k+1+2i) = -f\left(u_{\frac{k+1}{2}+i}\right),\\
f(u_{1+i}') &= (4k+2i) = -f\left(u'_{\frac{k+1}{2}+i}\right),\\
f(l_{1+i}') &= (6k-2+2i) = -f\left(l'_{\frac{k+1}{2}+i}\right),\\
f(e_{2+2i}) &= (4i+3) = -f(e_{k+1+2i})
\end{aligned}$$

and

$$f(e_{1+i}') = (4i+5) = -f\left(e'_{\frac{k+1}{2}+i}\right).$$

The induced vertex labeling are as follows:

$$f^*(a) = -2 = -f^*(w'_1),$$
$$f^*(c_{1,1}) = -1 = -f^*(x_1),\ f^*(c_{1,2}) = 3 = -f^*(w_1)$$

and

$$\begin{aligned}
f^*(x_2) &= -5 = -f^*(y_1), \quad \text{for } 1 \le i \le \frac{k-1}{2} \\
f^*(w_{1+i}) &= (7k+1+4i) = -f^*\left(w_{\frac{k+1}{2}+i}\right), \\
f^*(x_{2+2i}) &= (4i+3) = -f^*(x_{k+1+2i}), \\
f^*(y_{1+i}) &= (4i+5) = -f^*\left(y_{\frac{k+1}{2}+i}\right), \\
f^*(c_{1+i,i}) &= (5k+6+8i) = -f^*\left(c_{\frac{k+1}{2}+i,1}\right), \\
f^*(c_{1+i,2}) &= (9k+4+8i) = -f^*\left(c_{\frac{k+1}{2}+i,2}\right), \\
f^*(x_{1+2i}) &= (2k+2+2i) = -f^*(x_{k+2i})
\end{aligned}$$

and

$$f^*(w'_{1+i}) = (11k-3+4i) = -f^*\left(w'_{\frac{k+1}{2}+j}\right).$$

Then

$$\begin{aligned}
f^*(V(G)) = \{&\pm 1, \pm 2, \pm 3, \pm 5, \pm(2k+4), \pm(2k+6), \\
&\pm (2k+8), \ldots, \pm(3k+1), \pm(7k+5), \pm(7k+9), \\
&\pm (7k+13), \ldots, \pm(9k-1), \pm(11k+1), \pm(11k+5), \\
&\pm (11k+9), \ldots, \pm(13k-5), \pm(5k+14), \pm(5k+22), \\
&\pm (5k+30), \ldots, \pm(9k+2), \pm(9k+1), \pm 9, \pm 13, \\
&\pm 17, \ldots, \pm(2k+3)\}.
\end{aligned}$$

The example for the edge pair sum graph labeling of $WT(1:1)$ is shown in Figure 2.

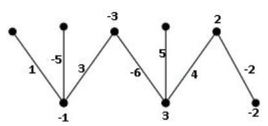

Figure 2:

Subcase (b). If $k \ge 1, n \ge 3$.

Define

$$f(e_1)=-1,\ f(e_2)=3,\ f(e_1')=-7,\ f(u_1)=-5,$$
$$f(u_1')=6,\ f(l_1')=-4,\ f(l_1)=8,\quad \text{for } 1\le i\le \frac{n-1}{2},$$
$$f(e_{2+i})=(7+2i)=-f\left(e_{\frac{n+3}{2}+i}\right)$$

and

$$f(e'_{1+i})=(n+6+2i)=-f\left(e'_{\frac{n+1}{2}+i}\right),\quad \text{for } 1\le i\le \left(\frac{k-1}{2}\right)$$

and $1\le j\le n$,

$$f(e_{(n+1)+i+1+j})=[2ni+5+2j]=-f\left(e_{(n+1)\left(\frac{k-1}{2}+i\right)+1+j}\right),$$

for $1\le i\le n\left(\frac{k-1}{2}\right)$,

$$f(e'_{n+i})=(nk+n+5+2i)=-f\left(e_{n\left(\frac{k+1}{2}\right)+i}\right),$$

for $1\le i\le \frac{k-1}{2}$, $f(e_{(n+1)i+1})=(2kn+4+2i)=-f\left(e_{(n+1)\left(\frac{k+1}{2}+i\right)+1}\right)$,

$$f(u_{1+i})=(2kn+k+2i+3)=-f\left(u_{\frac{k+1}{2}+i}\right),$$

$$f(u'_{1+i})=(2kn+2k+2+2i)=-f\left(u'_{\frac{k+1}{2}+i}\right),$$

$$f(l_{1+i})=(2kn+3k+1+2i)=-f\left(l_{\frac{k+1}{2}\,i}\right)$$

and

$$f(l'_{1+i})=(2kn+4k+2i)=-f\left(l'_{\frac{k+1}{2}+i}\right).$$

The induced vertex labeling are as follows:

$$
\begin{aligned}
f^*(a) &= -4 = -f^*(w_1'),\ f^*(x_1) = -1 = -f^*(w_1),\\
f^*(c_{1,1}) &= -3 = -f^*(x_2),\\
f^*(c_{1,2}) &= 7 = -f^*(y_1), \quad \text{for } 1 \le i \le \frac{k-1}{2}\\
f^*(x_{(n+1)i+1}) &= (2kn + 4 + 2i) = -f^*\left(x_{(n+1)\left(\frac{k-1}{2}+i\right)+1}\right),\\
f^*(w_{1+i}) &= (4kn + 3k + 5 + 4i) = -f^*\left(w_{\frac{k+1}{2}+i}\right),\\
f^*(w'_{1+i}) &= (4kn + 7k + 1 + 4i) = -f^*\left(w'_{\frac{k+1}{2}+i}\right),\\
f^*(c_{1+i,1}) &= (4kn + k + 11 + 3n^2 + 6n + (2n^2+4)(i-1))\\
&= -f^*\left(c_{\frac{k+1}{2}+i,1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
f^*(c_{1+i,2}) &= (4kn + 5k + 7 + n^2k + 6n + 2n^2 + (2n^2+4)(i-1))\\
&= -f^*\left(c_{\frac{k+1}{2}+i,2}\right), \quad \text{for } 1 \le i \le \frac{n-1}{2},\\
f^*(x_{2+i}) &= (7 + 2i) = -f^*\left(x_{\frac{n+3}{2}+i}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
f^*(y_{1+i}) &= (n + 6 + 2i)\\
&= -f^*\left(y_{\frac{n+1}{2}+i}\right), \quad \text{for } 1 \le i \le \left(\frac{k-1}{2}\right)
\end{aligned}
$$

and $1 \le j \le n$,

$$
\begin{aligned}
f^*(x_{(n+1)i+1+j}) &= (2ni + 5 + 2j)\\
&= -f^*\left(x_{(n+1)\left(\frac{k-1}{2}+i\right)+1+j}\right), \quad \text{for } 1 \le i \le n\left(\frac{k-1}{2}\right),\\
f^*(y_{n+i}) &= (nk + n + 5 + 2i) = -f^*\left(y_{n\left(\frac{k+1}{2}\right)+i}\right).
\end{aligned}
$$

From the above labeling we get

$$\begin{aligned} f^*(V(G)) = \{&\pm 1, \pm 3, \pm 4, \pm 7, \pm 9, \pm 11, \ldots, \\ &\pm (n+6), \pm(n+8), \pm(n+10), \pm(n+12), \ldots, \\ &\pm (2n+5), \pm(2n+7), \pm(2n+9), \pm(2n+11), \ldots, \\ &\pm (nk+n+5), \pm(nk+n+7), \pm(nk+_n+9), \pm(nk+n+11), \ldots, \\ &\pm (2nk+5), \pm(2nk+6), \pm(2nk+8), \pm(2nk+10), \ldots, \\ &\pm (2nk+k+3), \pm(4nk+3k+9), \pm(4nk+3k+17), \ldots, \\ &\pm (4nk+5k+3), \pm(4nk+7k+5), \pm(4nk+7k+9), \\ &\pm (4nk+7k+13), \ldots, \\ &\pm (4nk+9k-1), \pm(3n^2+4nk+k+6n+11), \\ &\pm (5n^2+4nk+k+6n+15), \ldots, \\ &\pm (n^2k+4nk+3k+6n+5), \pm(n^2k+4nk+5k+6n+2n^2+7), \\ &\pm (n^2k+4nk+5k+4n^2+6n+11), \ldots, \\ &\pm (4kn+2n^2k+7k+6n-n^2+1)\}. \end{aligned}$$

Hence f is an edge pair sum labeling.

Case (ii). If n, k are even.

$$\begin{aligned} f(e_1) &= 1 = -f(u_1'), \text{ for } 1 \le i \le \frac{n}{2}, \\ f(e_{1+i}) &= (2i+1) = -f\left(e_{\frac{n+2}{2}+i}\right), \\ f(e_i') &= (n+2i+3) = -f\left(e'_{\frac{n}{2}+i}\right), \\ f(e_{n+2+i}) &= (2n+2i+5) = -f\left(e_{\frac{3n+4}{2}+i}\right) \end{aligned}$$

and

$$\begin{aligned} f(e'_{n+i}) &= (3n+2i+9) = -f\left(e_{\frac{3n}{2}+i}\right), \\ f(u_1) &= (n+3) = -f(l_1), \; f(l_1') = (3n+8), \\ f(e_{n+2}) &= -(2n+5) = -f(u_2'), \; f(u_2) = -(3n+7) = -f(l_2), \\ f(l_2') &= -(3n+9), \text{ for } 1 \le i \le \frac{k-2}{2} \\ f(e_{n+(n+1)i+2}) &= (2nk+9+2i) \end{aligned}$$

$$
\begin{aligned}
&= -f\left(e_{n+(n+1)\left(\frac{k-2}{2}+i\right)+2}\right),\\
f(u_{2+i}) &= (5n+12+12i) = -f\left(u_{\frac{k+2}{2}+i}\right),\\
f(u'_{2+i}) &= (5n+k+10+2i) = -f\left(u'_{\frac{k+2}{2}+i}\right),\\
f(l_{2+i}) &= (2nk+k+7+2i) = -f\left(l_{\frac{k+2}{2}+i}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
f(l'_{2+i}) &= (2kn+2k+5+2i)\\
&= -f\left(l_{\frac{k+2}{2}+i}\right), \text{ for } 1 \le i \le \left(\frac{k-2}{2}\right)
\end{aligned}
$$

and $1 \le i \le n$,

$$
\begin{aligned}
f(e_{n+(n+1)i+2+j}) &= (2n+2ni+9+2j)\\
&= -f\left(e_{n+(n+1)\left(\frac{k-2}{2}+i\right)+2+j}\right),
\end{aligned}
$$

for $1 \le i \le n\left(\frac{k-2}{2}\right)$,

$$
f(e'_{2n+i}) = (nk+2n+9+2i) = -f\left(e'_{\frac{nk+2n}{2}+i}\right).
$$

The induced vertex labeling are as follows:

$$
\begin{aligned}
f^*(c_{1,1}) &= (n+4) = -f^*(c_{1,2}),\\
f^*(c_{2,1}) &= -(5n+12) = -f^*(c_{2,2}), \text{ for } 1 \le i \le \frac{k-2}{2}\\
f^*(w_{2+i}) &= (10n+k+22+4i) = -f^*\left(w_{\frac{k+2}{2}+i}\right),\\
f^*(x_{n+(n+1)i+2}) &= (2nk+9+2i)\\
&= -f^*\left(x_{n+(n+1)\left(\frac{k-2}{2}+i\right)+2}\right),\\
f^*(w'_{2+i}) &= (4kn+3k+12+4i)\\
&= -f^*\left(w'_{\frac{k+2}{2}+i}\right),\\
f^*(c_{2+i,1}) &= (2nk+5n^2+15n+25+(2n^2+4)(i-1))\\
&= -f^*\left(c_{\frac{k+2}{2}+i,1}\right)
\end{aligned}
$$

and

$$\begin{aligned}
f^*(c_{2+i,2}) &= (2nk + n^2k + 3n^2 + 15n + 2k + 21 + (2n^2+4)(i-1)) \\
&= -f^*\left(c_{\frac{k+2}{2}+i,2}\right), \\
f^*(x_{n+2}) &= -(2n+5) = -f^*(w_1'), \\
f^*(w_1) &= (n+2) = -f^*(w_2),\ f^*(w_2') = -2, \\
f^*(x_1) &= 1 = -f^*(a),\ \text{ for } 1 \le i \le \frac{n}{2}, \\
f^*(x_{1+i}) &= (2i+1) = -f^*\left(x_{\frac{n+2}{2}+i}\right), \\
f^*(y_i) &= (n+2i+3) = -f^*\left(y_{\frac{n}{2}+i}\right), \\
f^*(x_{n+2+i}) &= (2n+5+2i) = -f^*\left(x_{\frac{3n+4}{2}+i}\right)
\end{aligned}$$

and

$$f^*(y_{n+i}) = (3n+2i+9) = -f^*\left(y_{\frac{3n}{2}+i}\right),$$

for $1 \le i \le \left(\frac{k-2}{2}\right)$ and $1 \le j \le n$

$$\begin{aligned}
f^*(x_{n+(n+1)i+2+j}) &= (2n + 2ni + 9 + 2j) \\
&= -f^*\left(x_{n+(n+1)\left(\frac{k-2}{2}+i\right)+2+j}\right),
\end{aligned}$$

for $1 \le i \le n\left(\frac{k-2}{2}\right)$,

$$f^*(y_{2n+i}) = (nk + 2n + 9 + 2i) = -f^*\left(y_{\frac{nk+2n}{2}+i}\right).$$

From the above arguments we get

$$\begin{aligned}
f^*(V(G)) = \{&\pm 1, \pm 3, \pm 5, \pm 7, \ldots, \\
&\pm (n+1), \pm(n+2), \pm(n+4), \pm(2n+5), \pm(5n+12), \\
&\pm (n+5), \pm(n+7), \pm(n+9), \ldots, \\
&\pm (2n+3), \pm(2n+7), \pm(2n+9), \pm(2n+11), \ldots,
\end{aligned}$$

$$\begin{aligned}
&\pm (3n+5), \pm(3n+11), \pm(3n+13), \pm(3n+15), \dots,\\
&\pm (4n+9), (4n+11), \pm(4n+13), \pm(4n+15), \dots,\\
&\pm (nk+2n+9), \pm(2nk+11), \pm(2nk+13), \pm(2nk+15), \dots,\\
&\pm (2nk+k+7), \pm(nk+2n+11), \pm(nk+2n+13), \pm(nk+2n+15), \dots,\\
&\pm (2nk+9), \pm(10n+k+26), \pm(10n+k+30),\\
&\pm (10n+k+34), \dots, \pm(10n+3k+18), \pm(4kn+3k+16),\\
&\pm (4kn+3k+20), \pm(4kn+3k+24), \dots,\\
&\pm (4kn+5k+8), \pm(2nk+5n^2+15n+25),\\
&\pm (2nk+7n^2+15n+29), \dots, \pm(2nk+n^2+n^2k+15n+2k+17),\\
&\pm (2nk+n^2k+3n^2+15n+2k+21),\\
&\pm (2nk+n^2k+5n^2+15n+2k+25), \dots,\\
&\pm (2nk+2n^2k-n^2+15n+4k+13)\} \cup \{-2\}.
\end{aligned}$$

Hence f is an edge pair sum labeling.

Case (iii). If k is odd and n is even.

$$f(e_1) = 2,\ f(u_1) = -3 = -f(l_1'),\ f(u_1') = 4,\ f(l_1) = -6,$$

for $1 \le i \le \dfrac{k-1}{2}$,

$$\begin{aligned}
f(e_{(n+1)i+1}) &= (2nk+3+2i)\\
&= -f\left(e_{(n+1)\left(\frac{k-1}{2}+i\right)+1}\right),\\
f(u_{1+i}) &= (6+2i) = -f\left(u_{\frac{k+1}{2}+i}\right),\\
f(u'_{1+i}) &= (k+5+2i) = -f\left(u'_{\frac{k+1}{2}+i}\right),\\
f(l_{1+i}) &= (2nk+k+2+2i) = -f\left(l_{\frac{k+1}{2}+i}\right)
\end{aligned}$$

and

$$f(l'_{1+i}) = (2kn+2k+1+2i) = -f\left(l'_{\frac{k+1}{2}+i}\right),$$

for $1 \le i \le \dfrac{n}{2}$,

$$f(e_{1+i}) = (3+2i) = -f\left(e_{\frac{n+2}{2}+i}\right)$$

and

$$f(e_i') = (n+3+2i) = -f\left(e'_{\frac{n}{2}+i}\right)$$

for $1 \leq i \leq \left(\frac{k-1}{2}\right)$ and $1 \leq j \leq n$

$$f(e_{(n+1)i+1+j}) = (2ni + 3 + 2j) = -f\left(e_{(n+1)\left(\frac{k-1}{2}+i\right)+1+j}\right),$$

for $1 \leq i \leq n\left(\frac{k-1}{2}\right)$,

$$f(e'_{n+i}) = (nk + n + 3 + 2i) = -f\left(e_{n\left(\frac{k+1}{2}\right)+i}\right).$$

The induced vertex labeling are as follows:

$$\begin{aligned}
f^*(x_1) &= 2 = -f^*(c_{1,2}),\\
f^*(w_1) &= 1 = -f^*(c_{1,1}),\\
f^*(w'_1) &= -3 = -f^*(a), \text{ for } 1 \leq i \leq \frac{k-1}{2},\\
f^*(x_{(n+1)i+1}) &= (2nk + 3 + 2i)\\
&= -f^*\left(x_{(n+1)\left(\frac{k-1}{2}+i\right)+1}\right),\\
f^*(w_{1+i}) &= (k + 11 + 4i) = -f^*\left(w_{\frac{k+1}{2}+i}\right),\\
f^*(w'_{1+i}) &= (4kn + 3k + 3 + 4i)\\
&= -f^*\left(w'_{\frac{k+1}{2}+i}\right),\\
f^*(c_{1+i,1}) &= (2nk + 4n + 13 + 3n^2 + (2n^2 + 4)(i-1))\\
&= -f^*\left(c_{\frac{k+1}{2}+i,1}\right)
\end{aligned}$$

and

$$\begin{aligned}
f^*(c_{1+i,2}) &= (2nk + n^2k + 2n^2 + 2k + 4n + 11 + (2n^2 + 4)(i-1))\\
&= -f^*\left(c_{\frac{k+1}{2}+i,2}\right),
\end{aligned}$$

for $1 \leq i \leq \frac{n}{2}$,

$$f^*(x_{1+i}) = (3 + 2i) = -f^*\left(x_{\frac{n+2}{2}+i}\right)$$

and

$$f^*(y_i) = (n + 3 + 2i) = -f^*\left(y_{\frac{n}{2}+i}\right),$$

for $1 \leq i \leq \left(\dfrac{k-1}{2}\right)$ and $1 \leq j \leq n$

$$\begin{aligned} f^*(x_{(n+1)i+1+j}) &= (2ni+3+2j) \\ &= -f^*\left(x_{(n+1)\left(\frac{k-1}{2}+i\right)+1+j}\right), \end{aligned}$$

for $1 \leq i \leq n\left(\dfrac{k-1}{2}\right)$

$$f^*(y_{n+i}) = (nk+n+3+2i) = -f^*\left(y_{n\left(\frac{k+1}{2}\right)+i}\right).$$

From the above labeling we get

$$\begin{aligned} f^*(V(G)) = \{&\pm 1, \pm 2, \pm 3, \pm 5, \pm 7, \pm 9, \ldots, \\ &\pm (n+3), \pm(k+15), \pm(k+19), \pm(k+23), \ldots, \\ &\pm (3k+9), \pm(n+5), \pm(n+7), \pm(n+9), \ldots, \\ &\pm (2n+3), \pm(2n+5), \pm(2n+7), \pm(2n+9), \ldots, \\ &\pm (nk+n+3), \pm(2nk+5), \pm(2nk+7), \pm(2nk+9), \ldots, \\ &\pm (2nk+k+2), \pm(nk+n+5), \pm(nk+n+7), \pm(nk+n+9), \ldots, \\ &\pm (2nk+3), \pm(4nk+3k+7), \pm(4nk+3k+11), \\ &\pm (4nk+3k+15), \ldots, \pm(4nk+5k+1), \\ &\pm (2nk+3n^2+4n+13), \pm(2nk+5n^2+4n+17), \ldots, \\ &\pm (2nk+n^2k+2k+4n+7), \pm(2nk+n^2k+2n^2+2k+4n+11), \\ &\pm (2nk+n^2k+4n^2+2k+4n+15), \ldots, \\ &\pm (2nk+2n^2k-n^2+4n+4k+5)\}. \end{aligned}$$

Hence f is an edge pair sum labeling.

Case (iv). If k is even and n is odd.

Subcase (a). If k is even and $n = 1$.
Define

$$\begin{aligned} &f(e_1) = 1 = -f(u_1'),\ f(e_2) = 6 = -f(e_1'), \\ &f(u_1) = 4 = -f(l_1),\ f(l_1') = -9,\ f(e_3) = 13 = -f(u_2'), \\ &f(e_4) = 12 = -f(e_2'),\ f(u_2) = 10 = -f(l_2),\ f(l_2') = 8, \end{aligned}$$

for $1 \leq i \leq \dfrac{k-2}{2}$,

$$
\begin{aligned}
f(e_{3+2i}) &= (12+2i) = -f(e_{k+1+2i}),\\
f(e_{4+2i}) &= (k+10+2i) = -f(e_{k+2+2i}),\\
f(u_{2+i}) &= (5k-1+2i) = -f\left(u_{\frac{k+2}{2}+i}\right),\\
f(u'_{2+i}) &= (5k-1+2i) = -f\left(u_{\frac{k+2}{2}+i}\right),\\
f(u'_{2+i}) &= (6k-3+2i) = -f\left(u'_{\frac{k+2}{2}+i}\right),\\
f(e'_{2+i}) &= (2k+8+2i) = -f\left(l'_{\frac{k+2}{2}+i}\right).
\end{aligned}
$$

The induced vertex labelings are as follows:

$$
\begin{aligned}
&f^*(x_1) = 1 = -f^*(a),\ f^*(x_3) = 13 = -f^*(w'_1),\\
&f^*(w_1) = 3 = -f^*(w_2),\ f^*(w'_2) = -2,\\
&f^*(x_2) = 6 = -f^*(y_1),\ f^*(x_4) = 12 = -f^*(y_2),\\
&f^*(c_{1,1}) = 11 = -f^*(c_{1,2}),\ f^*(c_{2,1}) = 35 = -f^*(c_{2,2}),
\end{aligned}
$$

for $1 \leq i \leq k\dfrac{k-2}{2}$,

$$
\begin{aligned}
f^*(x_{3+2i}) &= (12+2i) = -f^*(x_{k+1+2i}),\\
f^*(w_{2+i}) &= (11k-4+4i) = -f^*\left(w_{\frac{k+2}{2}+i}\right),\\
f^*(w'_{2+i}) &= (7k+10+4i) = -f^*\left(w'_{\frac{k+2}{2}+i}\right),\\
f^*(x_{4+2i}) &= (k+10+2i) = -f^*(x_{k+2+2i}),\\
f^*(y_{2+i}) &= (2k+8+2i) = -f^*\left(y_{\frac{k+2}{2}+i}\right),\\
f^*(c_{2+i,1}) &= (6k+21+6i) = -f^*\left(c_{\frac{k+2}{2}+i,1}\right)
\end{aligned}
$$

and

$$
f^*(c_{2+i,2}) = (11k+11+6i) = -f^*\left(c_{\frac{k+2}{2}+i,2}\right).
$$

Then we get

$$\begin{aligned} f^*(V(G)) = \{&\pm 1, \pm 3, \pm 6, \pm 11, \pm 12, \pm 13,\\ &\pm 35, \pm 14, \pm 16, \pm 18, \ldots, \pm(k+10),\\ &\pm 11k, \pm(11k+4), \pm(11k+8)\pm, \ldots,\\ &\pm (13k-8), \pm(7k+14), \pm(7k+18),\\ &\pm (7k+22), \ldots, \pm(9k+6), \pm(k+12),\\ &\pm (k+14), \pm(k+16), \ldots, \pm(2k+8),\\ &\pm (2k+10), \pm(2k+12), \pm(2k+14), \ldots,\\ &\pm (3k+6), \pm(6k+27), \pm(6k+33), \pm(6k+39), \ldots,\\ &\pm (9k+15), \pm(11k+17), \pm(11k+23),\\ &\pm (11k+29), \ldots, \pm(14k+5)\} \cup \{-2\}. \end{aligned}$$

Hence f is an edge pair sum labeling.

Subcase (b). If k is even and $n \geq 3$.

$$\begin{aligned} f(e_1) &= 1 = -f(u'_1), \text{ for } 1 \leq i \leq n-1,\\ f(e_{1+i}) &= (2i+1) = -f(e'_i) \end{aligned}$$

and

$$\begin{aligned} f(e_{n+2+i}) &= -(2n+3+2i) = -f(e'_{n+1}),\\ f(e_{n+2}) &= -(2n+3) = -f(u'_2),\\ f(e_{n+1}) &= (2n-2) = -f(e'_n), f(u_1) = (2n+1) = -f(l_1),\\ f(l'_1) &= (4n+4), f(e_{2n+2}) = -4n = -f(e'_{2n}),\\ f(u_2) &= -(4n+3) = -f(l_2), f(l'_2) = -(4n+5), \end{aligned}$$

for $1 \leq i \leq \dfrac{k-2}{2}$,

$$\begin{aligned} f(e_{n+(n+1)i+2}) &= (4n+5+2i)\\ &= -f\left(e_{n+(n+1)\left(\frac{k-2}{2}+i\right)+2}\right),\\ f(l_{2+i}) &= (4n+k+3+2i) = -f\left(l_{\frac{k+2}{2}+i}\right),\\ f(u_{2+i}) &= (2kn+2k+2i) = -f\left(u_{\frac{k+2}{2}+i}\right),\\ f(u'_{2+i}) &= (2kn+3k-2+2i) = -f\left(u'_{\frac{k+2}{2}+i}\right) \end{aligned}$$

and

$$f(l'_{2+i}) = (2kn + 4k - 4 + 2i) = -f\left(l'_{\frac{k+2}{2}+i}\right),$$

for $1 \le i \le \left(\frac{k-2}{2}\right)$ and $1 \le j \le n$,

$$\begin{aligned} f(e_{n+(n+1)i+2+j}) &= (2n + 2ni + 2k + 1 + 2j) \\ &= -f\left(e_{n+(n+1)\left(\frac{k-2}{2}+i\right)+2+j}\right), \end{aligned}$$

for $1 \le i \le n\left(\frac{k-2}{2}\right)$

$$f(e'_{2n+i}) = (nk + 2n + 2k + 1 + 2i) = -f\left(e'_{\frac{nk+2n}{2}+i}\right).$$

The induced vertex labeling are as follows:

$$\begin{aligned} &f^*(x_1) = 1 = -f^*(a),\ f^*(x_{n+2}) = -(2n+3) = -f^*(w'_1), \\ &f^*(w_1) = 2n = -f^*(w_2),\ f^*(w'_2) = -2, \\ &f^*(c_{1,1}) = (n^2 + 4n - 1) = -f^*(c_{1,2}), \\ &f^*(c_{2,1}) = -(3n^2 + 10n + 3) = -f^*(c_{2,2}), \end{aligned}$$

for $1 \le i \le \frac{k-2}{2}$,

$$\begin{aligned} f^*(x_{n+(n+1)i+2}) &= (4n + 5 + 2i) \\ &= -f^*\left(x_{n+(n+1)\left(\frac{k-2}{2}+i\right)+2}\right), \\ f^*(w_{2+i}) &= (4kn + 5k - 2 + 4i) \\ &= -f^*\left(w_{\frac{k+2}{2}+i}\right), \\ f^*(w'_{2+i}) &= (2nk + 4n + 5k - 1 + 4i) \\ &= -f^*\left(w'_{\frac{k+2}{2}+i,1}\right), \\ f^*(c_{2+i,1}) &= (4nk + 5n^2 + 6n + 2k + 9 + (2n^2+4)(i-1)) \\ &= -f^*\left(c_{\frac{k+2}{2}+i,1}\right) \end{aligned}$$

and

$$\begin{aligned} f^*(c_{2+i,2}) &= (n^2k + 3n^2 + 4nk + 6n + 4k + 5 + (2n^2+4)(i-1)) \\ &= -f^*\left(c_{\frac{k+2}{2}+i,2}\right), \end{aligned}$$

for $1 \leq i \leq n-1$

$$f^*(x_{1+i}) = (2i+1) = -f^*(y_i)$$

and

$$\begin{aligned} f^*(x_{n+2+i}) &= -(2n+3+2i) = -f^*(y_{n+i}), \\ f^*(x_{n+1}) &= 2(n-1) = -f^*(y_n) \end{aligned}$$

and $f^*(x_{2n+2}) = -4n = -f^*(y_{2n})$, for $1 \leq i \leq \left(\dfrac{k-2}{2}\right)$ and $1 \leq j \leq n$,

$$\begin{aligned} f^*(x_{n+(n+1)i+2+j}) &= (2n+2ni+2k+1+2j) \\ &= -f^*\left(x_{n+(n+1)\left(\frac{k-2}{2}+i\right)+2+j}\right), \end{aligned}$$

for $1 \leq i \leq n\left(\dfrac{k-2}{2}\right)$,

$$\begin{aligned} f^*(y_{2n+i}) &= (nk+2n+2k+1+2i) \\ &= -f^*\left(y_{\frac{nk+2n}{2}+i}\right). \end{aligned}$$

We get

$$\begin{aligned} f^*(V(G)) = \{&\pm 1, \pm 2n, \pm 3, \pm 5, \pm 7, \ldots, (2n-1) \pm (2n+3), \\ &\pm (2n-2), \pm 4n, \pm(2n+5), \\ &\pm (2n+7), \ldots, \pm(4n+1), \pm[4n+7], \pm[4n+9], \ldots, \\ &\pm [4n+k+3], \pm[4kn+5k+2], \pm[4kn+5k+6], \ldots, \\ &\pm [4kn+7k-6], \pm(2nk+4n+5k+3), \\ &\pm (2nk+4n+5k+7), \ldots, \pm(2nk+4n+7k-5), \\ &\pm (4n+2k+3), \pm(4n+2k+5), \\ &\pm (4n+2k+7), \ldots, \pm(nk+2n+2k+1), \\ &\pm (kn+2k+2n+3), \pm(kn+2k+2n+5), \\ &\pm (kn+2k+2n+7), \ldots, \pm(2kn+2k+1), \\ &\pm (n^2+4n-1), \pm(3n^2+10n+3), \\ &\pm (4nk+5n^2+6n+2k+9), \pm(4nk+7n^2+6n+2k+13), \ldots, \\ &\pm (4nk+n^2k+n^2+6n+4k+1), \\ &\pm (4nk+n^2k+3n^2+6n+4k+5), \\ &\pm (4nk+n^2k+5n^2+6n+4k+9), \ldots, \\ &\pm (4kn+2n^2k-n^2+6n+6k-3)\} \cup \{-2\}. \end{aligned}$$

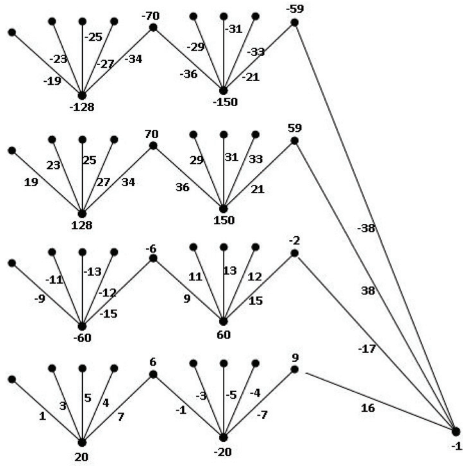

Figure 3:

From the above labeling f is an edge pair sum labeling. ■

The example for the edge pair sum graph labeling of $WT(3 : 4)$ is shown in Figure 3.

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