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# Exactness of Higher Order Nonlinear Ordinary Differential Equations

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**Abstract**

Given a vector field $\mathbf{F} \in C^1(R)$, where $R$ is simply connected domain in $I\!R^n$. We give conditions for the existence of a potential function $\Psi \in C^2(R)$, such that $\nabla\Psi = \mathbf{F}$, for all $\mathbf{x} \in R$. Based on this result, we define the concept of exactness for a class of a nonlinear ordinary differential equations. Using this concept, a higher order nonlinear differential equations can be solved.

**AMS subject classification:** 34A25, 34A30.
**Keywords:** Higher order differential equation, Exact equations, Potential function.

## 1. Introduction

The concept of exactness for a class of a second order nonlinear differential equations was presented [1] with a well-defined method of solution. The notion of integrating factor were introduced to convert differential equation that is not exact into an exact one.

Higher order nonlinear differential equations play an important role in Applied Mathematics, Physics, and Engineering [2, 3, 4, 5, 6]. To find the general solution of such equations is not an easy problem. In this paper, a class of higher order nonlinear differential equations will be solved.

The outline of the paper: we give mathematical formulation for the exactness of a class of higher order nonlinear differential equations. Also, we present a technique to solve the exact equations by reducing their order.

## 2. The Main Result

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n$ be the standard basis of $I\!R^n$, and let $R$ be a simply connected domain in $I\!R^n$. Define a function $\mathbf{F} : R \longrightarrow I\!R^n$ to be

$$\begin{aligned}\mathbf{F}(x_1, x_2, x_3, \ldots, x_n) &= \sum_{i=1}^{n} F_i(x_1, x_2, x_3, \ldots, x_n)\mathbf{e}_i \\ &= (F_1(x_1, x_2, x_3, \ldots, x_n), F_2(x_1, x_2, x_3, \ldots, x_n), \ldots, \\ &\quad F_n(x_1, x_2, x_3, \ldots, x_n)).\end{aligned}$$

Then

$$\begin{aligned}\nabla\Psi &= \sum_{i=1}^{n} \frac{\partial\Psi}{\partial x_i}(x_1, x_2, x_3, \ldots, x_n)\,\mathbf{e}_i \\ &= \left(\frac{\partial\Psi}{\partial x_1}(x_1, x_2, x_3, \ldots, x_n), \frac{\partial\Psi}{\partial x_2}(x_1, x_2, x_3, \ldots, x_n), \ldots, \right. \\ &\quad \left. \frac{\partial\Psi}{\partial x_n}(x_1, x_2, x_3, \ldots, x_n)\right).\end{aligned}$$

The following result gives the conditions for which a function $\Psi \in C^2(R)$ exists with $(\nabla\Psi)(\mathbf{x}) = \mathbf{F}(\mathbf{x})$, for all $\mathbf{x} \in R$.

**Theorem 2.1.** Assume that

i) $F_i(x_1, x_2, x_3, \ldots, x_n)$ and $\dfrac{\partial F_i}{\partial x_j}(x_1, x_2, x_3, \ldots, x_n)$ on $R$, for all $i, j = 1, 2, 3, \ldots, n$ are continuous in a simply connected domain $R \subseteq I\!R^n$, and

ii) $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$, for all $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, \ldots, i-1$.

Then there exists a function $\Psi \in C^2(R)$ satisfies that $(\nabla\Psi)(\mathbf{x}) = \mathbf{F}(\mathbf{x})$, for all $\mathbf{x} \in R$. The opposite direction is also correct.

*Proof.* Assume that $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$, for all $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, \ldots, i-1$. To construct a function $\Psi$ such that $\nabla\Psi = \mathbf{F}$, we fix $(t_1, t_2, \ldots, t_n) \in R$, and we define

$$\begin{aligned}\Psi(x_1, x_2, \ldots, x_n) &= \int_{t_1}^{x_1} F_1(\xi, x_2, x_3, \ldots, x_n)d\xi + \int_{t_2}^{x_2} F_2(t_1, \xi, x_3, \ldots, x_n)d\xi \\ &\quad + \int_{t_3}^{x_3} F_3(t_1, t_2, \xi, \ldots, x_n)d\xi + \cdots \\ &\quad + \int_{t_n}^{x_n} F_n(t_1, t_2, t_3, \ldots, t_{n-1}, \xi)d\xi.\end{aligned}$$

Clearly,

$$\frac{\partial \Psi}{\partial x_1} = F_1(x_1, x_2, x_3, \ldots, x_n).$$

By differentiating $\Psi$ with respect to $x_2$, and by using the assumption $\dfrac{\partial F_1}{\partial x_2} = \dfrac{\partial F_2}{\partial x_1}$, we get

$$\begin{aligned}
\frac{\partial \Psi}{\partial x_2}(x_1, x_2, x_3, \ldots, x_n) &= \int_{t_1}^{x_1} \frac{\partial F_1}{\partial x_2}(\xi, x_2, x_3, \ldots, x_n)d\xi + F_2(t_1, x_2, x_3, \ldots, x_n) \\
&= \int_{t_1}^{x_1} \frac{\partial F_2}{\partial \xi}(\xi, x_2, x_3, \ldots, x_n)d\xi + F_1(t_1, x_2, x_3, \ldots, x_n) \\
&= F_2(x_1, x_2, x_3, \ldots, x_n) - F_2(t_1, x_2, x_3, \ldots, x_n) \\
&\quad + F_2(t_1, x_2, x_3, \ldots, x_n) \\
&= F_2(x_1, x_2, x_3, \ldots, x_n).
\end{aligned}$$

In general, differentiate $\Psi$ with respect to $x_i$, and use the assumption $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$, $j = 1, 2, \ldots, i-1$, we get

$$\begin{aligned}
&\frac{\partial \Psi}{\partial x_i}(x_1, x_2, x_3, \ldots, x_n) \\
&\quad = \sum_{j=1}^{i-1} \int_{t_j}^{x_j} \frac{\partial F_j}{\partial x_i}(t_1, t_2, \ldots, t_{j-1}, \xi, x_{j+1}, \ldots, x_n)d\xi \\
&\qquad + F_i(t_1, t_2, \ldots, t_{i-1}, x_i, x_{i+1}, \ldots, x_n) \\
&\quad = \sum_{j=1}^{i-1} \int_{t_j}^{x_j} \frac{\partial F_i}{\partial \xi}(t_1, t_2, \ldots, t_{j-1}, \xi, x_{j+1}, \ldots, x_n)d\xi \\
&\qquad + F_i(t_1, t_2, \ldots, t_{i-1}, x_i, x_{i+1}, \ldots, x_n).
\end{aligned} \tag{2.1}$$

Since

$$\begin{aligned}
&\int_{t_j}^{x_j} \frac{\partial F_i}{\partial \xi}(t_1, t_2, \ldots, t_{j-1}, \xi, x_{j+1}, \ldots, x_n)d\xi \\
&= F_i(t_1, t_2, \ldots, t_{j-1}, x_j, x_{j+1}, \ldots, x_n) - F_i(t_1, t_2, \ldots, t_{j-1}, t_j, x_{j+1}, \ldots, x_n).
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{j=1}^{i-1} \int_{t_j}^{x_j} \frac{\partial F_i}{\partial \xi}(t_1, t_2, \ldots, t_{j-1}, \xi, x_{j+1}, \ldots, x_n)d\xi \\
&= \sum_{j=1}^{i-1} \left\{ F_i(t_1, t_2, \ldots, t_{j-1}, x_j, x_{j+1}, \ldots, x_n) - F_i(t_1, t_2, \ldots, t_{j-1}, t_j, x_{j+1}, \ldots, x_n) \right\} \\
&= F_i(x_1, x_2, \ldots, x_n) - F_i(t_1, t_2, \ldots, t_{i-1}, x_i, x_{i+1}, \ldots, x_n).
\end{aligned}$$

Substitute this in Eq. (2.1), we get

$$\frac{\partial \Psi}{\partial x_i}(x_1, x_2, x_3, \ldots, x_n) = F_i(x_1, x_2, x_3, \ldots, x_n).$$

Therefore

$$\begin{aligned}(\nabla\Psi)(x_1, x_2, x_3, \ldots, x_n) &= \sum_{i=1}^{n} \frac{\partial \Psi}{\partial x_i}(x_1, x_2, x_3, \ldots, x_n)\mathbf{e}_i \\ &= \sum_{i=1}^{n} F_i(x_1, x_2, x_3, \ldots, x_n)\mathbf{e}_i = \mathbf{F}(x_1, x_2, x_3, \ldots, x_n).\end{aligned}$$

The proof of the opposite direction is easy. In fact, If there exists function $\Psi \in C^2(R)$, so that

$$\nabla\Psi = \mathbf{F}.$$

Then

$$\frac{\partial \Psi}{\partial x_i}(x_1, x_2, x_3, \ldots, x_n) = F_i(x_1, x_2, x_3, \ldots, x_n),$$

which implies that

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial^2 \Psi}{\partial x_j \partial x_i} = \frac{\partial^2 \Psi}{\partial x_i \partial x_j} = \frac{\partial F_j}{\partial x_i}, \quad \text{for } i = 1, 2, \ldots, n, \text{ and } j = 1, 2, \ldots, i-1.$$

■

Now, we consider the following nonlinear ordinary differential equation of order $n$:

$$\begin{aligned}&F_n\left(t, y, y', y'', \ldots, y^{(n-1)}\right) y^{(n)} + F_{n-1}\left(t, y, y', y'', \ldots, y^{(n-1)}\right) y^{(n-1)} + \cdots \\ &+ F_1\left(t, y, y', y'', \ldots, y^{(n-1)}\right) y' + F_0\left(t, y, y', y'' \ldots, y^{(n-1)}\right) = 0. \end{aligned} \tag{2.2}$$

Let $R$ be a simply connected domain in $I\!R^{n+1}$. Assume there exists a function $\Psi(t, y, y', y'', \ldots, y^{(n-1)}) \in C^2(R)$ such that

$$\begin{aligned}\frac{\partial \Psi}{\partial t} &= F_0\left(t, y, y', y'' \ldots, y^{(n-1)}\right) \\ \frac{\partial \Psi}{\partial y} &= F_1\left(t, y, y', y'' \ldots, y^{(n-1)}\right) \\ \frac{\partial \Psi}{\partial y'} &= F_2\left(t, y, y', y'' \ldots, y^{(n-1)}\right) \\ &\vdots \\ \frac{\partial \Psi}{\partial y^{(n-1)}} &= F_n\left(t, y, y', y'' \ldots, y^{(n-1)}\right). \end{aligned} \tag{2.3}$$

Then, Eq. (2.2) becomes

$$\frac{\partial \Psi}{\partial y^{(n-1)}} y^{(n)} + \frac{\partial \Psi}{\partial y^{(n-2)}} y^{(n-1)} + \cdots + \frac{\partial \Psi}{\partial y} y' + \frac{\partial \Psi}{\partial t} = 0.$$

By applying the chain role to this equation, we get

$$\frac{d\Psi}{dt} = 0.$$

Hence, $\Psi\left(t, y, y', y'', \ldots, y^{(n-1)}\right) = c$ reduces Eq. (2.2) into a nonlinear ordinary differential equation of order $n - 1$.

**Definition 2.2.** A nonlinear differential equation (2.2) is called exact if there exists a function $\Psi\left(t, y, y', y'', \ldots, y^{(n-1)}\right) \in C^2(R)$, where $R$ is a simply connected domain in $I\!R^{n+1}$, such that (2.3) are satisfied.

**remark 2.3.** By using the above definition and Theorem 2.1, Eq. (2.2) is exact equation if

i) $F_0, F_1, F_2, \ldots, F_n$ are continuous with their first partial derivatives with respect to $t, y, y', \ldots, y^{(n-1)}$, on a simply connected domain $R$ in $I\!R^{n+1}$, and

ii) $\dfrac{\partial F_i}{\partial y^{(j-1)}} = \dfrac{\partial F_j}{\partial y^{(i-1)}}$, for all $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, \ldots, i-1$ (as a notation $\partial y^{(0)} = \partial y$), and $\dfrac{\partial F_i}{\partial t} = \dfrac{\partial F_0}{\partial y^{(i-1)}}$, for all $i = 1, 2, 3, \ldots, n$.

**Remark 2.4.** Consider the following third order differential equation

$$F_3(t, y, y', y'')y''' + F_2(t, y, y', y'')y'' + F_1(t, y, y', y'')y' + F_0(t, y, y', y'') = 0. \quad (2.4)$$

Then, this equation is exact if

i) $F_0, F_1, F_2$ and $F_3$ are continuous with their first partial derivatives with respect to $t, y, y'$, and $y''$ on a simply connected domain $R$ in $I\!R^4$, and

ii) the following equalities are satisfied:

$$\begin{aligned} &\partial_{y''} F_0 = \partial_t F_3,\ \ \partial_{y''} F_1 = \partial_y F_3,\ \ \partial_{y''} F_2 = \partial_{y'} F_3, \partial_{y'} F_0 = \partial_t F_2, \\ &\partial_{y'} F_1 = \partial_y F_2,\ \text{ and } \partial_y F_0 = \partial_t F_1, \end{aligned} \quad (2.5)$$

where $\partial_\eta F = \dfrac{\partial F}{\partial \eta}$.

In this case, the function $\Psi\left(t, y, y', y''\right)$ that reduces Eq. (2.4) into a second order differential equation is given by

$$\begin{aligned}\Psi\left(t, y, y', y''\right) &= \int_{t_0}^{t} F_0\left(\xi, y, y', y''\right) d\xi + \int_{y_0}^{y} F_1\left(t_0, \xi, y', y''\right) d\xi \\ &\quad + \int_{y'_0}^{y'} F_2\left(t_0, y_0, \xi, y''\right) d\xi \\ &\quad + \int_{y''_0}^{y''} F_3\left(t_0, y_0, y'_0, \xi\right) d\xi. \qquad (2.6)\end{aligned}$$

**Example 2.5.** Consider the following nonlinear third order differential equation:

$$\begin{cases} y''' + (3t^2 + y^2)y'' + (2yy' + 6t)y' + 6ty' + 6y = 0, \\ y(0) = 3,\ y'(0) = -9,\ y''(0) = 81. \end{cases} \qquad (2.7)$$

Then

$$F_3(t, y, y', y'') = 1,\ F_2(t, y, y', y'') = 3t^2 + y^2,\ F_1(t, y, y', y'') = 2yy' + 6t,$$

and

$$F_0(t, y, y', y'') = 6ty' + 6y.$$

It is easy to see that the conditions in (2.5) are satisfied. Hence, the third order differential equation (2.7) is exact. Using the formula (2.6), we get that

$$\begin{aligned}\Psi\left(t, y, y', y''\right) &= y'' + y^2y' + 3t^2y' + 6ty \\ &= y'' + (y^2 + 3t^2)y' + 6ty.\end{aligned}$$

Hence,

$$\Psi\left(t, y, y', y''\right) = y'' + (y^2 + 3t^2)y' + 6ty = c,$$

reduces Eq. (2.7) into the following nonlinear second order differential equation

$$y'' + (y^2 + 3t^2)y' + 6ty = c,$$

By applying the initial conditions, we get $c = 0$. Therefore, the above equation becomes

$$y'' + (y^2 + 3t^2)y' + 6ty = 0.$$

Following the same argument in [1], the above nonlinear second order differential equation is exact, and hence, it is reduced to the following first order differential equation:

$$y' + 3t^2y + \frac{y^3}{3} = 0.$$

This equation is Bernoulli equation and its solution is given by

$$y(t) = \frac{3\exp\{-t^3\}}{\left(1 + 6\int_0^t \exp\{-2\xi^3\}d\xi\right)^{\frac{1}{2}}}.$$

## 3. Conclusions and Remarks

In this paper, we imposed conditions on the following nonlinear differential equation of order $n$:

$$F_n\left(t, y, y', y'', \ldots, y^{(n-1)}\right) y^{(n)} + F_{n-1}\left(t, y, y', y'', \ldots, y^{(n-1)}\right) y^{(n-1)} + \cdots$$
$$+ F_1\left(t, y, y', y'', \ldots, y^{(n-1)}\right) y' + F_0\left(t, y, y', y'' \ldots, y^{(n-1)}\right) = 0, \qquad (3.1)$$

so that it is exact. In addition, we introduced a technique to reduce the order of this equation into an equation of order $n-1$. We also presented an example to solve a nonlinear third order differential equation. For further studies, if Eq. (3.1) is not exact, then it is reasonable to look for integrating factors that could transform it to an exact one.

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