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# Observations on the Quartic equation with six unknowns $(x^3 - y^3)z = (w^2 - p^2)R^2$

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**Abstract:**

We present different patterns of non-zero distinct integer solutions to the homogeneous quartic equation with six unknowns given by $(x^3 - y^3)z = (w^2 - p^2)R^2$. A few interesting properties among the solutions are also given.

**Keywords:** Homogeneous quartic, quartic with six unknowns, integer solutions.

**2010 Mathematical sub-classification:** 11D25

**Notations**:

$p_n^m$ - Pyramidal number of rank n with size m.

$T_{m,n}$ – polygonal number of rank n with size m

$p_n$ - Pronic number of rank n

$OH_n$ - Octahedral number of rank n

**Introduction**:
The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, biquadratic Diophantine equations, homogeneous and non-homogeneous

have aroused the interest of numerous mathematicians since antiquity [1-5]. In this context one may refer [6-9] for various problems on the biquadratic Diophantine equation with four variables and [10-14] for five variables and [15-16] for six variables. It is towards this end, this paper concerns with the problem of determining non-trival integral solutions of the homogeneous equation of degree four with six unknowns given by $(x^3-y^3)z=(w^2-p^2)R^2$.

**Method of Analysis:**

The homogeneous biquadratic equation to be solved is

$$(x^3-y^3)z=(w^2-p^2)R^2 \qquad (1)$$

We present below different patterns of integer solutions to (1)

**Pattern 1:**

Introduction of the transformations

$$\left.\begin{aligned} x&=au+bv\\ y&=au-bv\\ z&=2au\\ w&=auv+b\\ p&=auv-b \end{aligned}\right\} \qquad (2)$$

in (1) reduces it to

$$b^2v^2+3a^2u^2=R^2 \qquad (3)$$

Note that (3) is satisfied by

$$\left.\begin{aligned} u(a,b,T,S)&=2ab^2TS\\ v(a,b,T,S)&=ab^2[3T^2-S^2]\\ R(a,b,T,S)&=ab^2[3T^2+S^2] \end{aligned}\right\} \qquad (4)$$

Substituting the values of $u\ and\ v$ in (2), the values of $x,y,z,w\ and\ p$ are given by

$$\left.\begin{aligned} x(a,b,T,S)&=a^2b^2[2TS+3T^2-S^2]\\ y(a,b,T,S)&=a^2b^2[2TS-3T^2+S^2]\\ z(a,b,T,S)&=4a^2b^2TS\\ w(a,b,T,S)&=2a^4b^3TS[3T^2-S^2]+b\\ p(a,b,T,S)&=2a^4b^3TS[3T^2-S^2]-b \end{aligned}\right\} \qquad (5)$$

Thus (4) and (5) represent the integer solutions of (1)

**Properties:**

1. $x(a,b,T,S)+y(a,b,T,S)$ is expressed as different of two squares.
2. Each of the following expression is a Nasty Number.

(i) $x(a,b,6S,S)+y(a,b,6S,S)$

(ii) $6p_R^5[x(a,b,T,t_{3,R})+y(a,b,T,t_{3,R})]$.

(iii) $\frac{3b(x^2-y^2)}{w+p}$

**Remarkable Observation**:-

It is worth to note that

$$x^2-y^2=w^2-p^2$$

$$\Rightarrow x^2+p^2=y^2+w^2=N(say)$$

Here N represents second order Ramanujan number as it is expressed as the sum of two squares in two different ways. Numerical illustrations are exhibited in Table 1 below.

**Table 1: Numerical Examples**

| a | b | R | S | X | p | Y | w | N |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 1 | 15 | 43 | -7 | 45 | 2074 |
| 2 | 4 | 1 | 2 | 192 | -4100 | 320 | -4092 | 16846864 |
| 2 | 2 | 1 | 1 | 64 | 510 | 0 | 514 | 264196 |
| 1 | 2 | 1 | 2 | 12 | -34 | 20 | -30 | 1300 |
| 3 | 3 | 1 | 5 | -972 | -481143 | 2592 | -481137 | 2.31499312X$10^{11}$ |

**Pattern 2:**

Write (3) in the form of ratio as

$$\frac{R+bv}{au}=\frac{3au}{R-bv}=\frac{\alpha}{\beta},\quad \beta\neq 0 \tag{6}$$

The above equation (6) is equivalent to the system of equations

$$-au\alpha+\beta R+bv\beta=0$$

$$3au\beta-\alpha R+bv\alpha=0$$

Employing the method of cross multiplication, we have

$$u=2b\alpha\beta$$

$$v = a\alpha^2 - 3a\beta^2$$

$$R = 3ab\beta^2 + ba\alpha^2 \qquad (7)$$

Substituting the values of u and v in (2), the values of x,y,z,w and p are given by

$$\left.\begin{aligned} x &= 2ab\alpha\beta + ba\alpha^2 - 3ab\beta^2 \\ y &= 2ab\alpha\beta - ba\alpha^2 + 3ab\beta^2 \\ z &= 4ab\alpha\beta \\ w &= 2a^2b\alpha^3\beta - 6a^2b\alpha\beta^3 + b \\ p &= 2a^2b\alpha^3\beta - 6a^2b\alpha\beta^3 - b \end{aligned}\right\} \qquad (8)$$

Thus (7) and (8) represent the integer solutions to (1)

**Properties**:
Each of the following expression is a Nasty Number.

1. $6p_\alpha^5\left[x(a,a,\alpha,t_{3,\alpha}) + y(a,a,\alpha,t_{3,\alpha})\right]$.
2. $3\alpha p_\alpha^5\left[\frac{w+p}{x-y}\right]$.
3. $\frac{3bz(x-y)}{w+p}$

**Remark:**
It is to be noted that (3) may also be written in the form of ratios as given below

**Choice 1:**

$$\frac{R+bv}{3au} = \frac{au}{R-bv} = \frac{\alpha}{\beta}\ ,\beta \neq 0$$

**Choice 2:**

$$\frac{R+bv}{u} = \frac{3a^2u}{R-bv} = \frac{\alpha}{\beta}\ ,\beta \neq 0$$

**Choice 3:**

$$\frac{R+bv}{3a^2u} = \frac{u}{R-bv} = \frac{\alpha}{\beta}\ ,\beta \neq 0$$

**Choice 4:**

$$\frac{R+bv}{3u} = \frac{a^2u}{R-bv} = \frac{\alpha}{\beta}\ ,\beta \neq 0$$

**Choice 5:**

$$\frac{R+bv}{a^2u} = \frac{3u}{R-bv} = \frac{\alpha}{\beta} \quad ,\beta \neq 0$$

Following the procedure as presented above, the corresponding integer solutions to (1) are exhibited below:

**Solutions of Choice 1**:

$x = 2ab\alpha\beta - b\alpha\beta^2 + 3ab\alpha^2$

$y = 2ab\alpha\beta + b\alpha\beta^2 - 3ab\alpha^2$

z=4abαβ

$w = 6a^2b\alpha^3\beta - 2a^2b\alpha\beta^3 + b$

$p = 6a^2b\alpha^3\beta - 2a^2b\alpha\beta^3 - b$

$R = ab\beta^2 + 3ab\alpha^2$

**Solutions of Choice 2:**

$x = 2ab\alpha\beta + b\alpha^2 - 3a^2b\beta^2$

$y = 2ab\alpha\beta - b\alpha^2 + 3a^2b\beta^2$

z=4abαβ

$w = 2ab\alpha^3\beta - 6a^3b\alpha\beta^3 + b$

$p = 2ab\alpha^3\beta - 6a^3b\alpha\beta^3 - b$

$R = 3a^2b\beta^2 + b\alpha^2$

**Solutions of Choice 3:**

$x = 2ab\alpha\beta - b\beta^2 + 3a^2b\alpha^2$

$y = 2ab\alpha\beta + b\beta^2 - 3a^2b\alpha^2$

z=4abαβ

$w = 6a^3b\alpha^3\beta - 2ab\alpha\beta^3 + b$

$$p = 6\alpha^3 b\alpha^3\beta - 2ab\alpha\beta^3 - b$$

$$R = 3\alpha^2 b\alpha^2 + b\beta^2$$

**Solutions of Choice 4:**

$$x = 2ab\alpha\beta + 3b\alpha^2 - a^2 b\beta^2$$

$$y = 2ab\alpha\beta - 3b\alpha^2 + a^2 b\beta^2$$

z=4abαβ

$$w = 6ab\alpha^3\beta - 2\alpha^3 b\alpha\beta^3 + b$$

$$p = 6ab\alpha^3\beta - 2\alpha^3 b\alpha\beta^3 - b$$

$$R = \alpha^2 b\beta^2 + 3b\alpha^2$$

**Solutions of Choice 5:**

$$x = 2ab\alpha\beta - 3b\beta^2 + a^2 b\alpha^2$$

$$y = 2ab\alpha\beta + 3b\beta^2 - a^2 b\alpha^2$$

z=4abαβ

$$w = 2a^3 b\alpha^3\beta - 6ab\alpha\beta^3 + b$$

$$p = 2a^3 b\alpha^3\beta - 6ab\alpha\beta^3 - b$$

$$R = a^2 b\alpha^2 + 3b\beta^2$$

**Pattern 3:**
Consider the transformations

$$\left.\begin{array}{c} x = au + bv \\ y = au - bv \\ z = 2cu \\ w = buv + c \\ p = buv - c \end{array}\right\} \quad (9)$$

The substitution of (9) in (1) leads to the same equation(3)
Write, (3) as

$$b^2v^2 + 3a^2u^2 = R^2 = R^2 * 1 \quad (10)$$

Assume,

$$R = 4a^2b^2(\alpha^2 + 3\beta^2) \quad (11)$$

Write 1 as

$$1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \quad (12)$$

Using (11) and (12) in (10) and applying the method of factorization, define

$$\left(bv + i\sqrt{3}au\right) = \frac{1}{2}\left(1 + i\sqrt{3}\right)\left(2ab\alpha + i2ab\sqrt{3}\beta\right)^2 \quad (13)$$

Equating the real and imaginary parts of (13), we have

$$u = 2ab^2(\alpha^2 - 3\beta^2 + 2\alpha\beta)$$

$$v = 2a^2b(\alpha^2 - 3\beta^2 - 6\alpha\beta)$$

In view of (9), the corresponding solutions of (1) are given by

$$x = 2a^2b^2(2\alpha^2 - 6\beta^2 - 4\alpha\beta)$$

$$y = 2a^2b^2(8\alpha\beta)$$

$$z = 4ab^2c(\alpha^2 - 3\beta^2 + 2\alpha\beta)$$

$$w = 4a^3b^4[\alpha^4 - 18\alpha^2\beta^2 - 4\alpha^3\beta + 12\alpha\beta^3 + 9\beta^4] + c$$

$$p = 4a^3b^4[\alpha^4 - 18\alpha^2\beta^2 - 4\alpha^3\beta + 12\alpha\beta^3 + 9\beta^4] - c$$

along with (11)

**Properties:**

1. Each of the following expression is a Nasty Number.

(i). $6\left(\frac{y(a,b,\alpha,t_{3,\alpha})}{p_\alpha^5}\right)$

(ii). $2\left(\frac{y(a,b,\alpha,t_{\alpha+1})}{p_\alpha^5}\right)$

2. $t_{26,\beta} - \frac{y(a,b,1,\beta)}{a^2b^2} \equiv -1 \pmod 3$

3. $4 - \frac{y(a,b,1,\beta)}{a^2b^2}$ is a perfect square, for the values of $\beta$ given by

$$\beta_n = \frac{1}{6}\left[2 - \left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right] \forall n = 1,2,3, \dots \dots$$

**Remark**:
Note that inaddition to(12), 1 may also be written in the following two representations

**Representation 1:**

$$1 = \frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49}$$

**Representation 2:**

$$1 = \frac{(1+i15\sqrt{3})(1-i15\sqrt{3})}{26^2}$$

Following the procedure as in pattern (3), The solutions of (1) corresponding to representation 1 are given below:

$$x = 7a^2b^2(5\alpha^2 - 15\beta^2 - 22\alpha\beta)$$

$$y = 7a^2b^2(3\alpha^2 - 9\beta^2 + 26\alpha\beta)$$

$$z = 14ab^2c(4\alpha^2 - 12\beta^2 + 2\alpha\beta)$$

$$w = 49a^3b^4[4\alpha^4 - 72\alpha^2\beta^2 - 94\alpha^3\beta + 282\alpha\beta^3 + 36\beta^4] + c$$

$$p = 49a^3b^4[4\alpha^4 - 72\alpha^2\beta^2 - 94\alpha^3\beta + 282\alpha\beta^3 + 36\beta^4] - c$$

$$R = 49a^2b^2(\alpha^2 + 3\beta^2)$$

The solutions of (1) corresponding to representation 2 areas follows**:**

$$x = 26a^2b^2(16\alpha^2 - 48\beta^2 - 88\alpha\beta)$$

$$y = 26a^2b^2(14\alpha^2 - 42\beta^2 + 92\alpha\beta)$$

$$z = 52ab^2c(15\alpha^2 - 45\beta^2 + 2\alpha\beta)$$

$$w = 26^2a^3b^4[15\alpha^4 - 270\alpha^2\beta^2 - 1348\alpha^3\beta + 4044\alpha\beta^3 + 135\beta^4] + c$$

$$p = 26^2a^3b^4[15\alpha^4 - 270\alpha^2\beta^2 - 1348\alpha^3\beta + 4044\alpha\beta^3 + 135\beta^4] - c$$

$$R = 26^2a^2b^2(\alpha^2 + 3\beta^2)$$

**Conclusion:**
In this paper we have presented different patterns of non-zero distinct integer

solutions to the equation $(x^3 - y^3)z = (w^2 - p^2)R^2$. As quartic equations are rich in variety, one may search for other choices of quartic equations with variables greater than or equal to 6 and determine their corresponding properties.

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