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Direct Permutation Graphs

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Abstract

Using Permutations of a finite set, a new graph was constructed by Koh and Ree (see [1]). In this new paper, a variation of this graph is constructed by the author which is called ‘Direct Permutation Graph’. It is explained in this paper that these two graphs are indeed different. Also some basic, interesting properties are being obtained.

Key words : Permutation Graph , Direct Permutation Graph & Connected Components

AMS Subject Classification : 05C30

Introduction:

Permutation graphs are well studied in the past decade or so. This study perhaps originated from the study of 'perfect graphs' (chromatic numbers and clique numbers coincide for the graph and all its induced sub graphs). They have, for example, several applications in air traffic control. We have invented a new graph based on similar construction of permutation graphs, which can be termed a "Direct Permutation Graph". Some interesting fundamental properties of the direct permutation graphs are studied in this paper.

2. Permutation graphs

First, we describe permutation graphs, highlighting some results (without proofs, but with references) which have the platform for Direct Permutation Graphs in the next section.

2.1 Definition:

For any positive integer n, we denote by [n] the set {l, 2,..., n}. Let π be a permutation of the set [n], which is thus an element of the symmetric group S_n which contains n! elements.

We sometimes write $\pi = \pi(1)\pi(2) \ldots \pi(n)$ (the image integers are put in an ordered sequence even though this is not the best notation) or in terms of the (unique) cycle decomposition of π (which is standard).

The permutation graph $G_\pi = (V,E)$ corresponding to π is defined as follows : V = [n] and E = $\{\pi(i)\ \pi(j) \mid$ if and only if i and j satisfy the conditions $i < j \Rightarrow \pi(i) > \pi(j)\}$

This is equivalent to : $ij \in E \Leftrightarrow (i - j)\ (\pi(i)^{-1} - \pi(j)^{-1}) < 0$

But we prefer to follow the first formulation. Clearly G_π is a finite, simple, undirected graph.

Koh and Ree in [1] found some interesting properties of the graph G_π .We shall recall some of them, without proof, in order to compare similar situations in our new graph.

2.2 Proposition

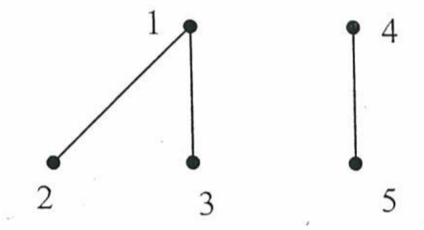
i) G_π is the null graph if and only if and only if π = identity

ii) G_π is the complete graph K_n if and only if π = **n (n-1) ...2.1.**

2.3 An example :

n = *5* and π= 23154.

Then G_π is the following graph.



2.4 Proposition

i) The complement graph $\overline{G_\pi}$ of G_π is also a permutation graph.

In fact, $\overline{G_\pi} = G_\sigma$ where σ is the permutation defined by σ(i) = π(n-i + 1) for i = 1,2,...,n.

ii) G_π and $G_{\pi^{-1}}$ are isomorphic.

3. Connected permutation graphs G_π

3.1 Proposition:

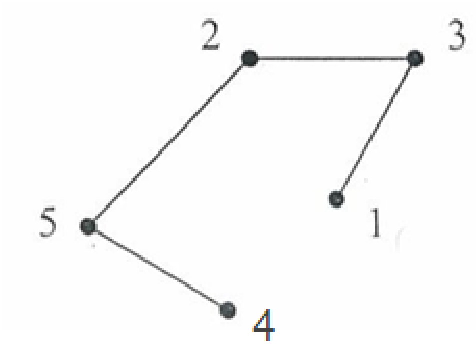
Let $\pi \in S_n$ such that n comes ahead of 1 in the arrangement

π(l),π(2),... , π(n). Then G_π is connected.

3.2 Remark:

The above condition is not necessary for the connectivity of G_π

For instance, π = 31524. (It is to be noted that 5 comes after 1) The graph G_π is drawn below which is connected.



3.3Proposition:

Let $\pi = \pi(1)\pi(2) \ldots \pi(n) \in S_n$. Then G_π is disconnected if and only if there exists i < n such that { π(l),π(2), ... , π(i) } = {l,2,...,i} and {π(i+l), π(i+2), ... , π(n) } = {i+1, i+2,...,n}.

4. Partitions vis-a-vis G_π.

4.1 Definition:

A partition λ of n is a set of position numbers whose sum is n; in notation $\lambda \mapsto$ n.

We usually write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r = 1^{w1} \; 2^{w2} \ldots n^{Wn}$ where w_i is the number of occurrences of part i in λ.

A composition of a number n is an ordered partition of positive numbers whose sum is n, denoted by λ n. One can easily check that the number of partitions and compositions of 4 are respectively 5 and 8.

Let w_λ be the number of different compositions by rearrangement of the parts λ_i 's of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$.

Then clearly $w_\lambda = \dfrac{r!}{w_1!w_2!\ldots w_n!}$

For instance $w_{(2,2,1)} = \dfrac{3!}{1!2!} = 3$

4.2. Theorem

$$n! = c_n + d_n = \sum_{\lambda \mapsto n} \left(\sum_{\lambda_i \in \lambda} c\lambda_i \right) = \sum_{\lambda \mapsto n} w_\lambda \left(\sum_{\lambda_i \in \lambda} c\lambda_i \right)$$

where c_n and d_n denote respectively the set of connected G_πs and disconnected G_πs as π runs through S_n.

5. Direct Permutation Graphs.

We now introduce our new concept of Direct Permutation Graphs.

5.1Definition:

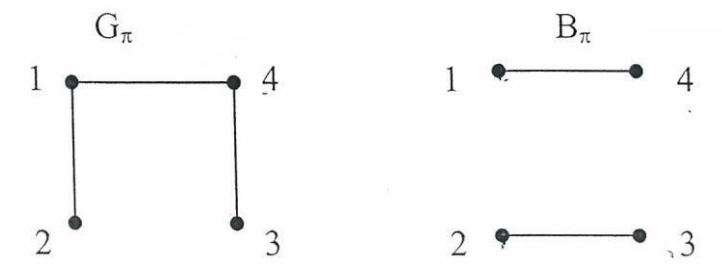
The direct permutation graph $B_\pi = (V,E)$ for any permutation π has $V = \{l,2,\ldots,n\}$ and $ij \in E$ if and only if $i + j = \pi(i) + \pi(j)$. Since the definition depends directly on the permutations we call the graph the Direct Permutation graph which should be the forerunner to the permutation graph in [1]. we remark that the original definition in [1] involves the inverse permutation in some form where as our definition involves the permutations directly and hence the name.

5.2. Remark:

The two graphs G_π and B_π are distinct in general.

Take n = 4 for example and $\pi = 2413$.

Then G_π and B_π are drawn below.



5.3. Proposition:
If $n > 2$, $B_\pi$is complete if and only if π = identity.

Proof:
First assume that π = identity.

Then for any pair of vertices i and j, $i+j = \pi(i) + \pi(j)$. Hence ij is an edge for all $i \neq j$ proving B_π is complete.

For the converse, assume B_π is complete and let that $\pi \neq$ identity

Case 1 :
In the unique cycle decomposition of π, assume that there is atleast one cycle $(i_1, i_2, \ldots, i_r)$, $r \geq 3$. Since B_π is complete, $i_1 i_2 \in E$. Hence $i_1 + i_2 = \pi(i_1) + \pi(i_2)$. But Since $\pi(i_1) = i_2$ and $\pi(i_2) = i_3$, the above gives $i_1 + i_2 = i_2 + i_3$, forcing $i_1 = i_3$, not correct since $r > 2$.

Case 2 :
$\pi$is a disjoint product of transpositions. Since disjoint transpositions commute, we can write $\pi = (ij)\ (ik) \ldots (rs)$.

Clearly $1 < j$ and we can take $i < k$. Since $n > 2$, i, j, k are distinct.

Now $1+i < j + i < j + k = \pi(l) + \pi(i)$.

This clearly shows that 1i is not an edge, contrary to the assumption that B_π is complete.

Therefore π must be the identity. Compare this result with the corresponding one for G_π.

Let $\boldsymbol{l(\pi)}$ denote the length of the smallest cycle in the (unique) cycle decomposition of π.

5.4 Proposition:
Let $n > 2$. Then, if B_π is the null graph, $l(\pi) > 2$.

Proof:
Assume that B_π has no edges.

Suppose $l(\pi) = 1$. If there are two or more vertices i and j such that $\pi(i) = i$ and $\pi(j) = j$, which clearly means $i + j = \pi(i) + \pi(j)$.

This shows that ij is an edge, violating our assumption that B_π is null. Next suppose there is a unique vertex i such that $\pi(i) = i$.

Let jk be a transposition in π.

Then $j + k = k + j = \pi(j) + \pi(k)$, proving that jk is an edge, again contrary to the assumption.

Thus $l(\pi) \geq 2$. if$l(\pi)= 2$, then π is a distinct product of transpositions, and as before, any transposition (jk) will give an edge, contradiction.

Hence $l(\pi) > 2$, proves the result.

5.5 Remark:

That $l(\pi) > 2$ need not imply that $\mathbf{B}_\pi$ is null and has been seen already in our example given earlier : $\pi= 2413 : 1 + 4 = \pi(l) + \pi(4)$.

6. Connectedness properties of B_π

6.1 Proposition:

Suppose B_π is connected and $\pi(i) = i$ for some i, then B_π must be complete.

Proof:

Let $B_\pi$is connected. First, let $\pi(i) = i$ for exactly one i. Now ij is an edge for some $j(\neq i)$, due to connectivity.

Then $i + j = \pi(i) + \pi(j) = i + \pi(j)$ gives $\pi(j) = j$, contrary to the assumption. Next suppose exactly two vertices are fixed by π.

Then for $k\neq i,j$, $i + k = \pi(i) + \pi(k) = i + \pi(k)$, forcing $\pi(k) = k$, contrary to the assumption.

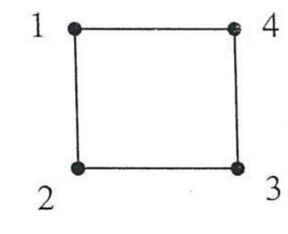
Next if exactly three vertices, say, i, j, k are fixed by π, then we get a triangle with vertices i, j and k in B_π, which is clearly a connected component, destroying connectivity of B_π. We continue this procedure until we arrive at $\pi(i) = i$ for every i, which means thatπ is the identity.

By proposition (5.3), $B_\pi$is complete.

Before seeing further connectivity properties, let us see the following simple examples.

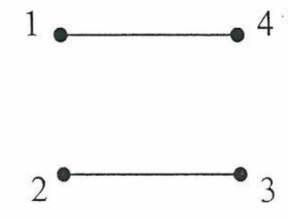
6.2 Some examples. Assume n = 4.

i) $\pi = (12)(34)$. B_π is the following graph.



B_π is connected

ii) $\pi = (14)(23)$, the graph B_π is drawn below, which is disconnected.



B_π is disconnected

The occurrence of 4 in the transposition is not accidental. We have the following powerful result.

6.3Theorem

Suppose π is a product of m transpositions where $m > 2$. Then, if (l m) is a part of π**,** then B_π is disconnected.

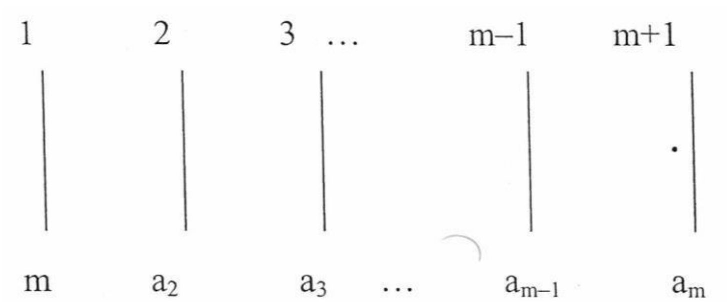
Proof:

Let $\mathbf{B}_\pi$ be connected.

Write $n = 2m$ and take (lm) as the first transposition occurring in π**.**

We can write $\pi = (la_1)(2a_2)(3a_3)...(m+l,a_m)$ (taking $a_1 = m$).

Then we get the following edges in $\mathbf{B}_\pi$ **:**



Clearly $a_i \geq m+2$ for all i = 2, **3 , . . m .**

Without loss of generality, we can assume that 1 is adjacent to some vertex $v_j \neq m$ (due to connectivity) we prove that v_j cannot be any vertex in the top row. In fact, suppose v_j = j for some j such that $2 \leq v_j \leq m+1$ ($v_j \neq m$).

Then $1 + v_j = \pi(1) + \pi(V_j) = m + w_j$ with w_j= one of the a_j s and is hence $\geq m+2$.

This is clearly not possible (as 1 < m and $v_j < w_j$). Hence 1 is adjacent to some a_j in the bottom row.

Then by adjacency condition, $1 + a_j = m+j$ (1) Also m+1 must be adjacent to some other a_t, due to connectivity. Hence $m + 1 + a_t = a_m + t$ (2) Already we have the following inequalities: $a_j > m+1$ for all i and $a_m > a_t$

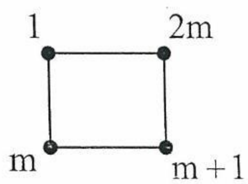
Case 1:

The bottom row numbers strictly increase from m to a_m. We then claim that 1 and m are not adjacent to any other vertex, that is, the edge lm is isolated, breaking the connectivity assumption.

Let $m = a_1 < a_2 < ... < a_j < ... < a_t < .. . < a_m$

This means that the last entry in the top row is m+1 and the last entry, a_m in the bottom row is 2m.

Hence $l+2m = \mathbf{m + m} + 1 = \pi(l) + \pi(2m)$, meaning that 1 and 2m are adjacent. Also by the same argument, m is adjacent to m+1. Hence we get the following subgraph in B_π.

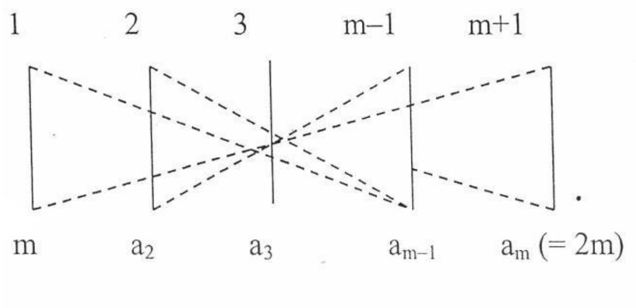


To ensure connectivity, 1 (and hence m) must also be adjacent to some other vertex (since m > 2):

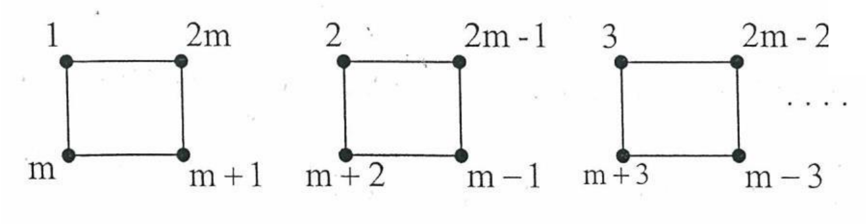
1 to a bottom row vertex m+y and m to the corresponding top row vertex y. But since the bottom row vertices steadily increase,

$1 + m + y = \pi(l) + \pi(m + y) = m + y$

whichis absurd. Hence in this case $B_\pi$must be disconnected. The final graph B_π looks like the following figure

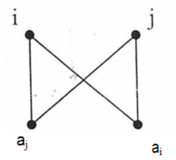


Redrawing, we get B_π as



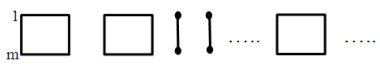
Clearly B_π is disconnected with several components, contrary to our original assumption that B_π is connected.

Case 2 :
There exist entries $\mathbf{a_i}$**,** a_j with bottom row and i, j in the top row such that $a_i > a_j$, $i < j$. In this case we get a component in B_π of the type



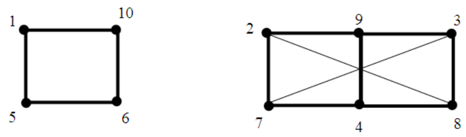
In the case $j+a_i = a_j+i$ and $i+a_j = a_i+j$ (in which there cannot be k different from i,j such that $\pi(k) = a_k = a_i$ or a_j) otherwise we will simply have disconnected edges ($i\ a_i$) and ($j\ a_j$) only

After renumbering (if necessary) we can write B_π as



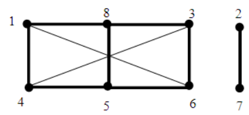
which is clearly disconnected contradicting the original assumption. Thus we have proved finally that $\mathbf{B_\pi}$ **must be disconnected.**

As an example we take n = 10 and π = (15)(27)(38)(49)(6 10) Then B_π is the following graph.



The above is a 'degenerate' case of our argument in case 1, but still B_π is disconnected. Similar degenerate case with single isolated edges can also occur.

As another example take n = 8 and π = (14)(27)(36)(58)



$\mathbf{B_\pi}$ is another degenerate case, but is disconnected nevertheless.

7. Number of connected components of $\mathbf{B_\pi}$

We shall give an algorithm to work out the number of connected components of B_π, through partitions.

7.1 Definition

If we write $n = n_1+n_2+...+n_r$ with $n_1 \geq n_2 \geq ... \geq n_r > 0$, then we say $\lambda = (n_1, n_2, ..., n_r)$ is a partition of n and we write this as $\lambda \mapsto$ n.

If $\mu = (a_1,a_2,..., a_{n_1})\ (a_{n_1+1}, ..., a_{n_2})\ (a_{n_{r-1}+1}, ..., a_{n_r})$ is the unique disjoint cycle decompositions ofπ, then the partition $n_1+n_2+...+n_r = n$ corresponds to μ.

The number of connected components of the graph B_π is given by the following interesting algorithm.

Let π denote the standard partition $\lambda = (1,2,...,n_1)(n_1+1,...,n_2)(n_2+1,..,n_3)...(n_{r-1}+1,\ldots,n_r)$ We simply denote this as$\lambda = (n_1, n_2, ..., n_r)$, $\Sigma n_i = n$.

7.2. Theorem :

The number of connected components of B_π is exactly equal to the number a+1**,** where a is the number of components arising from the adjacencies of vertices in the top rows **of** λ in the young diagram of λ such that $l(\lambda_i) \geq 2$.

Proof:

Let a denote the number of components arising out of these parts. Let the Young diagram corresponding toλ be as given below.

1	2	3			…	n_1
n_1+1	…		…	n_2		
…						
…		…				
	…	n_k				
n_{k+1}						
⋮						
n_k						

Suppose k of the top rows have length ≥ 2.

These k rows will account for $n_1+n_2+...+n_k$ vertices. We can draw the edges among these vertices according to our rule :ij is an edge if and only if $i + j = \pi(i) + \pi(j)$.

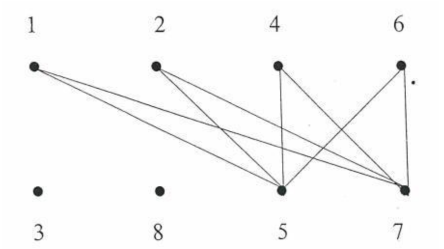
Then the rows **k + 1 , . . r** have just one vertex in each row.

Let **b** denote the number of components arising out of adjacencies between these vertices. This just means that whatever k, these isolated vertices will form one more connected component, i.e., b = 1.

Hence the total number of connected components equals a+1, which proves our theorem.

7.4 Example

π= (123) (45)(67)(8)



The number of components = **2 + 1 = 3**

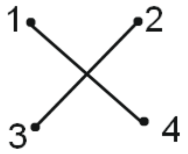
7.5 Remark

The above arguments open the gateway to character theory of the symmetric group S_n. It is well known that the number of (complex) irreducible characters = number of conjugacy classes = number of partitions of n.

The above algorithm therefore would have established a nice connection between the class of irreducible characters of S_n and the class of Direct Permutation Graphs corresponding to the standard partition n. But such a neat connection is not so easy to obtain as yet, for the simple reason that B_π is not invariant under conjugacy in general. In other words, $\pi,\sigma\in S_n$ then B_π and B $_{-1}$ need not be the same. This can be easily seen by the following example

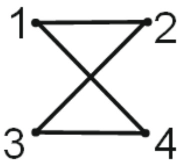
$\pi = (14)\,(23)$

B_π:



$\sigma = (1234)$
$\sigma\pi\sigma^{-1} = (12)\,(34)$

$B_{\sigma\pi\sigma^{-1}}$:



Conclusion :
The above difficulty may be circumverted by a ‘trick’ which we found was more or less the one used by R.C.Orellana in her recent work on ‘Centralizer Algebras and Kronecker Products’ (Conference on Non-Commutative Rings and Representation Theory, Pondicherry, India – 2010) work is in progress.

References :

[1] Koh and Ree , Connected Permutation Graphs , Discrete Mathematics , 307 (2007) , 2628 – 2635
[2] R.C.Orellana,‘Centralizer Algebras and Kronecker Products’ (Conference on Non-Commutative Rings and Representation Theory, Pondicherry, India – 2010)