Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 3 (2015), pp. 1265-1271

http://www.ripublication.com

Identities of generalized Carlitz's q-Bernoulli polynomials under the symmetric group of degree five

Jong Jin Seo

Department of Applied Mathematics,
Pukyong National University,
Busan 608-737, Republic of Korea.
E-mail: seo2011@pknu.ac.kr

Taekyun Kim

Department of Mathematics,
Kwangwoon National University,
Seoul 139-701, Republic of Korea.
E-mail: tkkim@kw.ac.kr

Abstract

The purpose of this paper is to give some new identities of symmetry for the Carlitz's q-Bernoulli polynomials under the symmetric group of degree five.

AMS subject classification:
Keywords: Carlitz's q-Bernoulli polynomials under the symmetric group of degree five.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normally defined as $|p|_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-1/p-1}$. For $d \in \mathbb{N}$ with $(d, p) = 1$, we set

$$X = \varprojlim_{N} (\mathbb{Z}/dp^N\mathbb{Z}), \;\; X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

and

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a(mod\ dp^N)\},\ (N \in \mathbb{N}),$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. The q-analogue of number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let χ be a Dirichlet's character with conductor d. Then the generalized Bernoulli polynomials attached to χ are defined by the generating function to be

$$\frac{t}{e^{dt}-1}\left(\sum_{a=0}^{d-1}\chi(a)e^{at}\right)e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x)\frac{t^n}{n!},\ \text{(see [1]-[9])}. \tag{1.1}$$

when $x = 0$, $B_{n,\chi} = B_{n,\chi}(0)$ are called the generalized Bernoulli numbers attached to χ.

Let $f(x)$ be uniformly differentiable function on $\mathbb{Z}_p$ (or X). Then the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu_q(x) &= \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x = \lim_{N\to\infty}\frac{1}{[dp^N]_q}\sum_{x=0}^{dp^N-1} f(x)q^x \\ &= \int_X f(x)d\mu_q(x),\ \text{(see [8])}.\end{aligned} \tag{1.2}$$

The generalized Carlitz's q-Bernoulli polynomials are given by Kim to be

$$\int_X \chi(y)e^{[x+y]_q t}d\mu_q(y) = \sum_{n=0}^{\infty}\beta_{n,\chi,q}(x)\frac{t^n}{n!},\ \text{(see [7])}. \tag{1.3}$$

Thus, by (1.3), we get

$$\int_X \chi(y)[x+y]_q^n d\mu_q(y) = \beta_{n,\chi,q}(x),\ (n \geq 0),\ \text{(see [7])}. \tag{1.4}$$

The purpose of this paper is to give some new identities of symmetry for the generalized Carlitz's q-Bernoulli polynomials attached to χ under S_5 which are derived from p-adic q-integral on $\mathbb{Z}_p$.

2. Symmetric identities for $\beta_{n,\chi,q}(x)$ under S_5

Let w_1, w_2, w_3, w_4, w_5 be natural numbers and let $d \in \mathbb{N}$. Then χ is a Dirichlet's character with conductor d. From (1.2), we have

$$
\begin{aligned}
&\int_X \chi(y) e^{[w_1w_2w_3w_4y+(\prod_{l=1}^5 w_l)x+w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_1w_2k+w_5w_3w_1w_2l]_q t} \\
&\qquad d\mu_{q^{w_1w_2w_3w_4}}(y) \\
&= \lim_{N\to\infty} \frac{1}{[dw_5p^N]_{q^{w_1w_2w_3w_4}}} \sum_{m=0}^{dw_5-1} \sum_{y=0}^{p^N-1} \chi(m) q^{w_1w_2w_3w_4(m+dw_5y)} \\
&\quad \times e^{[w_1w_2w_3w_4(m+dw_5y)+(\prod_{l=1}^5 w_l)x+w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_2w_1l]_q t}
\end{aligned} \tag{2.1}
$$

Thus, by (2.1), we get

$$
\begin{aligned}
&\frac{1}{[w_1w_2w_3w_4]_q} \sum_{i=0}^{dw_1-1} \sum_{j=0}^{dw_2-1} \sum_{k=0}^{dw_3-1} \sum_{l=0}^{dw_4-1} \chi(i)\chi(j)\chi(k)\chi(l) \\
&\quad \times q^{w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_1w_2l} \\
&\quad \times \int_X \chi(y) e^{[w_1w_2w_3w_4y+(\prod_{l=1}^5 w_l)x+w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_1w_2l]_q t} \\
&\qquad d\mu_{q^{w_1w_2w_3w_4}}(y) \\
&= \lim_{N\to\infty} \frac{1}{[dw_1w_2w_3w_4w_5p^N]} \\
&\quad \times \sum_{i=0}^{dw_1-1} \sum_{j=0}^{dw_2-1} \sum_{k=0}^{dw_3-1} \sum_{l=0}^{dw_4-1} \sum_{m=0}^{dw_5-1} \sum_{y=0}^{p^N-1} \chi(i)\chi(j)\chi(k)\chi(l)\chi(m) \\
&\quad \times q^{w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_2w_1l+q^{w_1w_2w_3w_4(m+dw_5y)}} \\
&\quad \times e^{[w_1w_2w_3w_4(m+dw_5y)+(\prod_{l=1}^5 w_l)x+w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_2w_1l]_q t}
\end{aligned} \tag{2.2}
$$

The equation (2.2) is an invariant for any $\sigma \in S_5$. Therefore, by (2.2), we get

$$\begin{aligned}
&\frac{1}{[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}]_q}\sum_{i=0}^{dw_{\sigma(1)}-1}\sum_{j=0}^{dw_{\sigma(2)}-1}\sum_{k=0}^{dw_{\sigma(3)}-1}\sum_{l=0}^{dw_{\sigma(4)}-1}\chi(i)\chi(j)\chi(k)\chi(l)\\
&\times q^{(w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(3)}w_{\sigma(2)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(3)}w_{\sigma(1)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(1)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(2)}w_{\sigma(1)}l)}\\
&\times\int_X\chi(y)e^{[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}y+w_1w_2w_3w_4w_5x+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(3)}w_{\sigma(2)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(3)}w_{\sigma(1)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(1)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(1)}w_{\sigma(2)}l]_qt}\\
&\quad d\mu_{q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}}(y)
\end{aligned}\tag{2.3}$$

are the same for any $\sigma \in S_5$.

It is easy to show that

$$\begin{aligned}
&\int_X\chi(y)e^{[w_1w_2w_3w_4y+(\prod_{l=1}^5 w_l)x+w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_1w_2l]_qt}\\
&\quad d\mu_{q^{w_1w_2w_3w_4}}(y)\\
&=\sum_{n=0}^{\infty}[w_1w_2w_3w_4]_q^n\\
&\times\int_X\chi(y)\left[y+w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}l\right]_{q^{w_1w_2w_3w_4}}^n d\mu_{q^{w_1w_2w_3w_4}}(y)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}[w_1w_2w_3w_4]_q^n\beta_{n,\chi,q^{w_1w_2w_3w_4}}(w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}l)\frac{t^n}{n!}.
\end{aligned}\tag{2.4}$$

From (2.4), we have

$$\begin{aligned}
&\int_X\chi(y)[w_1w_2w_3w_4y+(\prod_{l=1}^5 w_l)x+w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k\\
&\quad+w_5w_3w_1w_2l]_q^n d\mu_{q^{w_1w_2w_3w_4}}(y)\\
&=[w_1w_2w_3w_4]_q^n\beta_{n,\chi,q^{w_1w_2w_3w_4}}(w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}l),\ (n\geq 0).
\end{aligned}\tag{2.5}$$

By (2.3) and (2.5), we easily get the following equation (2.6):

$$[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}]_q^{n-1}\sum_{i=0}^{dw_{\sigma(1)}-1}\sum_{j=0}^{dw_{\sigma(2)}-1}\sum_{k=0}^{dw_{\sigma(3)}-1}\sum_{l=0}^{dw_{\sigma(4)}-1}\chi(i)\chi(j)\chi(k)\chi(l)$$
$$\times q^{w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(3)}w_{\sigma(2)}i+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(3)}w_{\sigma(1)}j+w_{\sigma(5)}w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(1)}k+w_{\sigma(5)}w_{\sigma(3)}w_{\sigma(2)}w_{\sigma(1)}l}$$
$$\times \beta_{n,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}}(w_{\sigma(5)}x+\frac{w_{\sigma(5)}}{w_{\sigma(1)}}i+\frac{w_{\sigma(5)}}{w_{\sigma(2)}}j+\frac{w_{\sigma(5)}}{w_{\sigma(3)}}k+\frac{w_{\sigma(5)}}{w_{\sigma(4)}}l) \tag{2.6}$$

are the same for any $\sigma \in S_5$.
We observe that

$$\left[y+w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}l\right]^n_{q^{w_1w_2w_3w_4}}$$
$$=\sum_{m=0}^{n}\binom{n}{m}\left(\frac{[w_5]_q}{[w_1w_2w_3w_4]_q}\right)^{n-m} \tag{2.7}$$
$$\times [w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k+w_3w_1w_2l]^{n-m}_{q^{w_5}}$$
$$\times q^{m(w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_1w_2l)}[y+w_5x]^m_{q^{w_1w_2w_3w_4}}.$$

From (2.7), we have

$$\int_X \chi(y)\left[y+w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}l\right]^n_{q^{w_1w_2w_3w_4}}d\mu_{q^{w_1w_2w_3w_4}}(y)$$
$$=\sum_{m=0}^{n}\binom{n}{m}\left(\frac{[w_5]_q}{[w_1w_2w_3w_4]_q}\right)^{n-m} \tag{2.8}$$
$$\times [w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k+w_3w_1w_2l]^{n-m}_{q^{w_5}}$$
$$\times q^{m(w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_2w_1l)}\beta_{m,\chi,q^{w_1w_2w_3w_4}}(w_5x).$$

By (2.8), we get

$$
\begin{aligned}
&[w_1w_2w_3w_4]_q^{n-1}\sum_{i=0}^{dw_1-1}\sum_{j=0}^{dw_2-1}\sum_{k=0}^{dw_3-1}\sum_{l=0}^{dw_4-1}\chi(i)\chi(j)\chi(k)\chi(l)\\
&\quad\times q^{w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_1w_2l}\\
&\quad\times\int_{\mathbb{Z}_p}\left[y+w_5x+\frac{w_5}{w_1}i+\frac{w_5}{w_2}j+\frac{w_5}{w_3}k+\frac{w_5}{w_4}l\right]_{q^{w_1w_2w_3w_4}}^{n}\chi(y)d\mu_{q^{w_1w_2w_3w_4}}(y)\\
&=\sum_{m=0}^{n}\binom{n}{m}[w_1w_2w_3w_4]_q^{m-1}[w_5]_q^{n-m}\beta_{m,\chi,q^{w_1w_2w_3w_4}}(w_5x)\sum_{i=0}^{dw_1-1}\sum_{j=0}^{dw_2-1}\sum_{k=0}^{dw_3-1}\\
&\quad\sum_{l=0}^{dw_4-1}\chi(i)\chi(j)\chi(k)\chi(l)q^{(m+1)(w_5w_4w_3w_2i+w_5w_4w_3w_1j+w_5w_4w_2w_1k+w_5w_3w_2w_1l)}\\
&\quad\times[w_4w_3w_2i+w_4w_3w_1j+w_4w_2w_1k+w_3w_2w_1l]_{q^{w_5}}^{n-m}\\
&=\sum_{m=0}^{n}\binom{n}{m}[w_1w_2w_3w_4]_q^{n-1}[w_5]_q^{n-m}\beta_{m,\chi,q^{w_1w_2w_3w_4}}(w_5x)\\
&\quad\times T_{n,\chi,q^{w_5}}(w_1,w_2,w_3,w_4\mid m),
\end{aligned}
\tag{2.9}
$$

where

$$
\begin{aligned}
T_{n,\chi,q}(w_1,w_2,w_3,w_4\mid m)&=\sum_{i=0}^{dw_1-1}\sum_{j=0}^{dw_2-1}\sum_{k=0}^{dw_3-1}\sum_{l=0}^{dw_4-1}\chi(i)\chi(j)\chi(k)\chi(l)\\
&\quad\times q^{(m+1)(w_4w_3w_2i+w_4w_3w_1j+w_4w_2w_1k+w_3w_2w_1l)}\\
&\quad\times[w_4w_3w_2i+w_4w_3w_1j+w_4w_2w_1k+w_3w_2w_1l]_q^{n-m}.
\end{aligned}
\tag{2.10}
$$

The equation (2.9) is also an invariant for any permutation $\sigma\in S_5$. Therefore, by (2.9), we obtain the following result:

$$
\begin{aligned}
&\sum_{m=0}^{n}\binom{n}{m}[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}]_q^{m-1}[w_{\sigma(5)}]_q^{n-m}B_{m,\chi,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}}}(w_{\sigma(5)}x)\\
&\quad\times T_{n,\chi,q^{w_{\sigma(5)}}}(w_{\sigma(1)},w_{\sigma(2)},w_{\sigma(3)},w_{\sigma(4)}\mid m)
\end{aligned}
$$

are the same for any $\sigma\in S_5$.

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