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Rainbow Coloring of Some Ladder Graphs

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Abstract

A rainbow coloring of a connected graph is a coloring of the edges of the graph, such that every pair of vertices is connected by at least one path in which no two edges are colored the same. Computing the rainbow connection number of a graph is *NP*- hard and it finds its applications to the secure transfer of classified information between agencies.
In this paper we compute rainbow coloring of ladder graph and circular ladder graph in a polynomial time.

Index Terms: Rainbow coloring, Rainbow connection number, Rainbow vertex connection number, Circular ladder graph, Ladder graph.

Introduction

All graphs considered in this paper are finite , undirected and simple. Let G be a nontrivial connected graph on which an edge-coloring $c : E(G) \rightarrow \{1,2, \cdots ,n\}$, $n \in N$, is defined, where adjacent edges may be colored the same. A path is rainbow if no two edges of it are colored the same. An edge-colored graph *G* is rainbow connected if every two distinct vertices are connected by at least one rainbow path. An edge-coloring under which *G* is rainbow connected is called a rainbow coloring. Thus, the rainbow connection number of a connected graph G, denoted by rc(G), is the smallest number of colors that are needed to make G rainbow connected. For example, the rainbow connection number of a complete graph is 1, that of a path is its length, and that of a tree is its number of edges.

Clearly, if a graph is rainbow connected, it must be connected. Conversely, every connected graph has a trivial edge-coloring that makes it rainbow connected by coloring edges with distinct colors. A vertex colouring of a graph *G* is said to be rainbow vertex connected if any pair of vertices in *G* are connected by at least one path whose internal vertices have distinct colors which was introduced by Krivelevich and Yuster in [6]. The rainbow vertex-connection number of a connected graph *G* denoted by rvc(G) is the smallest number of colors that are needed to make *G* rainbow vertex connected.

The rainbow connection number is not only a natural combinatorial measure, but it also has applications to the secure transfer of classified information between agencies and in communication networks.

An Overview of The Paper

The concept of rainbow colouring was introduced by Chartrand et al.,in [5]. Precise values of rainbow connection number for many special graphs like complete multipartite graphs, Peterson graph and wheel graph were also determined. It was shown in Chakraborty et al.,in [3], that computing the rainbow connection number of an arbitrary graph is *NP*-Hard. To rainbow color a graph it is enough to ensure that every edge of some spanning tree in the graph gets a distinct color. Hence order of the graph minus one is an upper bound for rainbow connection number.

In addition, the rainbow connection number can also be motivated by its interesting interpretation in the area of networking. Suppose that *G* represents a network (e.g., a cellular network). We wish to route messages between any two vertices in a pipeline, and require that each link on the route between the vertices (namely, each edge on the path) is assigned a distinct channel (e.g., a distinct frequency). Clearly, we want to minimize the number of distinct channels that we use in our network. There are also the concepts of strong rainbow connection or rainbow diameter, the rainbow connectivity, and the rainbow index. The rainbow connections of graphs are very new concepts. Recently, there has been great interest in these concepts and a lot of results have been published.

Chartrand et al. in [3] determined that the rainbow connection numbers of some graphs including trees, cycles, wheels, complete bipartite graphs and complete multipartite graphs. Caro et al. [4] observed that rc(G) can be bounded by a function $\delta(G)$, the minimum degree of G. Krivelvich and Yuster[6] also determined the behaviour of rc(G) as a function of $\delta(G)$: a connected graph G with n vertices has $rc(G) < 20n / \delta(G)$.

The above rainbow connection number involves edge colorings of graphs. Krivelevich and Yuster[6] are the first to introduce a new parameter corresponding to the rainbow connection number which is defined on a vertex colored graph. A vertex colored graph G is rainbow vertex connected if every two vertices are connected by a path whose internal vertices have distinct colors, and such a path is called a rainbow path. The rainbow vertex connection number of a connected graph G , denoted by rvc(G) is the smallest number of colors that are needed in order to make G rainbow vertex connected. We always have $rvc(G) \leq n-2$ if G is a graph of order n and $rvc(G) = 0$ if and only if G is a complete graph. Also $rvc(G) \geq diam(G) - 1$ and with equality if the diameter of G is 1 or 2. In [3],Chen, Li and Shi studied the computational complexity of rainbow connection and proved that computing rvc(G) is NP- hard.

Main Results

Rainbow connection number of a connected graph *G* is the minimum number of colors needed in order to make *G* rainbow connected. It is easy to see that diam(G) ≤ rc(G) for any connected graph G, where diam(G) is the diameter of G.

In this paper rainbow connection number of ladder graph and circular ladder graph has been computed.

Proposition 3.1:

Let *G* be a nontrivial connected graph of size m.

Then

(i) rc(G) = 1 if and only if G is a complete graph,

(ii) rc(G) = m if and only if G is a tree,

(iii) $rc(C_n) = \lceil n/2 \rceil$ for each integer $n \geq 4$, where C_n is a cycle with size *n*.

Definition 3.2:

Circular ladder graph CL_n: The circular ladder graph CL_n is the graph cartesian product $C_n \times K_2$, where K_2 is the complete graph on two vertices and C_n is the cycle graph on n vertices. It consists of two concentric n-cycles in which each of the n corresponding vertices is joined by an edge. It is a 3-regular simple graph.

Example: CL_4

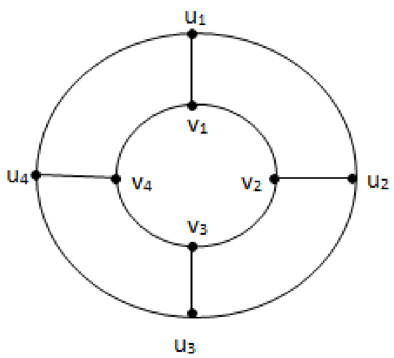

Figure 1: CL_4

Theorem 3.3: The circular ladder graph CL_n admits a rainbow coloring and

$$rc(CL_n) = \begin{cases} 2\ ;\ n = 3 \\ \dfrac{n}{2} + 1 ; n \text{ is even} \\ \dfrac{n+3}{2} ; n \text{ is odd} \end{cases}$$

where *n* is the number of vertices in each of the 2 concentric cycles.

Proof:

Case 1: If *n* = 3

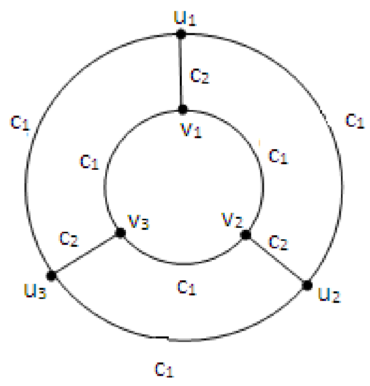

Figure 2: CL_3

It is obvious that minimum colors needed to color is 2. Hence rc(G) = 2

Case 2: If *n* is even

From the above proposition 3.1, $rc(G) \geq n/2$. We claim that rc(G) = n/2. Now label the vertices of the outer circle C_1 with $u_1, u_2, u_3,\ldots,u_n$ and the inner circle C_2 with $v_1, v_2, v_3,\ldots, v_n$ such that (u_i,v_i), $1 \leq i \leq n$ form spoke edges.

Color the outer edges (u_1,u_2), (u_2,u_3),.. .$(u_{n/2},u_{(n/2)+1})$ with c_1 ,$c_2,\ldots c_{n/2}$ and the edges $(u_{(n/2)+1},u_{(n/2)+2}),\ldots(u_n,u_1)$ with c_1, $c_2,\ldots c_{n/2}$ respectively. Color the inner edges (v_1,v_2), $(v_2,v_3),\ldots(v_{n/2},v_{(n/2)+1})$ with c_1, c_2, $c_3,\ldots c_{n/2}$ and the edges $(v_{(n/2)+1},v_{(n/2)+2}),\ldots(v_n,v_1)$ with c_1, $c_2,\ldots c_{n/2}$ respectively. Also the spoke edges (u_i,v_i) ,$1 \leq i \leq n/2$ are assigned with the colors $c_1,c_2,\ldots c_{n/2}$ and the edges (u_i,v_i) , $((n/2)+1) \leq i \leq n$ are colored with $c_1,c_2,\ldots c_{n/2}$ respectively.

Consider any distinct path in CL_n.

Case(i): If P_1 is any path connecting u_i and u_j, $1 \leq i, j \leq n$,

$i \neq j$, then the shortest path connecting the vertices u_i and u_j lies in the outer circle C_1 itself, forming a rainbow path.

Case(ii): Consider a path P_2 connecting v_i and v_j, $1\leq i, j \leq n$, $i \neq j$, then the shortest path connecting the vertices v_i and v_j lies in the inner circle C_2 itself, forming a rainbow path.

Case(iii): Consider a path P_3 connecting u_i and v_j, $1\leq i, j \leq n$, $i \neq j$. The path connecting such vertices must pass via spoke edges at least once which does not form a rainbow path.

Hence $rc(G) \neq n/2$.

To resolve this ,color all spoke edges with the color $c_{(n/2)+1}$, which will be a rainbow coloring of CL_n if *n* is even. Hence

rc(G)=(n/2)+1.

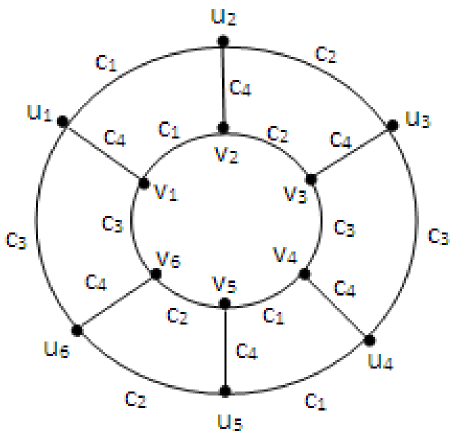

Figure 3: *CL_6*

In fig 3, we see that for CL_6, $rc(G) = 4$, we can also see that *diameter* = 4. Hence diam(G) ≤ rc(G).

Case 3: If n is odd

Claim: rc(G) = (n+3)/2

We color the edges as given below:

1. Color the outer circle edges (u_1,u_2), (u_2,u_3),…, $(u_{(n+1)/2}, u_{(n+3)/2})$ with colors c_1, c_2,…,$c_{(n+1)/2}$ respectively and the edges $(u_{(n+3)/2}, u_{(n+5)/2})$, …(u_n, u_1) with the colors c_1,c_2,…$c_{((n+1)/2)-1}$ respectively.
2. Assign the color $c_{(n+3)/2}$ to all the spoke edges (u_i,v_i), $1 \leq i \leq n$..
3. Color the inner circle edges (v_1,v_2), (v_2,v_3),…,$(v_{(n+1)/2}, v_{(n+3)/2})$ with c_1, c_2,…$c_{(n+1)/2}$ respectively and the edges $(v_{(n+3)/2}, v_{(n+5)/2})$, …(v_n,v_1) with the colors c_1, c_2,…$c_{((n+1)/2)-1}$ respectively. This will be a rainbow coloring of CL_n if n is odd.

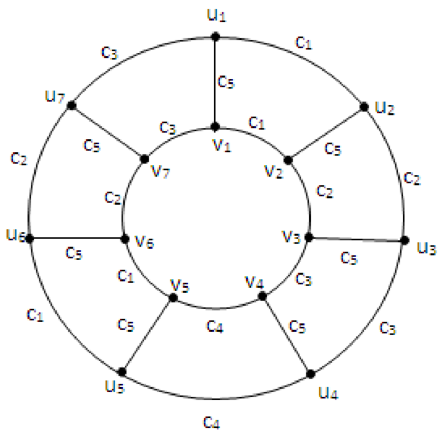

Figure 4: CL_7

In CL_7, we see that rc(G) = 5.

We can also see that diam(G) ≤ rc(G).

In the above proof the rainbow connection number of circular ladder graph has been computed using edge coloring. Next we shall find the rainbow vertex connection number of circular ladder graph using vertex coloring and also compare rc(G) and rvc(G).

Lemma 3.4:

The rainbow vertex connection number of the circular ladder graph CL_n is

$$rvc(CL_n) = \begin{cases} 1 & ; n = 3 \\ \frac{n}{2} & ; n \text{ is even and } n > 3 \\ \frac{n+1}{2} & ; n \text{ is odd and } n > 3 \end{cases}$$

Proof:

Diam(CL_3) = 2 , therefore rvc(CL_3) = 1.

Case 1: If n is even

Coloring Algorithm:

1. Color the inner circle vertices $v_1, v_2, v_3, \ldots, v_{n/2}$ with the colors $c_1, c_2, \ldots c_{n/2}$ respectively and the vertices $v_{(n/2)+1}, v_{(n/2)+2}, \ldots, v_n$ with the colors $c_1, c_2, \ldots c_{n/2}$ respectively.
2. Color the outer circle vertices $u_1, u_2, u_3, \ldots, u_{n/2}$ with the colors $c_1, c_2, \ldots c_{n/2}$ respectively and the vertices $u_{(n/2)+1}, u_{(n/2)+2}, \ldots u_n$ with the colors $c_1, c_2, \ldots c_{n/2}$ respectively.

Consider any distinct path in CL_n.

- Case(i): If P_1 is any path connecting u_i and u_j, $1 \leq i, j \leq n$, $i \neq j$, then the shortest path connecting the vertices u_i and u_j lies in the outer circle C_1 , forms a rainbow path.
- Case(ii): Consider a path P_2 connecting v_i and v_j, $1 \leq i, j \leq n$, $i \neq j$, then the shortest path connecting the vertices v_i and v_j lies in the inner circle C_2 ,forms a rainbow path.
- Case(iii): Consider a path P_3 connecting u_i and v_j, $1 \leq i, j \leq n$, $i \neq j$. The path connecting such vertices forms a rainbow path. Hence rc(G) = n/2.

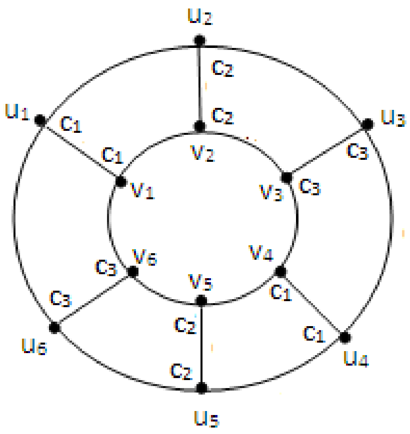

Figure 5: *CL_6*

Case 2: If n is odd

Coloring Algorithm:

1. Color the inner circle vertices $v_1,v_2,v_3,\ldots, v_{(n+1)/2}$ with the colors $c_1,c_2,\ldots, c_{(n+1)/2}$ respectively and the vertices $v_{((n+1)/2)+1}, v_{((n+1)/2)+2}, \ldots, v_n$ with the colors $c_1,c_2,\ldots c_{(n-1)/2}$ respectively.
2. Color the inner circle vertices $u_1,u_2,u_3,\ldots, u_{(n+1)/2}$ with the colors $c_1,c_2,\ldots, c_{(n+1)/2}$ respectively and the vertices $u_{((n+1)/2)+1}, u_{((n+1)/2)+2}, \ldots, u_n$ with the colors $c_1,c_2,\ldots c_{(n-1)/2}$ respectively.

We can see that any two vertices are connected by a rainbow path. This will be a rainbow coloring of CL_n if n is odd. Hence $rvc(CL_n) = (n+1)/2$.

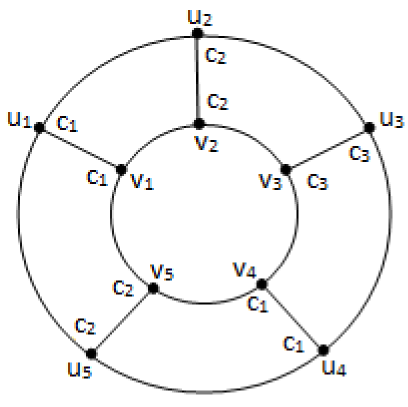

Figure 6: CL_5

From the above two results , we can see that
$rc(CL_n) > rvc(CL_n)$.

Definition 4.1:
Ladder graph L_n : Ladder graph is a planar undirected graph with 2n vertices and n+2(n-1) edges. It can be obtained as the cartesian product of two path graphs.

Example:L_3

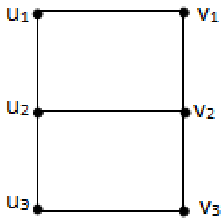

Figure 7: L_3

Theorem 4.2:
The Ladder graph L_n admits a rainbow coloring and $rc(G) = n$.

Proof:
Construction of rainbow coloring as follows:

1. Assign the color c_1 to the edges (u_i,v_i) , i =1, 2, …n.
2. The edges $(v_1,v_2),(v_2,v_3),\ldots(v_{n-1},v_n)$ can be colored as $c_2,c_3,\ldots c_n$ respectively
3. Also the edges $(u_1,u_2),(u_2,u_3),\ldots(u_{n-1},u_n)$ can be colored as $c_2,c_3,\ldots c_n$ respectively.

Consider any distinct path in L_n.
This will be a rainbow coloring of L_n.

- Case(i): If P_1 is any path connecting u_i and u_j, $1\leq i, j \leq n$, $i \neq j$, then the shortest path connecting the vertices u_i and u_j , forms a rainbow path.
- Case(ii): Consider a path P_2 connecting v_i and v_j,$1\leq i, j \leq n$, $i \neq j$, then the shortest path connecting the vertices v_i and v_j forms a rainbow path.
- Case(iii): Consider a path P_3 connecting u_i and v_j, $1\leq i, j \leq n$, $i \neq j$. The path connecting such vertices forms a rainbow path. Hence rc(G) = n.

Fig. 8 depicts rainbow coloring of L_4 and rc(G)=4.

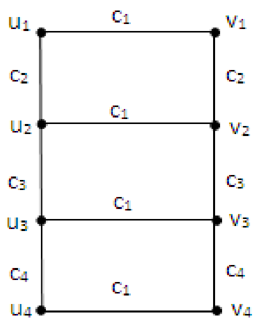

Figure 8: L_4

Lemma 4.3
The rainbow vertex connection number rvc(L_n) of the ladder graph L_n is n-1.

Proof:
Coloring Algorithm:

1. Color the vertices u_1, u_n, v_1, v_n with the color c_1.
2. Color the internal vertices $u_2, u_3, \ldots, u_{n-1}$ with the colors $c_2, c_3 \ldots c_{n-1}$ respectively.
3. Color the internal vertices $v_2, v_3, \ldots, v_{n-1}$ with the colors $c_2, c_3 \ldots c_{n-1}$ respectively.

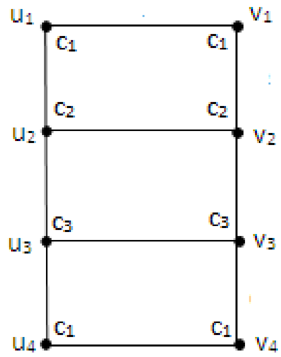

Figure 9: L_4

This will be the rainbow vertex coloring of L_n and the rainbow vertex connection number is $rvc(L_n) = n$-1

We can see that rc(L_n) > rvc(L_n).

Conclusion

It is known that computing the rainbow connection number of a graph is *NP*- hard. So it is interesting to compute *rc(G)* for any graph G. In this paper the rainbow coloring of circular ladder graph and ladder graph has been studied using edge coloring and vertex coloring and rc(G) > rvc(G) in both the graphs.

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