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Existence of Fuzzy Solutions for Nonlocal Impulsive Delay Differential Evolution Systems

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Abstract

In this paper, we prove the existence of solutions of fuzzy impulsive delay differential evolution equations with nonlocal condition. The results are obtained by using the fixed point principles.

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1. Introduction

In the mathematical description of many phenomena relative to scientific research, differential equations arise as an important tool to achieve a more adequate explanation and a faithful adjustment to the behavior of the particular magnitude of interest. It is well-known that determinism is not able, in general, to provide a complete and definite model for the analysis of a dynamical system [16, 17] due to the imperfections and vagueness of our perception of the system itself. To give an example, uncertainty is present in the study of intelligent systems from different points of view, such as soft computing or granular

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computing [20]. Thus, fuzziness constitutes an adequate mechanism to introduce the subjective factors which might influence the phenomena into the system. Fuzziness is a basic type of subjective uncertainty initiated by Zadeh [31] via membership function in 1965. It is a powerful tool for modeling of uncertainty and for processing vague or subjective information in mathematical models.

Fuzzy differential equation is a useful tool to model a dynamical system when information about its behavior is inadequate. It has been studied thoroughly in the last years as adequate models to predict the behavior of continuous processes susceptible to imprecision based on subjective choices. However, the meaning of a fuzzy differential equation strongly depends on the selection of the concept of fuzzy derivative [8]. Kandel and Byatt [10] introduced the concept of fuzzy differential equations. Several authors have studied the fuzzy differential equations by using the H-Differentiability for the fuzzy valued mappings of a real variable whose values are normal, convex, upper semicontinuous and compactly supported fuzzy sets in R^n.

Seikkala [28] defined the fuzzy derivative which is a generalization of the Hukuhara derivative in [22]. The existence theorem under compactness type conditions are investigated in [29] when the fuzzy valued mappings are satisfied with Lipschitz condition. Park et al [19] studied the fuzzy differential equation with nonlocal condition. Nieto [18] proved an existence theorem for fuzzy differentiable equation on the metric space(E^n, D). Balasubramaniam and Muralisankar [7] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integro differential equation with nonlocal conditions. There are several approaches to define a solution for fuzzy differential equation: Hukuhara approach [9], Differential inclusions [6], Quasiflows and differential equations in metric spaces [13]. For more on Fuzzy differential equation, we can refer [3], [9], [11], [21], [23], [25], [30].

Differential equations with impulses are a basic tool to study evolutionary processes that are subject to abstract changes in their state. Such equation arises naturally from a wide variety of applications, such as spacecraft control, inspection processes in operation research, drug administration, and threshold theory in biology. For the monographs of the theory of impulsive differential equations, we can refer the books of Bainov and Simenov [2], Lakshmikantham et al. [14], Samoilenko and Perestyuk [27].

The introduction of impulses in the equation is the key point in our procedure. This approach, far enough from representing a restriction in the field of applications, is clearly appropriate to analyze the evolution of real phenomena which depend on external factors, and, of course, it perfectly fits with control processes in many fields. This way, we illustrate the possibility to stabilize the solutions by using impulses, which allows to keep the solutions of the equation in a certain region or even provides the existence of periodic solutions.

The richness of the impulsive fuzzy differential equations even allows to extract continuous real solutions as (approximate) representations of discontinuous fuzzy solutions. Impulses allow to analyze the properties of the system during a certain period of time and change its behavior in order to, for instance, adjust the results in a previously established region. Besides, the way in which the system is impulsed possesses great interest and physical meaning, since it might be closely related to the mechanism of defuzzification selected, with the aim of 'translating' the conclusions obtained to the ordinary case.

Lakshmikantham and McRae [15] initiated the study of fuzzy impulsive differential equations. Some works on fuzzy impulsive differential equations were investigated in [4], [5] and [12]. Recently, Ramesh and Vengataasalam [24] analyzed the solutions of fuzzy impulsive delay integrodifferential equations with nonlocal condition. Vengataasalam and Ramesh [26] studied the fuzzy solutions for impulsive semilinear differential equations.

This work is focused on the study of fuzzy solutions for impulsive delay differential evolution equations, which makes sense following an impulsive formulation approach and has a very important meaning in the field of applications, being even natural in every control mechanism. Here in this paper, we prove the existence of solutions of fuzzy delay impulsive differential evolution equations with nonlocal conditions of the form

$$x'(t) = A(t)x(t) + f\big(t, x(\sigma_1(t)), x(\sigma_2(t)), \cdots, x(\sigma_n(t))\big), \quad t \in J = [0, a] \tag{1.1}$$

$$x(0) = x_0 + g(t_1, t_2, \cdots, t_p, x(.)), \tag{1.2}$$

$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, 3, \cdots, m \tag{1.3}$$

where $\sigma_i : J \to J, i = 1, 2 \cdots, n$ are continuous functions and $f : J \times E^{n^2} \to E^n$ is levelwise continuous function and $\sigma_i(t) \le t$ for all $t \in J$, $g : J^p \times E^n \to E^n$ satisfies the Lipschitz condition. The symbol $g(t_1, t_2, \ldots, t_p, x(.))$ is used in the sense that in the place of '.', we can substitute only elements of the set $\{t_1, t_2, \ldots, t_p\}$. For example $g(t_1, t_2, \ldots, t_p, x(.))$ can be defined by the formula

$$g(t_1, t_2, \cdots, t_p, x(.)) = c_1 x(t_1) + c_2 x(t_2) + \cdots + c_p x(t_p),$$

where $c_i (i = 1, 2 \cdots, p)$ are given constants and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at $t = t_k$ respectively.

2. Preliminaries

Let $P_K(R^n)$ denote the family of all non empty, compact, convex subsets of R^n. Addition and scalar multiplication in $P_K(R^n)$ are defined as usual. Let A nd B be two non empty bounded subsets of R^n. The distance between A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$d(A, B) = \max\left\{ \sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\| \right\},$$

where $\|.\|$ denote the usual Euclidean norm in R^n. Then it is clear that $\big(P_K(R^n), d\big)$ becomes a metric space. Let $I = [t_0, t_0 + a] \subset R(a > 0)$ be a compact interval and let E^n be the set of all $u : R^n \to [0, 1]$ such that u satisfies the following conditions:

(i) u is normal i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex, that is,

$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\} \quad \text{where} \quad x, y \in R^n \quad \text{and} \quad 0 \leq \lambda \leq 1,$$

(iii) u is upper semicontinuous

(iv) $[u]^0 = \text{cl}\{x \in R^n : u(x) > 0\}$ is compact.

If $u \in E^n$, then u is called a fuzzy number, and E^n is said to be a fuzzy number space. For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in R^n : u(x) \geq 0\}$. Then from (i)–(iv), it follows that the α-level set $[u]^\alpha \in P_K(R^n)$ for all $0 \leq \alpha \leq 1$.

If $g : R^n \times R^n \to R^n$ is a function, then by using Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the equation

$$\tilde{g}(u, v)(z) = \sup_{z=g(x,y)} \min\{u(x), v(y)\}.$$

It is well known that $[\tilde{g}(u, v)]^\alpha = g([u]^\alpha, [v]^\alpha)$ for all $u, v \in E^n, 0 \leq \alpha \leq 1$ and continuous function g. Further, we have $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$, $[ku]^\alpha = k[u]^\alpha$, where $k \in R$.
Define $D : E^n \times E^n \to [0, \infty)$ by the relation

$$D(u, v) = \sup_{0\leq\alpha\leq 1} d([u]^\alpha, [v]^\alpha),$$

where d is the Hausdorff metric defined in $P_K(R^n)$. Then D is a metric in E^n.
Further we know that [22]

(i) (E^n, D) is a complete metric space,

(ii) $D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E^n$,

(iii) $D(\lambda u, \lambda v) = |\lambda|\, D(u, v)$ for all $u, v \in E^n$ and $\lambda \in R$.

It can be proved that $D(u + v, w + z) \leq D(u, w) + D(v, z)$ for u, v, w and $z \in E^n$

Definition 2.1. [9] A mapping $F : I \to E^n$ is strongly measurable if for all $\alpha \in [0, 1]$ the set-valued map $F_\alpha : I \to P_K(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable when $P_K(R^n)$ has the topology induced by the Hausdorff metric d.

Definition 2.2. [9] A mapping $F : I \to E^n$ is said to be integrably bounded if there is an integrable function $h(t)$ such that $\|x(t)\| \leq h(t)$ for every $x(t) \in F_0(t)$.

Definition 2.3. The integral of a fuzzy mapping $F : I \to E^n$ is defined levelwise by $[\int_I F(t)dt]^\alpha = \int_I F_\alpha(t)dt$ = The set of all $\int_I f(t)dt$ such that $f : I \to R^n$ is a measurable selection for F_α for all $\alpha \in [0, 1]$.

Definition 2.4. [1] A strongly measurable and integrably bounded mapping $F : I \to E^n$ is said to be integrable over I if $[\int_I F(t)dt] \in E^n$.

Note that if $F : I \to E^n$ is strongly measurable and integrably bounded, then F is integrable. Further if $F : I \to E^n$ is continuous, then it is integrable.

Proposition 2.5. Let $F, G : I \to E^n$ be integrable and $c \in I, \lambda \in R$. Then

(i) $\int_{t_0}^{t_0+a} F(t)dt = \int_{t_0}^{c} F(t)dt + \int_{c}^{t_0+a} F(t)dt$,

(ii) $\int_I (F(t) + G(t))dt = \int_I F(t)dt + \int_I G(t)dt$,

(iii) $\int_I \lambda F(t)dt = \lambda \int_I F(t)dt$,

(iv) $D(F, G)$ is integrable,

(v) $D\left(\int_I F(t)dt, \int_I G(t)dt\right) \leq \int_I D(F(t), G(t))dt$.

Definition 2.6. A mapping $F : I \to E^n$ is Hukuhara differentiable at $t_0 \in I$, if for some $h_0 > 0$ the Hukuhara differences

$$F(t_0 + \Delta t) -_h F(t_0), F(t_0) -_h F(t_0 - \Delta t)$$

exist in E^n for all $0 < \Delta t < h_0$ and there exists an $F'(t_0) \in E^n$ such that

$$\lim_{\Delta t \to 0+} D((F(t_0 + \Delta t) -_h F(t_0))/\Delta t, F'(t_0)) = 0$$

and

$$\lim_{\Delta t \to 0+} D((F(t_0) -_h F(t_0 - \Delta t)/\Delta t, F'(t_0)) = 0.$$

Here $F'(t)$ is called the Hukuhara derivative of F at t_0.

Definition 2.7. A mapping $F : I \to E^n$ is called differentiable at a $t_0 \in I$, if for any $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at point t_0 with $DF_\alpha(t_0)$ and the family $\{DF_\alpha(t_0) : \alpha \in [0, 1]\}$ define a fuzzy number $F(t_0) \in E^n$.

If $F : I \to E^n$ is differentiable at $t_0 \in I$, then we say that $F'(t_0)$ is the fuzzy derivative of $F(t)$ at the point t_0.

Theorem 2.8. Let $F : I \to E^n$ be differentiable. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$. Then f_α and g_α are differentiable and $[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$.

Theorem 2.9. Let $F : I \to E^n$ be differentiable and assume that the derivative F' is integrable over I. Then, for each $s \in I$, we have

$$F(s) = F(a) + \int_a^s F'(t)dt.$$

Definition 2.10. A mapping $f : I \times E^n \to E^n$ is called levelwise continuous at a point $(t_0, x_0) \in I \times E^n$ provided, for any fixed $\alpha \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists a $\delta(\epsilon, \alpha) > 0$ such that

$$d([f(t, x)]^\alpha, [f(t_0, x_0)]^\alpha) < \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and $d([x]^\alpha, [x_0]^\alpha) < \delta(\epsilon, \alpha)$ for all $t \in I, x \in E^n$.

Corollary 2.11. [9] Suppose that $F : I \to E^n$ is continuous. Then the function

$$G(t) = \int_a^t F(s)ds, \quad t \in I$$

is differentiable and $G'(t) = F(t)$.

Now, if F is continuously differentiable on I, then we have the following mean value theorem

$$D(F(b), F(a)) \leq (b - a).\sup\{D(F'(t), \hat{0}), t \in I\}.$$

As a consequence, we have that

$$D(G(b), G(a)) \leq (b - a).\sup\{D(F'(t), \hat{0}), t \in I\}.$$

Theorem 2.12. Let X be a compact metric space and Y any metric space. A subset Ω of the space $C(X, Y)$ of continuous mappings of X into Y is totally bounded in the metric of uniform convergence if and only if Ω is equicontinuous on X, and $\Omega(x) = \{\phi(x) : \phi \in \Omega\}$ is totally bounded subset of Y for each $x \in X$.

3. Existence Results

Now, we concern with the existence of fuzzy solutions for the problem $(1.1) - (1.3)$.

Definition 3.1. A function $x : J \to E^n$ is a mild solution of the nonlocal impulsive evolution delay differential equation (1.1)–(1.3) if and only if it is levelwise continuous and satisfies the integral equation

$$\begin{aligned} x(t) &= S(t)x_0 + S(t)g(t_1, t_2, \cdots, t_p, x(.)) \\ &\quad + \int_0^t S(t - s) f(s, x(\sigma_1(s)), x(\sigma_2(s)), \cdots, x(\sigma_n(s)))ds \\ &\quad + \sum_{0<t_k<t} S(t - t_k)I_k(x(t_k)) \end{aligned}$$

for all $t \in J$. Let $Y = \{\xi \in E^n : H(\xi, x_0) \leq b\}$ be the space of continuous functions with $H(\xi, \psi) = \sup\limits_{0\leq t\leq \gamma} D(\xi(t), \psi(t))$ and b is a positive number.

Theorem 3.2. For the forthcoming analysis, we need the following assumptions:

(H1) The mapping $f : J \times Y \to E^n$ is levelwise continuous in t on J and there exists a constant l_0 such that

$$D(f(t, x_1, x_2, \cdots, x_n), f(t, y_1, y_2, \cdots, y_n)) \leq l_0 \sum_{i=1}^{n} D(x_i, y_i).$$

(H2) There exists a constant l_1 such that for all $x, y \in Y$ and $\sigma_i : J \to J, \quad i = 1, 2, \cdots, n$

$$D(x(\sigma_i(t)), y(\sigma_i(t))) \leq l_1 D(x(t), y(t)).$$

(H3) The mapping $g : J^p \times Y \to E^n$ is a function and there exists a constant $l_2 > 0$ such that

$$D(g(t_1, t_2, \cdots, t_p, x(.)), g(t_1, t_2, \cdots, t_p, y(.))) \leq l_2 D(x, y).$$

(H4) There exists a constant l_3 such that

$$D\bigg(I_k(x(t_k)), I_k(y(t_k))\bigg) \leq l_3 D(x, y), \quad k = 1, 2, \cdots, m.$$

(H5) Let $S(t)$ is a fuzzy number such that $|S(t)| \leq c$, $\forall t \in J$, where c is a constant. Then there exists a unique solution $x(t)$ of (1.1)–(1.3) defined on the interval $[0, \gamma]$ where

$$\begin{aligned}
\gamma &= \min\{a, (b - cl_5 - c\chi)/l_4 c, (1 - cl_2 + cl_3)/cl_0 l_1\}, \\
l_4 &= \max\ D(f(t, x(\sigma_1(t)), x(\sigma_2(t)), \cdots, x(\sigma_n(t))), \hat{0})) \\
l_5 &= D(g(t_1, t_2, \cdots, t_p, x(.)), \hat{0}) \quad \text{and} \\
\chi &= D\bigg(\sum_{0<t_k<t} I_k(x(t_k)), \hat{0}\bigg), \hat{0} \in E^n.
\end{aligned}$$

Proof. We show that the operator $\Phi : Y \to Y$ defined by

$$\begin{aligned}
\Phi x(t) &= S(t)x_0 + S(t)g(t_1, t_2, \cdots, t_p, x(.)) \\
&\quad + \int_0^t S(t-s) f(s, x(\sigma_1(s)), x(\sigma_2(s)), \cdots, x(\sigma_n(s)))ds \\
&\quad + \sum_{0<t_k<t} S(t - t_k) I_k(x(t_k))
\end{aligned}$$

has a fixed point x, which is the mild solution of the system (1.1) $-$ (1.3). First, we show that $\Phi : Y \to Y$ is continuous whenever $\xi \in Y$ and that $H(\Phi\xi, x_0) \leq b$. Since f is levelwise continuous and σ is continuous, we take

$$l_4 = \max\ D(f(t, x(\sigma_1(t)), x(\sigma_2(t)), \cdots, x(\sigma_n(t))), \hat{0}))$$

$$
\begin{aligned}
&D(\Phi\xi(t+h),\Phi\xi(t))\\
&= D\Big(S(t+h)x_0 + S(t+h)g(t_1,t_2,\cdots,t_p,\xi(.))\\
&\quad + \int_0^{t+h} S(t+h-s)f(s,\xi(\sigma_1(s)),\xi(\sigma_2(s)),\cdots,\xi(\sigma_n(s)))ds\\
&\quad + \sum_{0<t_k<t} S(t+h-t_k)I_k(\xi(t_k)), S(t)x_0 + S(t)g(t_1,t_2,\cdots,t_p,\xi(.))\\
&\quad + \int_0^{t} S(t-s)f(s,\xi(\sigma_1(s)),\xi(\sigma_2(s)),\cdots,\xi(\sigma_n(s)))ds\\
&\quad + \sum_{0<t_k<t} S(t-t_k)I_k(\xi(t_k))\Big)\\
&\le D\Big(S(t+h)x_0, S(t)x_0) + D(S(t+h)g(t_1,t_2,\cdots,t_p,\xi(.)),\\
&\quad S(t)g(t_1,t_2,\cdots,t_p,\xi(.)))\\
&\quad + D\Big(\int_0^{t} S(t+h-s)f(s,\xi(\sigma_1(s)),\xi(\sigma_2(s)),\cdots,\xi(\sigma_n(s)))ds,\\
&\quad \int_0^{t} S(t-s)f(s,\xi(\sigma_1(s)),\xi(\sigma_2(s)),\cdots,\xi(\sigma_n(s)))ds\Big)\\
&\quad + D\Big(\sum_{0<t_k<t} S(t+h-t_k)I_k(\xi(t_k)), \sum_{0<t_k<t} S(t-t_k)I_k(\xi(t_k))\Big)
\end{aligned}
$$

The right hand side tends to zero as $h \to 0$. Hence, the map Φ is continuous. Now

$$
\begin{aligned}
D(\Phi\xi(t),x_0) &= D\Big(S(t)x_0 + S(t)g(t_1,t_2,\cdots,t_p,\xi(.))\\
&\quad + \int_0^{t} S(t-s)f(s,\xi(\sigma_1(s)),\xi(\sigma_2(s)),\cdots,\xi(\sigma_n(s)))ds\\
&\quad + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), x_0\Big)\\
&\le cD(g(t_1,t_2,\cdots,t_p,\xi(.)),\hat{0})\\
&\quad + \int_0^{t} cD(f(s,\xi(\sigma_1(s)),\xi(\sigma_2(s)),\cdots,\xi(\sigma_n(s))),\hat{0})ds)\\
&\quad + cD\Big(\sum_{0<t_k<t} I_k(\xi(t_k)),\hat{0}\Big)\\
&= c(l_5 + l_4 t + \chi)
\end{aligned}
$$

Therefore,

$$
H(\Phi\xi,x_0) = \sup_{0\le t\le\gamma} D(\Phi\xi(t),x_0) \le c(l_5 + l_4\gamma + \chi) \le b.
$$

Thus Φ is a mapping from Y into Y. Since $C([0,\gamma], E^n)$ is a complete metric space with the metric H, we only show that Y is a closed subset of $C([0,\gamma], E^n)$.

Let define $\{\psi_n\}$ be a sequence in Y such that $\psi_n \to \psi \in C([0,\gamma], E^n)$ as $n \to \infty$. Then

$$D(\psi(t), x_0) \le D(\psi(t), \psi_n(t)) + D(\psi_n(t), x_0),$$

That is,

$$\begin{aligned} H(\psi, x_0) = \sup_{0\le t\le\gamma} D(\psi(t), x_0) &\le H(\psi, \psi_n) + H(\psi_n, x_0) \\ &\le \epsilon + b \end{aligned}$$

for sufficiently large n and arbitrary $\epsilon > 0$. Thus, $\psi \in Y$. This shows that Y is closed subset of $C([0,\gamma], E^n)$. Therefore Y is a complete metric space.

From hypotheses $(H1) - (H5)$, we will deduce that Φ is a contraction mapping. For $\xi, \psi \in Y$,

$$\begin{aligned} &D(\Phi\xi(t), \Phi\psi(t)) \\ &= D\Big(S(t)x_0 + S(t)g(t_1, t_2, \cdots, t_p, \xi(.)) \\ &\quad + \int_0^t S(t-s)f(s, \xi(\sigma_1(s)), \xi(\sigma_2(s)), \cdots, \xi(\sigma_n(s)))ds \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(\xi(t_k)), S(t)x_0 + S(t)g(t_1, t_2, \cdots, t_p, \psi(.)) \\ &\quad + \int_0^t S(t-s)f(s, \psi(\sigma_1(s)), \psi(\sigma_2(s)), \cdots, \psi(\sigma_n(s)))ds \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(\psi(t_k))\Big) \\ &\le cD(g(t_1, t_2, \cdots, t_p, \xi(.)), g(t_1, t_2, \cdots, t_p, \psi(.))) \\ &\quad + \int_0^t cD(f(s, \xi(\sigma_1(s)), \xi(\sigma_2(s)), \cdots, \xi(\sigma_n(s)))ds, \\ &\quad f(s, \psi(\sigma_1(s)), \psi(\sigma_2(s)), \cdots, \psi(\sigma_n(s)))ds\Big) \\ &\quad + cD\big(I_k(\xi(t_k)), I_k(\psi(t_k))\big) \\ &\le cl_2D(\xi(.), \psi(.)) + c\int_0^t l_0l_1D(\xi(s), \psi(s))ds + cl_3D(\xi, \psi) \end{aligned}$$

Then we obtain

$$\begin{aligned} H(\Phi\xi, \Phi\psi) &\le c\sup_{t\in\gamma}\left\{l_2D(\xi(.), \psi(.)) + \int_0^t l_0l_1D(\xi(s), \psi(s))ds + l_3D(\xi, \psi)\right\} \\ &\le cl_2D(\xi(.), \psi(.)) + c\gamma l_0l_1D(\xi(t), \psi(t)) + cl_3D(\xi, \psi) \\ &\le c(l_2 + l_0l_1\gamma + l_3)H(\xi, \psi). \end{aligned}$$

Since $c(\gamma l_0 l_1 + l_2 + l_3) < 1$, Φ is a contraction map. Hence, Φ has a unique fixed point $x \in C([0, \gamma], E^n)$ such that $\Phi x = x$, that is,

$$\begin{aligned} x(t) &= S(t)x_0 + S(t)g(t_1, t_2, \cdots, t_p, x(.)) \\ &+ \int_0^t S(t-s) f(s, x(\sigma_1(s)), x(\sigma_2(s)), \cdots, x(\sigma_n(s)))ds \\ &+ \sum_{0<t_k<t} S(t - t_k) I_k(x(t_k)). \end{aligned}$$

■

Theorem 3.3. Let f, σ and g be as in Theorem 3.1. Denote by $x(t, x_0)$, $y(t, y_0)$ the solutions of equation $(1.1) - (1.3)$ corresponding to x_0, y_0 respectively. Then there exists constant $\eta > o$ such that

$$H(x(., x_0), y(., y_0)) \le \eta\, [D(x_0, y_0)]$$

for any $x_0, y_0 \in E^n$ and $\eta = c/1 - c(l_2 + \gamma l_0 l_1 + l_3)$.

Proof. Let $x(t, x_0)$, $y(t, y_0)$ be solutions of equations $(1.1) - (1.3)$ corresponding to x_0, y_0 respectively. Then

$$\begin{aligned} & D(x(t, x_0), y(t, y_0)) \\ &= D\Big(S(t)x_0 + S(t)g(t_1, t_2, \cdots, t_p, x(., x_0)) \\ &\quad + \int_0^t S(t-s) f(s, x(\sigma_1(s)), x(\sigma_2(s)), \cdots, x(\sigma_n(s)))ds \\ &\quad + \sum_{0<t_k<t} S(t-t_k) I_k(x(t_k, x_0)), S(t)y_0 + S(t)g(t_1, t_2, \cdots, t_p, y(., y_0)) \\ &\quad + \int_0^t S(t-s) f(s, y(\sigma_1(s)), y(\sigma_2(s)), \cdots, y(\sigma_n(s)))ds \\ &\quad + \sum_{0<t_k<t} S(t-t_k) I_k(y(t_k, y_0))\Big) \end{aligned}$$

$$\begin{aligned} &\le cD(x_0, y_0) + cD\big(g(t_1, t_2, \cdots, t_p, x(., x_0)), g(t_1, t_2, \cdots, t_p, y(., y_0))\big) \\ &\quad + \int_0^t cD\big(f(s, x(\sigma_1(s)), x(\sigma_2(s)), \cdots, x(\sigma_n(s))), \\ &\qquad\qquad f(s, y(\sigma_1(s)), y(\sigma_2(s)), \cdots, y(\sigma_n(s)))\big)ds \\ &\quad + cD\big(I_k(x(t_k, x_0)), I_k(y(t_k, y_0))\big) \\ &\le cD(x_0, y_0) + cl_2 D(x(.), y(.)) + \int_0^t c l_0 l_1 D(x(s), y(s))ds + cl_3 D(x, y) \end{aligned}$$

Thus,

$$H(x(.,x_0),y(.,y_0)) \leq cD(x_0,y_0) + c(l_2+\gamma l_0 l_1 + l_3)H(x(.,x_0),y(.,y_0))$$

that is,

$$H(x(.,x_0),y(.,y_0)) \leq c/1 - c(l_2+\gamma l_0 l_1 + l_3)\,[D(x_0,y_0)]\,.$$

Taking $\eta = c/1 - c(l_2+\gamma l_0 l_1 + l_3)$, then we get

$$H(x(.,x_0),y(.,y_0)) \leq \eta\,[D(x_0,y_0)]\,.$$

This completes the proof of the theorem.

Now, we generalize the above theorem for the fuzzy delay impulsive differential equation (1.1)–(1.3) with nonlocal condition. ■

Theorem 3.4. If $f : J \times E^{n^2} \to E^n$ is levelwise continuous and bounded, $\sigma_i : J \to J \quad i = 1,2,\cdots,n)$ are continuous and $g : J^p \times E^n \to E^n$ is continuous, then the initial value problem (1.1)–(1.3) possesses atleast one solution on the interval J.

Proof. Since f is continuous and bounded and g is a continuous function there exists $r \geq 0$ such that

$$D(f(t,x(\sigma_1(t)),x(\sigma_2(t)),\cdots,x(\sigma_n(t)),\hat{0}) \leq r, \quad t \in J, x \in E^n.$$

Let B be a bounded set in $C(J,E^n)$. The set $\Phi B = \{\Phi x : x \in B\}$ is totally bounded if and only if it is equicontinuous and for every $t \in J$, the set $\Phi B(t) = \{\Phi x(t) : t \in J\}$ is a totally bounded subset of E^n. For every $t_0, t_1 \in J$ with $t_0 \leq t_1$, and $x \in B$ we have that

$$\begin{aligned}
D(\Phi x(t_0),\Phi x(t_1)) = D\Big(& S(t)x_0 + S(t)g(t_1,t_2,\cdots,t_p,x(.)) \\
&+ \int_0^{t_0} S(t-s)f(s,x(\sigma_1(s)),x(\sigma_2(s)),\cdots,x(\sigma_n(s)))ds \\
&+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_0)), S(t)x_0 + S(t)g(t_1,t_2,\cdots,t_p,x(.)) \\
&+ \int_0^{t_1} S(t-s)f(s,x(\sigma_1(s)),x(\sigma_2(s)),\cdots,x(\sigma_n(s)))ds \\
&+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_1))\Big)
\end{aligned}$$

$$\leq cD\Big(\int_0^{t_0} f(s, x(\sigma_1(s)), x(\sigma_2(s)), \cdots, x(\sigma_n(s)))ds,$$
$$\int_0^{t_1} f(s, x(\sigma_1(s)), x(\sigma_2(s)), \cdots, x(\sigma_n(s)))ds\Big)$$
$$+ cl_3 D(x, y)$$
$$\leq \int_{t_0}^{t_1} cD(f(s, x(\sigma_1(s)), x(\sigma_2(s)), \cdots, x(\sigma_n(s))), \hat{0})ds$$
$$+ cl_3 D(x, y)$$
$$\leq c|t_1 - t_0|.\sup\{D(f(t, x(\sigma_1(t)), x(\sigma_2(t)), \cdots, x(\sigma_n(t))), \hat{0})$$
$$+ cl_3 D(x, y)\}$$
$$\leq c|t_1 - t_0|.r. + cl_3$$

This shows that ΦB is equicontinuous. Now, for $t \in J$ fixed. We have

$$D(\Phi x(t), \Phi x(t')) \leq c|t - t'|.r + cl_3,$$

for every $t' \in J,\ x \in B.$

Hence, the set $\{\Phi x(t) : x \in B\}$ is totally bounded in E^n. Thus, we conclude that ΦB is a relatively compact subset of $C(J, E^n)$ by Ascoli theorem. Then Φ is compact, that is, Φ transforms bounded sets in to relatively compacts sets.

We know that $x \in C(J, E^n)$ is a solution of $(1.1) - (1.3)$ if and only if x is a fixed point of the operator Φ. Now, in the metric space $(C(J, E^n), H)$. Consider the ball

$$B = \{\xi \in C(J, E^n), H(\xi, \hat{0}) \leq m\},\ m = a.r.$$

Thus, $\Phi B \subset B$. Indeed, for $x \in C(J, E^n)$,

$$\begin{aligned}
D(\Phi x(t), \Phi x(0)) &= D\Big(S(t)x_0 + S(t)g(t_1, t_2, \cdots, t_p, x(.)) \\
&\quad + \int_0^t S(t-s)f(s, x(\sigma_1(s)), x(\sigma_2(s)), \cdots, x(\sigma_n(s)))ds, \\
&\quad + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k)), S(t)x_0 + S(t)g(t_1, t_2, \cdots, t_p, x(.))\Big) \\
&\leq \int_0^t cD\left(f(s, x(\sigma_1(s)), x(\sigma_2(s)), \cdots, x(\sigma_n(s)), \hat{0}\right) ds \\
&\quad + cD\Big(\sum_{0<t_k<t} I_k(x(t_k)), \hat{0}\Big) \\
&\leq c|t|.r + c\chi \\
&\leq c(a.r + \chi)
\end{aligned}$$

Therefore, defining $\hat{0} : J \to E^n$, $\hat{0}(t) = \hat{0},\ t \in J$, we have

$$H(\Phi x, \Phi\hat{0}) = \sup\{D(\Phi x(t), \Phi\hat{0}(t)) : t \in J\}$$

Therefore, Φ is compact and in consequence, it has a fixed point $x \in B$. This fixed point is a solution of the initial value problem $(1.1) - (1.3)$. ■

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