Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 3 (2015), pp. 1193-1199

http://www.ripublication.com

Results for a New Subclass of Spirallike Functions

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Abstract

This paper introduces and investigates a new subclass $S_\beta(\alpha, \lambda)$ of spirallike functions. Coefficient inequalities and subordination results are proved.

AMS subject classification: 30C45.
Keywords: Analytic functions, starlike functions, convex functions, spriallike functions, subordination, subordinating factor sequence, coefficient inequalities.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| \leq 1\}$. Let S denote the subclass of A consisting of analytic and univalent functions in U. Also let S^* and K denote the subclasses of S that are respectively starlike and convex.

A function $f \in A$ is said to be in the class of α-spirallike functions of order β in U, denoted by $S^*(\alpha, \beta)$ if

$$Re\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > \beta\cos\alpha, \quad z \in U,$$

for $0 \leq \beta < 1$ and for some real α with $|\alpha| < \pi/2$. The class $S^*(\alpha, \beta)$ was studied by R.J. Libera [2] and F.R. Keogh and E.P. Merkes [1]. We observe that $S^*(\alpha, 0)$ is the class of spirallike functions introduced by L. Spacek [4], $S^*(0,\beta) = S^*(\beta)$ is the class of starlike functions of order β and $S^*(0,0) = S^*$ is the familiar class of starlike functions.

Definition 1.1. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are analytic in U, then the Hadamard product or convolution of f and g is

$$f * g = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Definition 1.2. (Subordination Principle) Let $f(z)$ and $g(z)$ be analytic in U. Then we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write

$$f \prec g$$

or

$$f(z) \prec g(z)$$

if there exists a Schwartz function $\omega(z)$, analytic in U with

$$\omega(0) = 0, |\omega(z)| < 1 \ \ (z \in U),$$

such that

$$f(z) = g(\omega(z)) \ \ (z \in U).$$

In particular, if the function $g(z)$ is univalent in U, then

$$f(z) \prec g(z) \ \ (z \in U) \Leftrightarrow f(0) = g(0) \ \text{ and } \ f(U) \subset g(U).$$

Definition 1.3. (Subordinating factor sequence) A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z) \ \ (z \in U; a_1 = 1).$$

Lemma 1.4. [5] The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if

$$Re\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0 \ \ (z \in U).$$

Motivated by [3], we now introduce a new subclass of spirallike functions $S_\beta(\alpha,\lambda)$.

2. Definitions and Results

Definition 2.1. Let $F(z) = \dfrac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}$, $f(z) \in S$, $0 \le \lambda \le 1$. When $\lambda = 0$, $F(z)$ reduces to $\dfrac{zf'(z)}{f(z)}$ and when $\lambda = 1 F(z) = 1 + \dfrac{zf''(z)}{f'(z)}$.

A function $f(z) \in S$ is said to be in the class $S_\beta(\alpha, \lambda)$ if it satisfies the inequality

$$\left| \frac{1}{e^{i\beta} F(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad z \in U. \tag{2.1}$$

for some real β and $0 < \alpha < 1$.

Theorem 2.2. $f(z) \in S_\beta(\alpha, \lambda)$ iff $Re\left(e^{i\beta} \dfrac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \right) > \alpha$.

Proof. Let $F(z) = \dfrac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}$ for $f(z) \in S$. If $f(z) \in S_\beta(\alpha, \lambda)$ then

$$\begin{aligned}
&\left| \frac{2\alpha - e^{i\beta}F(z)}{2\alpha e^{i\beta}F(z)} \right| < \frac{1}{2\alpha} \\
&\Leftrightarrow \left| \frac{2\alpha - e^{i\beta}F(z)}{2\alpha e^{i\beta}F(z)} \right|^2 < \left(\frac{1}{2\alpha} \right)^2 \\
&\Leftrightarrow (2\alpha - e^{i\beta}F(z))[\overline{2\alpha - e^{i\beta}F(z)}] < [e^{-i\beta}\overline{F(z)}]e^{i\beta}F(z) \\
&\Leftrightarrow (2\alpha - e^{i\beta}F(z))[\overline{2\alpha - e^{i\beta}F(z)}] < [e^{-i\beta}\overline{F(z)}]e^{i\beta}F(z) \\
&\Leftrightarrow 4\alpha^2 - 2\alpha e^{-i\beta}\overline{F(z)} - 2\alpha e^{i\beta}F(z) + F(z)\overline{F(z)} < F(z)\overline{F(z)} \\
&\Leftrightarrow 4\alpha^2 - 2\alpha(e^{-i\beta}\overline{F(z)} + e^{i\beta}F(z)) < 0 \\
&\Leftrightarrow 2\alpha - 2Re(e^{i\beta}F(z)) < 0 \\
&\Leftrightarrow Re(e^{i\beta}F(z)) > \alpha \\
&\Leftrightarrow Re\left(e^{i\beta} \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \right) > \alpha.
\end{aligned}$$

This completes the proof. ■

Theorem 2.3. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \{n + |n - 2\alpha e^{-i\beta}|\} |1 + \lambda(n-1)| |a_n| \le 1 - |1 - 2\alpha e^{-i\beta}| \tag{2.2}$$

for some $|\beta| < \pi/2$, $0 < \alpha < \cos\beta$ and $0 \le \lambda \le 1$, then $f(z) \in S_\beta(\alpha, \lambda)$.

Proof. It is enough if we show that $\left|\frac{2\alpha - e^{i\beta}F(z)}{e^{i\beta}F(z)}\right| < 1$, for some $|\beta| < \pi/2$, $0 < \alpha < \cos\beta$ and $0 \le \lambda \le 1$, where $F(z) = \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}$.

Now,

$$\begin{aligned}
\left|\frac{2\alpha - e^{i\beta}F(z)}{e^{i\beta}F(z)}\right| &= \left|\frac{2\alpha - e^{i\beta}\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}}{e^{i\beta}\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}}\right| \\
&= \left|\frac{2\alpha e^{-i\beta}((1-\lambda)f(z) + \lambda z f'(z)) - (zf'(z) + \lambda z^2 f''(z)}{zf'(z) + \lambda z^2 f''(z)}\right| \\
&= \left|\frac{(1 - 2\alpha e^{-i\beta} + \sum_{n=2}^{\infty}(2\alpha e^{-i\beta}(1-\lambda+\lambda n) - n(1-\lambda(n-1))|a_n||z|^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}(1+\lambda(n-1))}\right| \\
&\le \frac{|1 - 2\alpha e^{-i\beta}| + \sum_{n=2}^{\infty}|1+\lambda(n-1)||n - 2\alpha e^{-i\beta}||a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty}|n||a_n||z|^{n-1}|1+\lambda(n-1)|} \\
&< \frac{|1 - 2\alpha e^{-i\beta}| + \sum_{n=2}^{\infty}|n - 2\alpha e^{-i\beta}||1+\lambda(n-1)||a_n|}{1 - \sum_{n=2}^{\infty} n|a_n||1+\lambda(n-1)|} \qquad (2.3)
\end{aligned}$$

Hence, if

$$\sum_{n=2}^{\infty}\{n + |n - 2\alpha e^{-i\beta}|\}|1+\lambda(n-1)||a_n| \le 1 - |1 - 2\alpha e^{-i\beta}|$$

for some $|\beta| < \pi/2$, $0 < \alpha < \cos\beta$ and $0 \le \lambda \le 1$, then

$$\begin{aligned}
&\sum_{n=2}^{\infty}|n - 2\alpha e^{-i\beta}||1+\lambda(n-1)||a_n| \\
&\le 1 - \sum_{n=2}^{\infty} n|a_n||1+\lambda(n-1)| - |1 - 2\alpha e^{-i\beta}| \qquad (2.4)
\end{aligned}$$

Using (2.4) in (2.3), we get

$$\begin{aligned}
\left|\frac{2\alpha - e^{i\beta}F(z)}{e^{i\beta}F(z)}\right| &< \frac{|1 - 2\alpha e^{-i\beta}| + 1 - \sum_{n-2}^{\infty} n|a_n||1+\lambda(n-1)| - |1 - 2\alpha e^{-i\beta}|}{1 - \sum_{n=2}^{\infty} n|a_n||1+\lambda(n-1)|} \\
&= 1,
\end{aligned}$$

which implies $f(z) \in S_\beta(\alpha, \lambda)$ for some $|\beta| < \pi/2$, $0 < \alpha < \cos\beta$ and $0 \le \lambda \le 1$. ■

When $\beta = \pi/4$ in Theorem 2.3 we have the following result.

Corollary 2.4. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty}\{n+\sqrt{n^2-2\sqrt{2}\alpha n+4\alpha^2}\}|1+\lambda(n-1)||a_n| \leq 1-\sqrt{1-2\sqrt{2}\alpha+4\alpha^2}$$

for some $0<\alpha<\dfrac{\sqrt{2}}{2}$, then $f(z) \in S_{\pi/4}(\alpha,\lambda)$.

Corollary 2.5. Theorem 2.3 reduces to Theorem 2 of [3], when $\lambda = 0$ and Theorem 7 when $\lambda = 1$.

Let $f \in H_\lambda(\alpha,\beta)$, the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ whose coefficients satisfy the conditon (2.2).

Theorem 2.6. Let $f \in H_\lambda(\alpha,\beta)$. Then

$$\frac{1+|1-2\alpha e^{-i\beta}|}{2(2+|1-2\alpha e^{-i\beta}|)(1+\lambda)}(f*g)(z) \prec g(z) \tag{2.5}$$

for $z \in U$ and for every function $g(z)$ in the class K.

In particular,

$$Re\ f(z) > -\frac{[2+|1-2\alpha e^{-i\beta}|](1+\lambda)}{1+|1-2\alpha e^{-i\beta}|} \text{ for } z \in U.$$

The constant $\dfrac{1+|1-2\alpha e^{-i\beta}|}{2[2+|1-2\alpha e^{-i\beta}|](1+\lambda)}$ cannot be replaced by any larger one.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $H_\lambda(\alpha,\beta)$ and let $g(z) = z + \sum_{n=2}^{\infty} c_n z^n$ be in K. Then

$$\frac{1+|1-2\alpha e^{-i\beta}|}{2(2+|1-2\alpha e^{-i\beta}|)(1+\lambda)}(f*g)(z) = \frac{1+|1-2\alpha e^{-i\beta}|}{2(2+|1-2\alpha e^{-i\beta}|)(1+\lambda)}\left(z+\sum_{n=2}^{\infty} a_n c_n z^n\right).$$

By Definition 1.3, the theorem will hold if the sequence $\left(\dfrac{\{1+|1-2\alpha e^{-i\beta}|\}a_n}{2(2+|1-2\alpha e^{-i\beta}|)(1+\lambda)}\right)_{n=1}^{\infty}$ is subordinating factor sequence with $a_1 = 1$. In view of Lemma 1.4, this is true iff

$$Re\left\{1+2\sum_{n=2}^{\infty}\frac{1+|1-2\alpha e^{-i\beta}|}{2(2+|1-2\alpha e^{-i\beta}|)(1+\lambda)}a_n z^n\right\} > 0, \quad \text{for } z \in U.$$

we have

$$Re\left[1+\sum_{n=1}^{\infty}\frac{1+|1-2\alpha e^{-i\beta}|}{(2+|1-2\alpha e^{-i\beta}|)(1+\lambda)}a_nz^n\right]$$

$$= Re\left[1+\frac{1+|1-2\alpha e^{-i\beta}|}{(2+|1-2\alpha e^{-i\beta}|)(1+\lambda)}z\right.$$

$$\left.+\frac{1}{(2+|1-2\alpha e^{-i\beta}|)(1+\lambda)}\sum_{n=2}^{\infty}(1+|1-2\alpha e^{-i\beta}|)a_nz^n\right].$$

But $1+|1-2\alpha e^{-i\beta}| \le n+|n-2\alpha e^{-i\beta}| \ \forall \ n\ge 2, |\beta|<\pi/2$ and $0<\alpha<\cos\beta$, we get

$$Re\left[1+\sum_{n=1}^{\infty}\frac{1+|1-2\alpha e^{-i\beta}|}{(2+|1-2\alpha e^{-i\beta}|)(1+\lambda)}a_nz^n\right]$$

$$> \left[1-\frac{1+|1-2\alpha e^{-i\beta}|}{[2+|1-2\alpha e^{-i\beta}|](1+\lambda)}r\right.$$

$$\left.-\frac{1-|1-2\alpha e^{-i\beta}|}{(2+|1-2\alpha e^{-i\beta}|)(1+\lambda)}\sum_{n=2}^{\infty}\frac{n+|n-2\alpha e^{-i\beta}|}{1-|1-2\alpha e^{-i\beta}|}|a_n|r^n\right]$$

$$\ge \left[1-\frac{1+|1-2\alpha e^{-i\beta}|}{[2+|1-2\alpha e^{-i\beta}|](1+\lambda)}r-\frac{1-|1-2\alpha e^{-i\beta}|r}{(2+|1-2\alpha e^{-i\beta}|)(1+\lambda)}\right]$$

$$= \left[1-\frac{r}{[2+|1-2\alpha e^{-i\beta}|](1+\lambda)}(1+|1-2\alpha e^{-i\beta}|+1-|1-2\alpha e^{-i\beta}|)\right]$$

$$= 1-\frac{2r}{[2+|1-2\alpha e^{-i\beta}|](1+\lambda)} > 0.$$

Hence

$$Re\ f(z) > -\frac{[2+|1-2\alpha e^{-i\beta}|](1+\lambda)}{1+|1-2\alpha e^{-i\beta}|}$$

for $f(z)\in H_\lambda(\alpha,\beta)$, by taking $g(z)=\dfrac{z}{1-z}$ in (2.4). For proving the sharpness of the constant $\dfrac{1+|1-2\alpha e^{-i\beta}|}{2[2+|1-2\alpha e^{-i\beta}|](1+\lambda)}$ we consider the function

$$f_0(z) = z-\frac{1}{1+|1-2\alpha e^{-i\beta}|}z^2 \quad (|\beta|<\pi/2, 0<\alpha<\cos\beta),$$

$f_0(z)\in H_\lambda(\alpha,\beta)$. Hence from (2.4) we get $\dfrac{1+|1-2\alpha e^{-i\beta}|}{2[2+|1-2\alpha e^{-i\beta}|](1+\lambda)}f_0(z)\prec\dfrac{z}{1-z}$.

Also $\min\limits_{|z|<1} Re\left[\dfrac{1+|1-2\alpha e^{-i\beta}|}{2[2+|1-2\alpha e^{-i\beta}|](1+\lambda)}f_0(z)\right]=-\dfrac{1}{2}$. ■

Corollary 2.7. Theorem 2.2 reduces to Theorem 2.3 given by [3], when $\lambda = 0$ and Theorem 2.6 when $\lambda = 1$.

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