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Perturbation of n - Dimensional Additive Functional Equation in Banach Spaces and Banach Algebra

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Abstract

The stability of functional equation $\sum_{i=1}^{n} f\left(\sum_{j=1}^{i} x_j\right) = \sum_{i=1}^{n} (n-i+1) f(x_i)$ in Banach spaces and Banach algebra was investigated through this paper using Hyers direct and fixed point approch.

Keywords: Additive functional equation, additive derivation, generalized additive derivation, Ulam - Hyers stability.

Introduction

Stability of functional equation initially sustains through the question of S.M. Ulam [17]. For the question of S.M.Ulam respective opportune respond is obtained by Hyers [9] for Banach space with brief statement. Ensuing T. Aoki [2] in his research generalized, the Hyers result for additive equation. Progressively more probe acquire by [1,4,7-8, 10-12, 14-16].

The principal equation in the study of stability of functional equation is the equation

$$f(x+y) = f(x) + f(y). \tag{1}$$

which is additive functional equation having solution $f(x) = cx$. Many researchers have their results about the stability of (1) in various spaces. Recently M. Arun kumar and T. Namachivayam [3,5] introduced the functional equation

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{i} x_j\right) = \sum_{i=1}^{n} (n-i+1) f\left(x_i\right) \tag{2}$$

have noted that it is stable in Random normed space and provideed the solution which is additive. Through this paper, the authors investigate how the stability of equation (2) in Banach spaces and Banach algebra, with the help of Hyers direct and fixed point method.

Stability Results: Hyers Direct Method

Consider E be a normed space, F be a Banach space and the mapping $\Upsilon_f : E^n \to F$ such that

$$\Upsilon_f(\mathrm{x}) = \Upsilon_f(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} f\left(\sum_{j=1}^{i} x_j\right) - \sum_{i=1}^{n} (n-i+1) f\left(x_i\right), \ \forall\ \mathrm{x} = (x_1, x_2, \cdots, x_n) \in E^n$$

The subsequent theorem explore the generalized Ulam-Hyers stability of the functional equation (2).

Theorem 2.1

Let $f : E \to F$ *and* $\zeta : E^n \to [0, \infty)$ *be a functions such that*

$$\lim_{m \to \infty} \frac{1}{2^{mj}} \zeta(2^{mj}\mathrm{x}) = 0, \tag{3}$$

$$\left\| \Upsilon_f(\mathrm{x}) \right\| \le \zeta(\mathrm{x}), \qquad \forall\, \mathrm{x} = (x_1, x_2, \cdots, x_n) \in E^n. \tag{4}$$

where $j = \pm 1$. *Then* $\exists\ A : E \to F$ *satisfying (2) which is unique additive function and*

$$\| f(x) - A(x) \| \le \sum_{k=\frac{1-j}{2}}^{\infty} \frac{1}{2^{kj}} \xi(2^{kj} x), \qquad \forall\ x \in E, \tag{5}$$

where $\xi(2^{kj} x)$ *and* $A(x)$ *are such that*

$$\xi(2^{kj} x) = \frac{1}{\left(n - \frac{1-(-1)^n}{2}\right)} \zeta(2^{kj} x, 2^{kj} x, 2^{kj}(-x), 2^{kj} x, 2^{kj}(-x), 2^{kj} x, \cdots, 2^{kj}(-1)^n x) \tag{6}$$

and

$$A(x) = \lim_{k \to \infty} \frac{1}{2^{kj}} f(2^{kj} x), \qquad \forall\ x \in E. \tag{7}$$

Proof: Take $j=1$. Changing $\mathrm{x}=(x_1,x_2,x_3,x_4,x_5,x_6,\cdots,x_n)$ into $(x,x,-x,x,-x,x,\cdots,(-1)^n x)$ in (4) and by the additive property of f, we get

$$\left\|\frac{\left(n-\frac{1-(-1)^n}{2}\right)}{2}f(2x)-\left(n-\frac{1-(-1)^n}{2}\right)f(x)\right\|\le\zeta(x,x,-x,x,-x,x,\cdots,(-1)^n x), \tag{8}$$

$$\Rightarrow\left\|\frac{f(2x)}{2}-f(x)\right\|\le\xi(x) \text{ where } \xi(x)=\frac{1}{\left(n-\frac{1-(-1)^n}{2}\right)}\zeta(x,x,-x,x,-x,x,\cdots,(-1)^n x), \tag{9}$$

$\forall\ x\in E$. Now changing x into $2x$ and multiply by $\frac{1}{2}$ in (9), we arrive

$$\left\|\frac{f(2^2x)}{2^2}-\frac{f(2x)}{2}\right\|\le\frac{\xi(2x)}{2},\quad \forall\ x\in E. \tag{10}$$

Using (9) and (10), we get

$$\left\|\frac{f(2^2x)}{2^2}-f(x)\right\|\le\left\|\frac{f(2x)}{2}-f(x)\right\|+\left\|\frac{f(2^2x)}{2^2}-\frac{f(2x)}{2}\right\|\le\xi(x)+\frac{\xi(2x)}{2},. \tag{11}$$

$\forall\ x\in E$. Repeting the above process m - times, we obtain

$$\left\|\frac{f(2^m x)}{2^m}-f(x)\right\|\le\sum_{k=0}^{m-1}\frac{\xi(2^k x)}{2^k}\le\sum_{k=0}^{\infty}\frac{\xi(2^k x)}{2^k},\quad \forall\ x\in E\cdot \tag{12}$$

For to prove the convergence of $\left\{\frac{f(2^m x)}{2^m}\right\}$, replace x by $2^l x$ and multiplying by $\frac{1}{2^l}$ in (12), for any $m,l>0$, we deduce

$$\left\|\frac{f(2^{m+l}x)}{2^{(m+l)}}-\frac{f(2^l x)}{2^l}\right\|=\frac{1}{2^l}\left\|\frac{f(2^m\cdot 2^l x)}{2^m}-f(2^l x)\right\|\le\sum_{k=0}^{\infty}\frac{\xi(2^{k+l}x)}{2^{k+l}}\to 0\ \ as\ \ m\to\infty,$$

$\forall x\in E$. Then, we conclude $\left\{\frac{f(2^m x)}{2^m}\right\}$ is a Cauchy sequence. Since F is complete, $\exists\ A:E\to F$ such that $A(x)=\lim_{m\to\infty}\frac{f(2^m x)}{2^m},\ \forall\ x\in E.$ By taking $m\to\infty$ in (12), we can conclude that (5) holds. To establish that A satisfies (2), taking $(x_1,x_2,x_3,\cdots,x_n)$ by $(2^m x_1,2^m x_2,2^m x_3,\cdots,2^m x_n)$ and multiplying by $\frac{1}{2^m}$ in (4), we find $\frac{1}{2^m}\left\|\Upsilon_f(2^m\mathrm{x})\right\|\le\frac{1}{2^m}\zeta(2^m\mathrm{x})$ for all $\mathrm{x}=(x_1,x_2,x_3,\cdots,x_n)\in E^n$. Letting $m\to\infty$, we see that $\Upsilon_A(x_1,x_2,\cdots,x_n)=0.$ Then we conclude A satisfy (2) for

$x = (x_1, x_2, x_3, \cdots, x_n) \in E^n$. Now we prove the uniqueness of A, let B be one more additive function satisfying (2) and (5), then

$$\|A(x) - B(x)\| \le \frac{1}{2^m}\left\{ \|A(2^m x) - f(2^m x)\| + \|f(2^m x) - B(2^m x)\| \right\} \le \sum_{k=0}^{\infty} \frac{\xi(2^{k+m} x)}{2^{(k+m)}} \to 0 \ \ as\ m \to \infty$$

This shows the uniqueness of A. For the case $j = -1$, we can show alike stability result.

The subsequent Corollary shows that the results of different substutions of control function $\zeta(x)$ in Theorem 2.1 relating to the stability of (2).

Corollary 2.2

For the positive real numbers ρ and s, the function $f : E \to F$ satisfies the inequality

$$\|\Upsilon_f(x)\| \le \begin{cases} \rho, & \\ \rho \sum_{i=1}^{n} \| x_i \|^s, & s < 1 \quad or \quad s > 1; \\ \rho \prod_{i=1}^{n} \| x_i \|^s, & s < \frac{1}{n} \quad or \quad s > \frac{1}{n}; \\ \rho \left\{ \prod_{i=1}^{n} \| x_i \|^s + \sum_{i=1}^{n} \| x_i \|^{ns} \right\}, & s < \frac{1}{n} \quad or \quad s > \frac{1}{n}; \end{cases} \tag{13}$$

for $x = (x_1, x_2, \cdots, x_n) \in E^n$. Then $\exists$ $A : E \to F$ satisfying (2) which is unique additive function such that

$$\|f(x) - A(x)\| \le \begin{cases} \dfrac{2\rho}{\left(n - \frac{1-(-1)^n}{2}\right)}, \\ \dfrac{2n\rho \| x \|^s}{\left(n - \frac{1-(-1)^n}{2}\right) | 2 - 2^s |}, \\ \dfrac{2\rho \| x \|^{ns}}{\left(n - \frac{1-(-1)^n}{2}\right) | 2 - 2^{ns} |} \\ \dfrac{2(n+1)\rho \| x \|^{ns}}{\left(n - \frac{1-(-1)^n}{2}\right) | 2 - 2^{ns} |} \end{cases}, \quad \forall x \in E \tag{14}$$

Stability Results: Fixed Point Method

With the help of B. Margolis and J.B. Diaz contraction principle and alternative fixed point theorem [13], we arrive the stability of functional equation (2) it follows.

Theorem 3.1

Let $f : E \to F$ and $\zeta : E^n \to [0, \infty)$ be a functions such that

$$\lim_{k\to\infty}\frac{1}{\kappa_i^k}\zeta(\kappa_i^k \mathrm{x}) = 0, \tag{15}$$

$$\left\|\Upsilon_f(\mathrm{x})\right\| \le \zeta(\mathrm{x}),\ \forall\ \mathrm{x} = (x_1, x_2 \cdots, x_n) \in E^n, \tag{16}$$

$$\text{where}\ \ \kappa_i = \begin{cases} 2 & if \quad i = 0; \\ \frac{1}{2} & if \quad i = 0, \end{cases}. \tag{17}$$

If $\exists\ \ L = L(i) < 1$ *such that the mapping*

$$x \to \psi(x) = \xi\left(\frac{x}{2}\right) = \frac{2}{\left(n - \frac{1-(-1)^n}{2}\right)}\zeta\left(\frac{x}{2}, \frac{x}{2}, \frac{-x}{2}, \frac{x}{2}, \frac{-x}{2}, \frac{x}{2}, \cdots, (-1)^n\frac{x}{2}\right)$$

with the condition $L\psi(x) = \frac{1}{\kappa_i}\psi\left(\kappa_i x\right), \quad \forall x \in E$. (18)

Then $\exists\ \ A: E \to F$ *satisfying (2) which is unique additive function such that*

$$\left\| f(x) - A(x)\right\| \le \frac{L^{1-i}}{1-L}\psi(x), \qquad \forall x \in E. \tag{19}$$

proof: Consider, $\Gamma = \{p/p: E \to F,\ p(0) = 0\}$ and define the metric on Γ,

$$d(p,q) = \inf\{K \in (0,\infty) : \left\|p(x) - q(x)\right\| \le K\psi(x), x \in E\}.$$

Obviously, (Γ, d) is complete. Define $\chi : \Gamma \to \Gamma$ by $\chi_p(x) = \frac{1}{\kappa_i}p(\kappa_i x),\ \forall x \in E$. For $p, q \in \Gamma$,

$$d(p,q) \le K \Rightarrow \left\|p(x) - q(x)\right\| \le K\psi(x), x \in E.$$

$$\Rightarrow \left\|\frac{1}{\kappa_i}p(\kappa_i x) - \frac{1}{\kappa_i}q(\kappa_i x)\right\| \le \frac{1}{\kappa_i}K\psi(\kappa_i x), x \in E,$$

$$\Rightarrow \left\|\frac{1}{\kappa_i}p(\kappa_i x) - \frac{1}{\kappa_i}q(\kappa_i x)\right\| \le LK\psi(x), x \in E,$$

$$\Rightarrow \left\|\chi_p(x) - \chi_q(x)\right\| \le LK\psi(x), x \in E,$$

$$\Rightarrow d(p,q) \le LK.$$

Hence, $d(\chi_p, \chi_q) \le Ld(p,q), \forall\ p, q \in \Gamma$, i.e., χ is a strictly contractive mapping on Γ with Lipschitz constant L.

Changing $\mathrm{x} = (x_1, x_2, x_3, x_4, x_5, x_6, \cdots, x_n)$ into $(x, x, -x, x, -x, x, \cdots, (-1)^n x)$ in (16) and multiplying by 2, we get

$\| f(2x) - 2f(x) \| \le \xi(x)$

where $\xi(x) = \dfrac{2}{\left(n - \dfrac{1-(-1)^n}{2}\right)} \zeta(x, x, -x, x, -x, x, \cdots, (-1)^n x), \ \forall x \in E$. (20)

From (18) for $i = 0$ it trim down to $\left\| \dfrac{f(2x)}{2} - f(x) \right\| \le L\psi(x), \ \forall x \in E$,

i.e., $d(\chi_f, f) \le L \Rightarrow d(\chi_f, f) \le L = L^1 < \infty.$

Now substuting $x = \dfrac{x}{2}$ in (20), we get

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \le \xi\left(\frac{x}{2}\right), \qquad \forall x \in E \tag{21}$$

From (18) for $i = 1$ the above inequality condense to $\left\| f(x) - 2f\left(\dfrac{x}{2}\right) \right\| \le \psi(x), \ \forall x \in E$

i.e., $d(f, \chi_f) \le 1 \Rightarrow d(f, \chi_f) \le 1 = L^0 < \infty.$

Hence $d(f, \chi_f) \le L^{1-i}$. By fixed point alternative theorem, there exists a fixed point A of χ in Γ with

$$A(x) = \lim_{k \to \infty} \frac{f(\kappa_i^k x)}{\kappa_i^k}, \ \forall \ x \in E \tag{22}$$

It is clear that $A : E \to F$ is additive function and it satisfies the functional equation (2) for all $x_1, x_2, \cdots, x_n \in E$, by taking $k \to \infty$ after changing x into $\kappa_i^k x$ and dividing by κ_i^k in (17).

Again using the fixed point alternative theorem, we conclude A is the fixed point of χ in the set $\Delta = \{A \in \Gamma : d(f, A) < \infty\}$, then A unique with

$\| f(x) - A(x) \| \le K\psi(x), \ \forall \ x \in E$

and $K > 0$. Finally, $$d(f, A) \le \frac{1}{1-L} d(f, \chi_f)$$

this implies $$d(f, A) \le \frac{L^{1-i}}{1-L}$$

which yields $$\| f(x) - A(x) \| \le \frac{L^{1-i}}{1-L} \psi(x), \ \forall x \in E.$$

Hence the theorem.

The subsequent Corollary shows that the results of different substutions for control function $\zeta(\mathrm{x})$ in Theorem 3.1 relating to the stability of (2).

Corollary 3.2

For the positive real numbers ρ and s, the function $f:E\to F$ satisfies the inequality

$$\left\|\Upsilon_f(\mathrm{x})\right\| \le \begin{cases} (i) & \rho, \\ (ii) & \rho\sum_{j=1}^{n}\|x_j\|^s, & s<1 \quad or \quad s>1; \\ (iiii) & \rho\prod_{j=1}^{n}\|x_j\|^s, & s<\frac{1}{n} \quad or \quad s>\frac{1}{n}; \\ (iv) & \rho\left\{\prod_{j=1}^{n}\|x_j\|^s+\sum_{j=1}^{n}\|x_j\|^{ns}\right\}, & s<\frac{1}{n} \quad or \quad s>\frac{1}{n}; \end{cases},$$

$\forall\, x=(x_1,x_2,\cdots,x_n)\in E^n$. *Then* $\exists\ \ A:E\to F$ *satisfying (2) which is unique additive function such that*

$$\|f(x)-A(x)\| \le \begin{cases} (i) & \dfrac{\rho}{\left(n-\frac{1-(-1)^n}{2}\right)}, \\ (ii) & \dfrac{n\rho\|x\|^s}{\left(n-\frac{1-(-1)^n}{2}\right)|2-2^s|}, \\ (iii) & \dfrac{\rho\|x\|^{ns}}{\left(n-\frac{1-(-1)^n}{2}\right)|2-2^{ns}|} \\ (iv) & \dfrac{(n+1)\rho\|x\|^{ns}}{\left(n-\frac{1-(-1)^n}{2}\right)|2-2^{ns}|} \end{cases}, \ \forall\, x\in E. \qquad (24)$$

proof:

Assume $\zeta(\mathrm{x})=\begin{cases}\rho, \\ \rho\sum_{j=1}^{n}\|x_j\|^s, \\ \rho\prod_{j=1}^{n}\|x_j\|^s, \\ \rho\left\{\prod_{j=1}^{n}\|x_j\|^s+\sum_{j=1}^{n}\|x_j\|^{ns}\right\},\end{cases}$, $\forall\, x=(x_1,x_2,\cdots,x_n)\in E^n$.

Hence

$$\frac{1}{\kappa_i^k}\zeta(\kappa_i^k \mathrm{x}) = \begin{cases} \dfrac{\rho}{\kappa_i^k}, \\ \dfrac{\rho}{\kappa_i^k}\sum_{j=1}^{n} \| \kappa_i^k x_j \|^s, \\ \dfrac{\rho}{\kappa_i^k}\prod_{j=1}^{n} \| \kappa_i^k x_j \|^s, \\ \dfrac{\rho}{\kappa_i^k}\left\{\prod_{j=1}^{n} \| \kappa_i^k x_j \|^s + \sum_{j=1}^{n} \| \kappa_i^k x_j \|^{ns}\right\}, \end{cases} = \begin{cases} \to 0 \; as \;\; k \to \infty, \\ \to 0 \; as \;\; k \to \infty, \\ \to 0 \; as \;\; k \to \infty, \\ \to 0 \; as \;\; k \to \infty. \end{cases}$$

This shows that, (15) is true for our assumption. Also given that $\psi(x) = \xi\left(\frac{x}{2}\right)$ such that $L\psi(x) = \frac{1}{\kappa_i}\psi\left(\kappa_i x\right), \; \forall\, x \in E$.

Therefore

$$\begin{aligned} \psi(x) &= \xi\left(\frac{x}{2}\right) \\ &= \frac{2}{\left(n - \frac{1-(-1)^n}{2}\right)}\zeta\left(\frac{x}{2}, \frac{x}{2}, \frac{-x}{2}, \frac{x}{2}, \frac{-x}{2}, \frac{x}{2}, \cdots, \frac{(-1)^n x}{2}\right) \\ &= \begin{cases} \dfrac{2\rho}{\left(n - \frac{1-(-1)^n}{2}\right)} \\ \dfrac{2n\rho}{2^s\left(n - \frac{1-(-1)^n}{2}\right)} \| x \|^s, \\ \dfrac{2\rho}{2^{ns}\left(n - \frac{1-(-1)^n}{2}\right)} \| x \|^{ns}, \\ \dfrac{2(n+1)\rho}{2^{ns}\left(n - \frac{1-(-1)^n}{2}\right)} \| x \|^{ns}. \end{cases} \end{aligned}$$

$$\Rightarrow \quad \frac{1}{\kappa_i}\psi(\kappa_i x)=\begin{cases}\kappa_i^{-1}\dfrac{2\rho}{\left(n-\dfrac{1-(-1)^n}{2}\right)}\\ \kappa_i^{s-1}\dfrac{2n\rho}{2^s\left(n-\dfrac{1-(-1)^n}{2}\right)}\|x\|^s,\\ \kappa_i^{ns-1}\dfrac{2\rho}{2^{ns}\left(n-\dfrac{1-(-1)^n}{2}\right)}\|x\|^{ns},\\ \kappa_i^{ns-1}\dfrac{2(n+1)\rho}{2^{ns}\left(n-\dfrac{1-(-1)^n}{2}\right)}\|x\|^{ns}.\end{cases}=\begin{cases}\kappa_i^{-1}\psi(x)\\ \kappa_i^{s-1}\psi(x),\\ \kappa_i^{ns-1}\psi(x),\\ \kappa_i^{ns-1}\psi(x).\end{cases}$$

.

The above equation shows that, (18) is true for the assumption of $\zeta(\mathrm{x})$ in particular, the values of L, considered as $L=\begin{cases}\frac{1}{2}, & for\ \ i=0\\ 2, & for\ \ i=1\end{cases}$, $L=\begin{cases}2^{s-1}, & for\ \ i=0\\ \frac{1}{2^{s-1}}, & for\ \ i=1\end{cases}$, $L=\begin{cases}2^{ns-1}, & for\ \ i=0\\ \frac{1}{2^{ns-1}}, & for\ \ i=1\end{cases}$, and $L=\begin{cases}2^{ns-1}, & for\ \ i=0\\ \frac{1}{2^{ns-1}}, & for\ \ i=1\end{cases}$ for the condition (i),(ii),(iii) and (iv) respectively.

The subsequent cases shows the condition (i) by using (19).

Case:1 If $i=0 \Rightarrow L=\frac{1}{2}$

$$\|f(x)-A(x)\|\le\frac{\left(2^{-1}\right)^{1-0}}{1-2^{-1}}\psi(x)=\frac{2\rho}{\left(n-\dfrac{1-(-1)^n}{2}\right)}.$$

Case:2 If $i=1 \Rightarrow L=2$

$$\|f(x)-A(x)\|\le\frac{(2)^{1-1}}{1-2}\psi(x)\le\frac{2\rho}{\left(n-\dfrac{1-(-1)^n}{2}\right)}.$$

The subsequent cases shows the condition (ii) result by using (19).

Case: 1 If $i=0 \Rightarrow L=2^{s-1}$ for $s<1$

$$\| f(x) - A(x) \| \le \frac{\left(2^{(s-1)}\right)^{1-0}}{1-2^{(s-1)}} \psi(x) = \frac{1}{2-2^s} \frac{2n\rho}{\left(n - \frac{1-(-1)^n}{2}\right)} \| x \|^s = \frac{2n\rho \| x \|^s}{\left(n - \frac{1-(-1)^n}{2}\right)(2-2^s)}.$$

Case:2 If $i = 1 \Rightarrow L = \frac{1}{2^{s-1}}$ for $s > 1$

$$\| f(x) - A(x) \| \le \frac{\left(\frac{1}{2^{(s-1)}}\right)^{1-1}}{1-\frac{1}{2^{(s-1)}}} \psi(x) = \frac{2n\rho \| x \|^s}{\left(n - \frac{1-(-1)^n}{2}\right)(2^s-2)}.$$

Using (19), we get the result of condition (iii) for the below cases .

Case:1 If $i = 0 \Rightarrow L = 2^{ns-1}$ for $s < \frac{1}{n}$

$$\| f(x) - A(x) \| \le \frac{\left(2^{(ns-1)}\right)^{1-0}}{1-2^{(ns-1)}} \psi(x) = \frac{1}{2-2^{ns}} \frac{2\rho}{\left(n - \frac{1-(-1)^n}{2}\right)} \| x \|^{ns}$$

$$= \frac{2\rho \| x \|^{ns}}{\left(n - \frac{1-(-1)^n}{2}\right)(2-2^{ns})}.$$

Case:2 If $i = 1 \Rightarrow L = \frac{1}{2^{(2n+1)s-1}}$ for $s > \frac{1}{n}$

$$\| f(x) - A(x) \| \le \frac{\left(\frac{1}{2^{(ns-1)}}\right)^{1-1}}{1-\frac{1}{2^{(ns-1)}}} \psi(x) = \frac{2\rho \| x \|^{ns}}{\left(n - \frac{1-(-1)^n}{2}\right)(2^{ns}-2)}.$$

Finally, using the same equation (19) we get the result of condition (iv) for the below cases .

Case:1 If $i = 0 \Rightarrow L = 2^{ns-1}$ for $s < \frac{1}{n}$

$$\| f(x) - A(x) \| \le \frac{\left(2^{(ns-1)}\right)^{1-0}}{1-2^{(ns-1)}} \psi(x) = \frac{1}{2-2^{ns}} \frac{2(n+1)\rho}{\left(n - \frac{1-(-1)^n}{2}\right)} \| x \|^{ns}$$

$$= \frac{2(n+1)\rho \| x \|^{ns}}{\left(n - \frac{1-(-1)^n}{2}\right)(2-2^{ns})}.$$

Case:2 If $i = 1 \Rightarrow L = \frac{1}{2^{ns-1}}$ for $s > \frac{1}{n}$

$$\| f(x) - A(x) \| \le \frac{\left(\frac{1}{2^{(ns-1)}}\right)^{1-1}}{1-\frac{1}{2^{(ns-1)}}} \psi(x) = \frac{2(n+1)\rho \| x \|^{ns}}{\left(n - \frac{1-(-1)^n}{2}\right)(2^{ns}-2)}.$$

This complete the proof.

Stability Results: Direct Method in Banach Algebra

Let X be a normed algebra and Y be a Banach Algebra. For to provide the stability results of (2) in Banach algebra we are using the definition of ***Additive Derivation*** and ***Generalized Additive Derivation*** defined on [6].

The subsequent theorem discover the generalized Ulam-Hyers stability of the functional equation (2) in Banach algebra.

Theorem 4.1

Let $j = \pm 1$ and $f : X \to Y$ be an satisyfing

$$\| \Upsilon_f(\mathrm{x}) \|_Y \le \zeta(\mathrm{x}) \tag{27}$$

and

$$\| f(x_1 x_2 \cdots x_n) - f(x_1)(x_2 x_3 \cdots x_n) - \cdots \; -(x_1 x_2 \cdots x_{n-1}) f(x_n) \|_Y \le \rho(\mathrm{x}) \tag{28}$$

where $\rho, \zeta : X^n \to [0, \infty)$ such that $\lim_{m\to\infty} \frac{1}{2^{mj}} \zeta(2^{mj}\mathrm{x}) = 0,$ (29)

$$\lim_{m\to\infty} \frac{1}{2^{nmj}} \rho(2^{mj}\mathrm{x}) = 0, \tag{30}$$

$\forall \mathrm{x} = (x_1, x_2, \cdots, x_n) \in X^n$. *Then* $\exists$ $A : X \to Y$ *which is unique generalized additive derivatiion function satisfying (2) and*

$$\| f(x) - A(x) \|_Y \le \sum_{k=\frac{1-j}{2}}^{\infty} \frac{1}{2^{kj}} \xi(2^{kj} x), \; \forall x \in X \tag{31}$$

where $\xi(x)$ & $A(x)$ are defined in (6) & (7) correspondingly.

Proof:

Using Theorem 2.1, we conclude $A(x)$ satisfying (2) which is unique additive function. Now using (28) and (30),

$$\| A(x_1 x_2 \cdots x_n) - A(x_1)(x_2 x_3 \cdots x_n) - \cdots - (x_1 x_2 \cdots x_{n-1}) A(x_n) \|_Y$$

$$= \frac{1}{2^{nm}} \| f(2^m x_1 2^m x_2 \cdots 2^m x_n) - f(2^m x_1)(2^m x_2 2^m x_3 \cdots 2^m x_n) - \cdots - (2^m x_1 2^m x_2 \cdots 2^m x_{n-1}) f(2^m x_n) \|_Y$$

$$\le \frac{1}{2^{nm}} \rho(2^m \mathrm{x}) \to 0.$$

This complete the proof.

Corollary 4.2

For the positive real numbers λ and s, the function $f : X \to Y$ satisfies the inequalities

$$\left\| \Upsilon_f(\mathrm{x}) \right\|_Y \le \begin{cases} \lambda, & \\ \lambda \sum_{i=1}^{n} \| x_i \|_X^s, & s < 1 \quad or \quad s > 1; \\ \lambda \prod_{i=1}^{n} \| x_i \|_X^s, & s < \frac{1}{n} \quad or \quad s > \frac{1}{n}; \\ \lambda \left\{ \prod_{i=1}^{n} \| x_i \|_X^s + \sum_{i=1}^{n} \| x_i \|_X^s \right\}, & s < \frac{1}{n} \quad or \quad s > \frac{1}{n}; \end{cases} \tag{32}$$

$$\left\| f(x_1 x_2 \cdots x_n) - f(x_1)(x_2 x_3 \cdots x_n) - \cdots - (x_1 x_2 \cdots x_{n-1}) f(x_n) \right\|_Y$$

$$\le \begin{cases} \lambda, & \\ \lambda \sum_{i=1}^{n} \| x_i \|_X^s, & s < 1 \quad or \quad s > 1; \\ \lambda \prod_{i=1}^{n} \| x_i \|_X^s, & s < \frac{1}{n} \quad or \quad s > \frac{1}{n}; \\ \lambda \left\{ \prod_{i=1}^{n} \| x_i \|_X^s + \sum_{i=1}^{n} \| x_i \|_X^s \right\}, & s < \frac{1}{n} \quad or \quad s > \frac{1}{n}; \end{cases}, \tag{33}$$

$\forall \mathrm{x} = (x_1, x_2, \cdots, x_n) \in X^n$. *Then* $\exists \; A : X \to Y$ *satisfying (2) which is unique generalized additive derivative function and*

$$\left\| f(x) - A(x) \right\|_Y \le \begin{cases} \dfrac{2\lambda}{\left(n - \frac{1-(-1)^n}{2} \right)}, \\ \dfrac{2n\lambda \| x \|_X^s}{\left(n - \frac{1-(-1)^n}{2} \right) | 2 - 2^s |}, \\ \dfrac{2\lambda \| x \|_X^{ns}}{\left(n - \frac{1-(-1)^n}{2} \right) | 2 - 2^{ns} |} \\ \dfrac{2(n+1)\lambda \| x \|_X^{ns}}{\left(n - \frac{1-(-1)^n}{2} \right) | 2 - 2^{ns} |} \end{cases}, \qquad \forall \; x \in X. \tag{34}$$

Stability Results: Fixed Point Method In Banach Algebra

Theorem 6.1

Let $f: X \to Y$ and $\rho, \zeta : X^n \to [0,\infty)$ be a functions such that

$$\lim_{k\to\infty} \frac{1}{\kappa_i^k} \zeta(\kappa_i^k \mathrm{x}) = 0, \tag{35}$$

$$\lim_{k\to\infty} \frac{1}{\kappa_i^{nk}} \rho(\kappa_i^k \mathrm{x}) = 0, \tag{36}$$

$$\left\| \Upsilon_f(\mathrm{x}) \right\| \le \zeta(\mathrm{x}) \tag{37}$$

and

$$\left\| f(x_1 x_2 \cdots x_n) - f(x_1)(x_2 x_3 \cdots x_n) - \cdots \ -(x_1 x_2 \cdots x_{n-1}) f(x_n) \right\|_Y \le \rho(\mathrm{x}) \tag{38}$$

where

$$\kappa_i = \begin{cases} 2 & if \quad i = 0; \\ \frac{1}{2} & if \quad i = 0, \end{cases} . \tag{39}$$

If $\exists \ L = L(i) < 1$ such that the mapping

$$x \to \psi(x) = \xi\left(\frac{x}{2}\right) = \frac{2}{\left(n - \frac{1-(-1)^n}{2}\right)} \zeta\left(\frac{x}{2}, \frac{x}{2}, \frac{-x}{2}, \frac{x}{2}, \frac{-x}{2}, \frac{x}{2}, \cdots, (-1)^n \frac{x}{2}\right)$$

with the condition

$$L\psi(x) = \frac{1}{\kappa_i} \psi\left(\kappa_i x\right), \qquad \forall x \in X . \tag{40}$$

Then $\exists \ A: X \to Y$ which is unique generalized additive derivative function satisfying (2) such that

$$\left\| f(x) - A(x) \right\| \le \frac{L^{1-i}}{1-L} \psi(x), \qquad \forall x \in X . \tag{41}$$

Proof:

Obvisly, we conclude $A(x)$ satisfying (2) which is unique additive function by using Theorem 3.1. It follows form (38) and (36) we can prove $A(x)$ is gendralized additive derivation.

Corollary 6.2

Consider $f: X \to Y$ be a function such that

$$\left\|\Upsilon_f(\mathrm{x})\right\|_Y \le \begin{cases} (i) & \lambda, \\ (ii) & \lambda\sum_{j=1}^{n}\|x_j\|_X^s, & s<1 \quad or \quad s>1; \\ (iiii) & \lambda\prod_{j=1}^{n}\|x_j\|_X^s, & s<\frac{1}{n} \quad or \quad s>\frac{1}{n}; \\ (iv) & \lambda\left\{\prod_{j=1}^{n}\|x_j\|_X^s+\sum_{j=1}^{n}\|x_j\|_X^{ns}\right\}, & s<\frac{1}{n} \quad or \quad s>\frac{1}{n}; \end{cases} \tag{40}$$

and $\left\| f(x_1x_2\cdots x_n)-f(x_1)(x_2x_3\cdots x_n)-\cdots-(x_1x_2\cdots x_{n-1})f(x_n)\right\|_Y$

$$\le \begin{cases} (i) & \lambda, \\ (ii) & \lambda\sum_{j=1}^{n}\|x_j\|_X^s, & s<1 \quad or \quad s>1; \\ (iiii) & \lambda\prod_{j=1}^{n}\|x_j\|_X^s, & s<\frac{1}{n} \quad or \quad s>\frac{1}{n}; \\ (iv) & \lambda\left\{\prod_{j=1}^{n}\|x_j\|_X^s+\sum_{j=1}^{n}\|x_j\|_X^{ns}\right\}, & s<\frac{1}{n} \quad or \quad s>\frac{1}{n}; \end{cases} \tag{41}$$

for all $x=(x_1,x_2,\cdots,x_n)\in X^n$ and λ *and* s *are non nagative real constant*. Then $\exists\ A: X\to Y$ satisfying (2) which is unique generalized additive derivative function and

$$\left\| f(x)-A(x)\right\|_Y \le \begin{cases} (i) & \dfrac{2\lambda}{\left(n-\frac{1-(-1)^n}{2}\right)}, \\ (ii) & \dfrac{2n\lambda\|x\|_X^s}{\left(n-\frac{1-(-1)^n}{2}\right)|2-2^s|}, \\ (iii) & \dfrac{2\lambda\|x\|_X^{ns}}{\left(n-\frac{1-(-1)^n}{2}\right)|2-2^{ns}|} \\ (iv) & \dfrac{2(n+1)\lambda\|x\|_X^{ns}}{\left(n-\frac{1-(-1)^n}{2}\right)|2-2^{ns}|} \end{cases}, \qquad \forall\ x\in X. \tag{42}$$

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