Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 1 (2015), pp. 527-535

http://www.ripublication.com/gjpam.htm

On the Distribution of Poles and Zeros of Padé Approximants for Elliptic Functions

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Abstract

Padé approximant (PA) has been studied through the investigations of the pattern formation generated on the complex plane. This paper deals with any analytical functions which are suspected to be elliptic and we are successful in applying the PA to display their global singularity structure. The basic idea is to investigate the distribution of the poles and zeros of the computed PA. One useful application is to numerically characterize the unknown general solutions of many ordinary differential equations of physical interest.

AMS subject classification: 30D30, 30D35, 37M05.
Keywords: Elliptic functions, singularity structure, Padé approximants.

1. Introduction

1.1. Singularity structure of the elliptic function

The class of first-order ordinary differential equations $E(u, u', x) = 0$, with E polynomial in u and u', analytic in x, defines one and only one function [9], from the general solution of the equation

$$u'^2 = 4u^3 - g_2 u - g_3, \qquad (g_2, g_3) \text{ complex constants.} \tag{1.1}$$

This is the well-known elliptic function introduced by Karl Weierstrass and is frequently named as $\wp$-function (or $\wp$ for short). The Weierstrass elliptic function which is preciously the particular solution of eqn. (1.1) has the following Laurent series expansion:

$$\wp(x, g_2, g_3) = x^{-2} + \frac{g_2}{20}x^2 + \frac{g_3}{28}x^4 + \frac{g_2^2}{1200}x^6 + \frac{3g_2g_3}{6160}x^8 + O(x^{10}). \tag{1.2}$$

The $\wp$-function which admits a double pole at the origin is an even function. In complex analysis, elliptic function has many interesting properties and, in particular, we are interested in the most fundamental property which is their structure of singularities. Elliptic functions are doubly periodic meromorphic functions, and there exists an entire function σ, whose $-\wp$ is the second logarithmic derivative: $\wp = -\zeta'$, $\zeta = (\log \sigma)'$, $\zeta''^2 + 4\zeta'^3 - g_2\zeta' + g_3 = 0$. The only singularities of the general solution $\wp(x - x_0, g_2, g_3)$ of (1.1) come from the zeros of σ and are a lattice of movable double poles located at $x_0 + 2m\omega_1 + 2n\omega_2$, with m and n integers, ω_1, ω_2 the two half-periods.

The Laurent series expansion (1.2) motivated P. Hoyer and S. Kowalevski [7] to investigate further the possibility for the general solution of an ODE to be represented by the Laurent series with a finite principal part. This assumption consists in checking the existence of the Laurent series and its ability to represent the general solution, i.e., to contain enough arbitrary parameters. But since a Laurent series is defined only inside its annulus of convergence, the study is still local.

1.2. Padé approximants

Padé approximants (PA) are a particular type of rational function approximation to the value of a function throughout the whole complex plane. More precisely, a Padé approximant is the ratio of two polynomials constructed from the coefficients of the formal Taylor series expansion of a function. In particular, PA can be used to display the global information about the singularity structure of the function from its Taylor series coefficients. We first review the definitions of the Padé approximants, and the Padé table.

Definitions 1.1. We denote the Padé approximant to a formal power series $S(x)$ by

$$[L, M] = \frac{a_0 + a_1 x + \cdots + a_L x^L}{1 + b_1 x + \cdots + b_M x^M}, \tag{1.3}$$

where the two non-negative integers L and M are given. The partial sum $S_N = \sum_{j=0}^{N} c_j x^j$ $(c_0 \neq 0)$ of the formal power series $S(x)$ uniquely determines the $L + M + 1$ coefficients $a_0, a_1, \cdots, a_L, b_1, b_2, \cdots, b_M$ by the condition

$$S_N(x) - [L, M] = O(x^{N+1}), \qquad L + M = N. \tag{1.4}$$

The basic idea of (1.4) is to match the series expansion to the highest possible order. By the *Padé table* we mean the two-dimensional array of Padé approximants to the function represented by the power series. In particular, the partial sums of the Taylor series occupy the first column of the table.

Multiplying (1.4) by $Q_M = 1 + b_1 x + \cdots + b_M x^M$ and equating the coefficients on both sides gives the $L+M+1$ linear equations for $L+M+1$ unknowns:

$$\begin{cases} c_0 & = \ a_0, \\ c_1 + c_0 b_1 & = \ a_1, \\ c_2 + c_1 b_1 + c_0 b_2 & = \ a_2, \\ \quad \vdots & \quad \vdots \\ c_L + c_{L-1} b_1 + \cdots + c_0 b_L & = \ a_L, \\ c_{L+1} + c_L b_1 + \cdots + c_{L-M+1} b_M & = \ 0, \\ \quad \vdots & \quad \vdots \\ c_{L+M} + c_{L+M-1} b_1 + \cdots + c_L b_M & = \ 0. \end{cases} \tag{1.5}$$

In general a solution for the coefficients $(a_0, a_1, \ldots, a_L)$ is known after substitution of a solution for the $(b_1, b_2, \ldots, b_M)$ in the left hand side of the first $L+1$ equations of (1.5). So the crucial point is to solve the remaining linear system of M equations in the M unknown coefficients of Q_M.

In Sect. 2, we shall explain in more details on how to use Padé approximants to display the singularity structure of the Weierstrass elliptic function, for an illustration. This can be done by observing the distribution of poles and zeros of the Padé approximants on the complex plane. In Sect. 3, we shall confine our attention to the general elliptic function and would perform a different kind of investigation with Padé approximants which seems to disclose the global singularity structure of any analytic functions which are suspected to be elliptic. A concrete example will be given.

2. Mathematical Formulation

2.1. Detection of singularities from Padé approximants

Padé approximants have been used to determine quantitative results about functions when the analytic properties are qualitatively known [3, 5, 6]. PA can be particularly used to deduce global information about the singularity structure of a function from its Taylor series coefficients. If $f(x)$ has a simple pole, then a simple zero in the denominator of the Padé approximant near the pole is expected. If $f(x)$ has a multiple pole, a cluster of zeros of the Padé denominator is expected. In principle, the Padé approximants of the function cannot escape from the original singularity structure. The picture of the distribution of the zeros and poles of the Padé approximants on the complex plane will be useful to analyse the singularity structure of the target function.

Given a function we can formally find its Laurent series. Based on the natural extension of PA to Laurent series, we can compute the Padé approximants lying in at least a broad band about the central diagonal of the Padé table. The purpose is to examine the distribution of poles and zeros by displaying the Padé patterns for different values of parameters. Our general procedure consists of the following steps:

(1) To compute the Laurent series of which the coefficients are rational numbers. For the construction of Padé information we need the corresponding Taylor series by

considering Taylor$(x) = x^p \cdot$ Laurent(x), where p is the order of the pole.

(2) To construct the Padé table by using the MAPLE package `pade` for the computed Taylor series. In most cases only the approximants P_M/Q_M $(M = 1, 2, \cdots)$ in the diagonal of the Padé table will be computed. Our calculations so far are all "exact" in the sense that the coefficients of both the Laurent series and the PA are all rational numbers.

(3) To restore the Padé table for the Laurent series by

$$\left(P_M/Q_M\right)_L = x^{-p}\left(P_M/Q_M\right)_T = P_M/\left(x^p Q_M\right).$$

(4) To compute all the zeros of both numerator and denominator of each Padé approximant $[M, M]$.

(5) To display the zeros and poles of the PA on the complex plane with the symbols $\circ$ and $\times$, respectively.

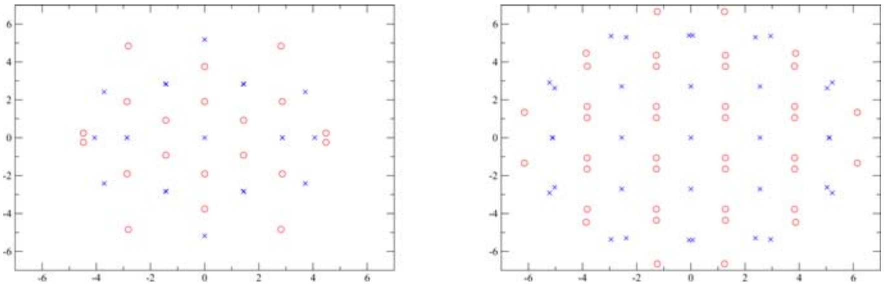

Figure 1: (Left) The elliptic function $\wp(g_2 = 1, g_3 = 1; \Delta = -26)$ (Right) $\wp(g_2 = 4, g_3 = 1/3; \Delta = 61)$ Padé patterns = doubly periodic.

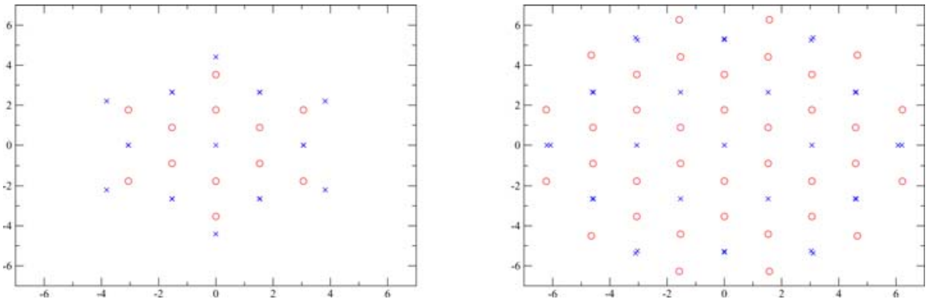

Figure 2: The elliptic function $\wp(g_2 = 0, g_3 = 1; \Delta = -27)$ (Left: [20, 20]; Right: [40, 40]) Padé patterns = doubly periodic.

2.2. Distribution of zeros and poles of PA for the $\wp$-function

The Weierstrass elliptic function has the formal Laurent series expansion (1.2). In fact we may formally compute the Laurent series's coefficients of any finite (and sufficiently large) order in terms of g_2 and g_3 symbolically. We then apply the MAPLE package and construct the Padé table for the Taylor series $f(x, g_2, g_3) = x^2 \wp(x, g_2, g_3)$. The Padé approximants $[M, M]$ will be computed for different values of M and with different values of the parameters g_2 and g_3. See Fig. 1 for the displayed distributions of the zeros and poles for the cases of $g_2 = 1$, $g_3 = 1$, $\Delta := g_2^3 - 27g_3^2 < 0$ and $g_2 = 4$, $g_3 = 1/3$, $\Delta > 0$. See Fig. 2 for the distributions when $M = 20, 40$ and $g_2 = 0$, $g_3 = 1$, $\Delta < 0$ are considered.

3. Expressing any elliptic functions in terms of $\wp$ and $\wp'$

Let $u(x)$ be any elliptic function and $\wp(x)$ be the Weierstrass elliptic function formed with the same periods $2\omega_1, 2\omega_2$. In general, we may write

$$u(x) = \big[u(x)/2 + u(-x)/2\big] + \big[\{u(x)/2 - u(-x)/2\}\{\wp'(x)\}^{-1}\big]\,\wp'(x).$$

Since both the functions in the square brackets represent even elliptic functions (whereas $\wp'$ is odd) which can further be rewritten as two rational functions of $\wp$. Given any elliptic function, in summary, there exists two rational functions R_1 and R_2 such that

$$\begin{aligned} u(x) &= R_1\big(\wp(x, g_2, g_3)\big) + R_2\big(\wp(x, g_2, g_3)\big)\,\wp'(x, g_2, g_3) \\ &= \underbrace{\frac{\mathrm{Poly}_{N_1}(\wp)}{\mathrm{Poly}_D(\wp)}}_{\text{even part}} + \frac{\mathrm{Poly}_{N_2}(\wp)}{\mathrm{Poly}_D(\wp)}\sqrt{4\wp^3 - g_2\wp - g_3}, \end{aligned} \tag{3.1}$$

where Poly_D denotes a polynomial of degree at most D. The interesting question here and now is how to determine the degrees N_1, N_2 and D. In the subsequent section we make use of the Padé approximants of the local Laurent series of the even part solution. Such numerical analysis could suggest the degrees of N_1 and D. As a result, the odd part elliptic solution could also be analytically determined.

3.1. Numerical observation: Padé patterns in $\wp^{-1}$-space

Consider an equation whose solutions are suspected to be elliptic. Using the local analysis one can compute the Laurent series expansion $u_L(x)$ of the solution $u(x)$. Since we are interested in looking for the even part of the solution in (3.1) we try to hunt for the degrees N_1 and D by observing the Padé patterns which are generated by our symbolic computations.

Now we consider the even part of the Laurent series and first write the Laurent series into $u_L = u_L^{\text{even}} + u_L^{\text{odd}}$, respectively the even and the odd series. It could be shown that we may invert (1.2), the Laurent series of $\wp$, into a $(\wp^{-1})$-series of x^2 such that

$$\begin{aligned} x^2 \;=\; \wp^{-1}\Bigg[& 1 + \frac{g_2}{20}(\wp^{-1})^2 + \frac{g_3}{28}(\wp^{-1})^3 + \frac{7g_2^2}{1200}(\wp^{-1})^4 + \frac{29g_2g_3}{3080}(\wp^{-1})^5 \\ & + \left(\frac{11g_2^3}{12480} + \frac{5g_3^2}{1274}\right)(\wp^{-1})^6 + \frac{167g_2^2g_3}{73920}(\wp^{-1})^7 \\ & + \left(\frac{77g_2^4}{509184} + \frac{669g_2g_3^2}{340340}\right)(\wp^{-1})^8 + \mathcal{O}\big((\wp^{-1})^9\big)\Bigg], \end{aligned} \tag{3.2}$$

where the coefficients can be computed, in principle, as many as we want. By (3.2) we may rewrite the even series u_L^{even} as a series in $\wp^{-1}$:

$$\begin{aligned} u_L^{\text{even}} &= x^{-2p}(u_0 + u_1x^2 + u_2x^4 + \cdots) \\ &= \text{Laurent}(\wp^{-1}), \\ \wp &= \wp(x, g_2, g_3),\ p > 0. \end{aligned}$$

By denoting $\tilde{\wp} = \wp^{-1}$, we again consider

$$\hat{u}_L^{\text{even}}(\tilde{\wp}) := \tilde{\wp}^p \cdot u_L^{\text{even}}(\tilde{\wp}) = \text{Taylor}\,(\tilde{\wp})\,,\ p > 0$$

and compute the diagonal Padé approximants for $\hat{u}_L^{\text{even}}$. However, the coefficients of $\tilde{\wp}$-series of x^2 in (3.2) are dependent of (g_2, g_3) and therefore the $\tilde{\wp}$-series of $\hat{u}_L^{\text{even}}$. Practically, the dependence of (g_2, g_3) prevents us to compute the Padé approximants since our original Padé algorithm doesn't accept any non-numerical coefficients. So we must choose some fixed values of (g_2, g_3) before computing the Padé approximants.

Our general procedure can be outlined in the following steps:

Step 1. To find the formal Laurent series $u_L(x)$ of the potentially elliptic solution $u(x)$ for a target equation $E(u, x, \vec{\alpha}) = 0$ and to fix a parameter set $\vec{\alpha}$ (if any) such that $u_L^{\text{even}} = \text{Laurent}\,(x^2)$ whose coefficients are only numerical values.

Step 2. To make use of the series (3.2) and to expand the even part of $u_L(x)$ in $\tilde{\wp}$-series:

$$u_L^{\text{even}} = \text{Laurent}(\tilde{\wp}, g_2, g_3).$$

Step 3. To fix the values of the parameters of (g_2, g_3) so that the even series in Step 2 acts as an input

$$\left(\hat{u}_L^{\text{even}}(\tilde{\wp}) = \tilde{\wp}^p\, u_L^{\text{even}}(\tilde{\wp})\right)$$

in the `pade` package of MAPLE to compute the Padé approximants of some order M. One needs at least $2M + 1$ coefficients of $\hat{u}_L^{\text{even}}(\tilde{\wp})$ (or equivalently $2M + 1$ coefficients of (3.2) and $2M + 1$ coefficients of $u_L^{\text{even}}(x^2)$) in order to achieve the Padé order M.

Step 4. To generate the Padé patterns in $\tilde{\wp}$-space and to determine the number of poles and zeros appearing in each fundamental period parallelogram in the Padé pattern, up to any cancellations of poles and zeros. To conclude the degrees N_1 and D in (3.1) based on the fact that

$$\text{Padé}(u_L^{\text{even}})_{[M,M]} = \wp^p \cdot \text{Padé}(\hat{u}_L^{\text{even}})_{[M,M]}.$$

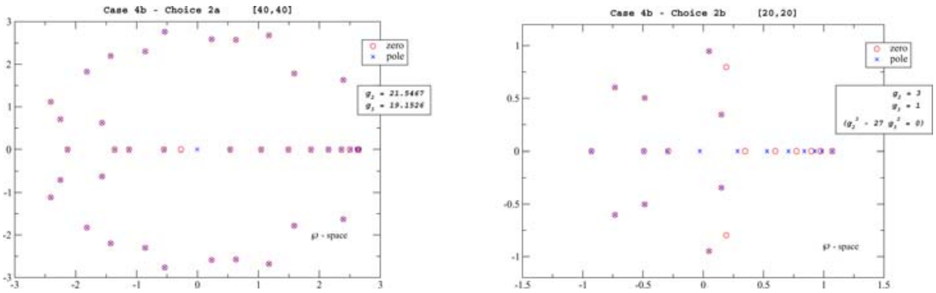

Figure 3: KS ODE (Case 4b) ($\mu = 1$, $\nu = 1/16$, $b = 1$, $A = -128$) Padé patterns = simply periodic.

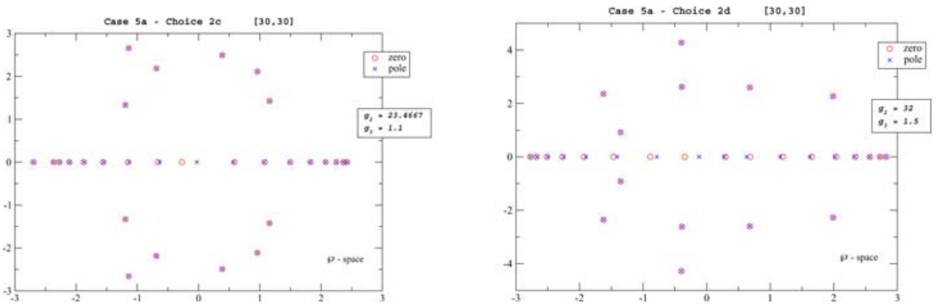

Figure 4: KS ODE (Case 5a) ($\mu = 16$, $\nu = 1$, $b = 16$, $A = -52168$) Padé patterns = doubly periodic.

3.2. A chaotic third-order ordinary differential equation

Now we shall use the traveling wave reduction of the Kuramoto-Sivashinsky (KS) equation as an illustration:

$$\nu u''' + bu'' + \mu u' + \frac{u^2}{2} + A \;=\; 0, \qquad (\nu, b, \mu, A) \text{ real constants}, \qquad \nu \neq 0. \quad (3.3)$$

The KS ODE is chaotic [8]. Currently, only five cases of all the analytic solutions have been known on the KS ODE. In fact, the analytic solutions can either be elliptic in x, rational in e^{ax}, or rational in x. We shall apply the Padé investigation to display the singularity structure of (3.3). There exists a unique convergent Laurent series solution for the KS ODE [4] and the even part of the Laurent series is given by

$$u_L^{\text{even}}(x) \;=\; -15b\,x^{-2}+\frac{b}{608\,\nu^2}\big(56\mu\nu-13b^2\big)-\frac{b}{23104\,\nu^4}\big(10\mu\nu-3b^2\big)^2x^2+\cdots .$$

For any good values of (g_2, g_3), our result of the Padé pattern in $\wp$-plane turns out to agree with the reality. The number of zeros and poles in our examined cases in $\wp$-space correctly displays the correct degrees of N_1 and D in (3.1). Furthermore, the stability of dependence on (g_2, g_3) should be an important issue. This has been investigated by perturbing the values of (g_2, g_3), see Fig. 3 for the two cases of different choices of perturbed values of (g_2, g_3) and the Padé pattern is simply periodic. See also Fig. 4 for the doubly periodic Padé patterns. The details of the two cases are: Case 4b (number of periods = 1): $(\nu = 1/16, b = 1, \mu = 1, A = -128)$; Case 5a (number of periods = 2): $(\nu = 1, b = 16, \mu = 16, A = -52168)$.

One useful investigation with Padé approximants is the high feasibility of establishing necessary numerical evidence that the unknown general analytic solution of some ordinary differential equation is suspected to be elliptic. For the KS ODE (3.3), on various generic cases when $b^2/(\mu\nu)$ = arbitrary, we find that the generated Padé patterns of the unknown general analytic solution are all doubly periodic [10].

4. Conclusion

Padé approximants can be used to analyse the singularity structure of any analytical functions. This can be done by displaying the distribution of poles and zeros of the Padé approximants to the Laurent series expansion of the function on the complex plane. The Weierstrass elliptic function is used for illustration. Even though, in general, a differential equation is nonintegrable, it may nevertheless admits particular solutions which are globally analytic. For those equations possessing solutions which are suspected to be elliptic, Padé approximants can be used to characterize the doubly periodic structure. The traveling wave reduction of the Kuramoto-Sivashinsky equation has been investigated. The investigations in this paper will provide new insights to the singularity structure of the analytical solutions of many ordinary differential equations of physical interest.

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