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On The Generalized Order and Generalized Type of Laplace-Stieltjes Transformation Convergent in the Right Half-Plane

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Abstract

In the present paper we obtain some important relationships between maximum modulus and maximum term of the analytic Dirichlet series defined by Laplace-Stieltjes transformation. We have obtained the characterizations for their generalized order and generalized type.

Key words: Laplace-Stieltjes Transformations, Maximum modulus, Maximum term, Generalized order, Generalized type.

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1. INTRODUCTION

Consider the Laplace-Stieltjes transformations defined as

$$F(s)=\int_0^{\infty}\exp(-sx)\,d\alpha(x),(s=\sigma+it) \tag{1.1}$$

where $\alpha(x)$ is a function of bounded variation on any finite interval $[0,X]$, $(0< X <+\infty),\sigma$ and t are real variables. We choose a sequence $\{\lambda_n\}$ of real numbers satisfying the following conditions:

$$0 =\lambda_1<\lambda_2<\lambda_3<\cdots<\lambda_n\uparrow+\infty, \tag{1.2}$$

$$\lim_{n\to\infty}\sup(\lambda_{n+1}-\lambda_n)<+\infty,\quad \lim_{n\to\infty}\sup\frac{n}{\lambda_n}=D<+\infty. \tag{1.3}$$

We put $A_n^* = \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{-ity} d\alpha(y) \right|$.

In [3], Jiarong obtained Valiron-Knopp-Bohr formula stated as follows:

Theorem A. Suppose that the Laplace-Stieltjes transformations (1.1) satisfy

$$\lim_{n\to\infty} \sup(\lambda_{n+1} - \lambda_n) < +\infty, \text{ and } \lim_{n\to\infty} \sup \frac{\ln n}{\lambda_n} < +\infty,$$

and σ_μ^F denotes the abscissa of uniform convergence of (1.1).Then

$$\lim_{n\to\infty} \sup \frac{\ln A_n^*}{\lambda_n} \le \sigma_\mu^F \le \lim_{n\to\infty} \sup \frac{\ln A_n^*}{\lambda_n} + \lim_{n\to\infty} \sup \frac{\ln n}{\lambda_n}. \tag{1.4}$$

Suppose that

$$\lim_{n\to\infty} \sup \frac{\ln A_n^*}{\lambda_n} = 0. \tag{1.5}$$

Then by (1.3) ,(1.4) and (1.5), it follows that $\sigma_\mu^F = 0$ and $F(s)$ is analytic in the right half plane $\sigma > 0$ .The maximum modulus, the maximum term and the index of maximum term (usually called the rank) of (1.1) can now be defined as:

$$M(\sigma,F) = \sup_{-\infty<t<+\infty} |F(\sigma+it)|,\ M_\mu(\sigma,F) = \sup_{0<x\le+\infty, -\infty<t<+\infty} \left| \int_0^x e^{-sy} d\alpha(y) \right|, s = \sigma + it, \sigma > 0,$$

$$\mu(\sigma,F) = \max_{n \varepsilon N} \left\{ A_n^* e^{-\lambda_n \sigma} \right\}, \sigma > 0 \text{ and } N(\sigma,F) = \max\left\{ \lambda_n : \mu(\sigma,F) = A_n^* e^{-\lambda_n \sigma} \right\}.$$

Following the definition of order and type for an analytic function of slow growth represented by classical Dirichlet series, the order ρ of $F(s)$ is defined as

$$\rho = \lim_{\sigma\to 0^+} \sup \frac{\ln^+ \ln^+ M_\mu(\sigma, F)}{-\ln \sigma} \tag{1.6}$$

where for $C > 0, \ln^+ C = \max\{\log C, 0\}$. If $\rho \in (0, +\infty)$, the type τ can be defined as,

$$\tau = \lim_{\sigma\to 0^+} \sup \frac{\ln^+ M_\mu(\sigma, F)}{e^{\rho/\sigma}}.$$

Let Λ denote the class of functions $h(x)$, satisfying the following conditions:

i. $h(x)$ is defined on $[a,\infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

ii. $\lim_{x\to\infty} \frac{d(h(x))}{d(\ln^{[p]} x)} = k \in (0,\infty),\ p \in N^+,$

where $\ln^{[0]} x = x,\ \ln^{[1]} x = \ln x,\ \ln^{[p]} x = \ln^{[p-1]} \ln x.$

Huo and Kong ([2], Lemma 2.2**)** obtained the following result:

Lemma A. ([2], Lemma 2.2**):** Suppose $\alpha(x) \in \Lambda,$ then,

$$a)\quad \lim_{x\to\infty} \frac{\alpha(cx)}{\alpha(x)} = 1,\ \lim_{x\to\infty} \frac{\alpha(c+x)}{\alpha(x)} = 1, \tag{1.7}$$

where $c > 0$ is a constant.

$b)$ When $p = 1,$ for every $A \in \mathbb{R},$ there is $C \in \mathbb{R}$ such that

$$\lim_{x\to\infty} \frac{\alpha\left(x^A\right)}{\alpha(x)} = A. \tag{1.8}$$

$c)$ When $p = 2,3,\ldots, \forall\, 1 \le A_1, A_2 < B,$

$$\lim_{x\to\infty} \frac{\alpha^{-1}\left[A_1\alpha(x)\right]\alpha^{-1}\left[A_2\alpha(x)\right]}{\alpha^{-1}\left[B\alpha(x)\right]} = 0. \tag{1.9}$$

Further, during the course of proof of Theorem 1, [5,pp. 199-200], Yinying and Daochun have shown that for $\varepsilon > 0$,

$$\frac{1}{3}\mu(\sigma,F) \le M_u(\sigma,F) \le K(\varepsilon)\frac{\mu((1-\varepsilon)\sigma,F)}{\sigma}$$

where $K(\varepsilon)$ depends on ε only. Consequently, for analytic function F of finite order,

$$\ln \mu(\sigma,F) \simeq \ln M_u(\sigma,F) \text{ as } \sigma \to 0+. \tag{1.10}$$

Using the above, we define the generalized order and generalized type of Laplace-Stieltjes Transformations $F(s),$ where $F(s) = \int_0^\infty \exp(-sy)\, \mathrm{d}\alpha(y).$

Definition. Let $\alpha(x) \in \Lambda,$ the generalized order ρ of the function $F(s)$ is defined as

$$\rho = \lim_{\sigma\to 0^+} \sup \frac{\alpha\left(\ln M_\mu(\sigma,F)\right)}{\alpha\left(1/\sigma\right)},$$

If $\rho \in (0,\infty),$ the generalized type τ is defined as,

$$\tau = \lim_{\sigma\to 0^+} \sup \frac{\alpha\left(M_\mu(\sigma,F)\right)}{\left[\alpha\left(e^{1/\sigma}\right)\right]^\rho}.$$

In this paper we obtain some significant relationships between maximum modulus and maximum term of the analytic Dirichlet series defined by Laplace-Stieltjes transformations. For further details of Laplace-Stieltjes transformations we refer to ([1],[4],[5]).

2. Main Results.

We now prove

Theorem 1: Let $F(s)=\int_0^{\infty}\exp(-sy)\,\mathrm{d}\alpha(y)$ be Laplace-Stieltjes transformations, then

i. $$\limsup_{\sigma\to 0^+}\frac{\alpha\left(\ln M_\mu(\sigma,F)\right)}{\alpha(1/\sigma)}+1=\limsup_{n\to\infty}\frac{\alpha(\lambda_n)}{\alpha\left(\lambda_n/\ln A_n^*\right)} \text{ for } p=1.$$

ii. $$\limsup_{\sigma\to 0^+}\frac{\alpha\left(\ln M_\mu(\sigma,F)\right)}{\alpha(1/\sigma)}=\limsup_{n\to\infty}\frac{\alpha(\lambda_n)}{\alpha\left(\lambda_n/\ln A_n^*\right)}>1 \text{ for } p=2,3,\ldots.$$

Proof: We will prove the result in two parts, for p =1 and for p > 1. Let us denote

$$\limsup_{n\to\infty}\frac{\alpha(\lambda_n)}{\alpha\left(\lambda_n/\ln A_n^*\right)}=Q \tag{2.1}$$

for both the cases. In view of (1.10), we can replace $\ln M_\mu(\sigma,F)$ by $\ln\mu(\sigma,F)$ in (i) and (ii) above

Case I. We take p =1. From (2.1), for a given $\delta>0$ and for n sufficiently large, we have

$$\alpha(\lambda_n)<(Q+\delta)\alpha\left(\lambda_n/\ln A_n^*\right),$$

or, $$\ln\left(A_n^*e^{-\lambda_n\sigma}\right)\le\lambda_n\left[\frac{1}{\alpha^{-1}\left[\alpha(\lambda_n)/(Q+\delta)\right]}-\sigma\right].$$

By ([2], Lemma 2.2, p-177, inequality (2.6)) we have

$$\exp\left\{\frac{(k-\varepsilon)\ln\lambda_n}{(Q+\delta)(k+\varepsilon)}\right\}<\alpha^{-1}\left[\frac{\alpha(\lambda_n)}{(Q+\delta)}\right]<\exp\left\{\frac{(k+\varepsilon)\ln\lambda_n}{(Q+\delta)(k-\varepsilon)}\right\} \tag{2.2}$$

or $$\lambda_n^{\left\{\frac{(k-\varepsilon)}{(Q+\delta)(k+\varepsilon)}\right\}}<\alpha^{-1}\left[\frac{\alpha(\lambda_n)}{(Q+\delta)}\right].$$

We have by above equation,

$$\ln\left(A_n^*e^{-\lambda_n\sigma}\right)\le\lambda_n\left[\lambda_n^{\left\{\frac{-(k-\varepsilon)}{(Q+\delta)(k+\varepsilon)}\right\}}-\sigma\right].$$

The maximum value of right hand side expression is obtained for

$$\lambda_n=\left(\frac{(Q+\delta)(k+\varepsilon)\sigma}{(Q+\delta)(k+\varepsilon)-(k-\varepsilon)}\right)^{-\frac{(Q+\delta)(k+\varepsilon)}{(k-\varepsilon)}}.$$

Hence we get

$$\ln\left(A_n^*e^{-\lambda_n\sigma}\right)\le C\sigma^{1-\frac{(Q+\delta)(k+\varepsilon)}{(k-\varepsilon)}},$$

where $C = \dfrac{k-\varepsilon}{(Q+\delta)(k+\varepsilon)-(k-\varepsilon)}\left(\dfrac{(Q+\delta)(k+\varepsilon)}{(Q+\delta)(k+\varepsilon)-(k-\varepsilon)}\right)^{-\frac{(Q+\delta)(k+\varepsilon)}{(k-\varepsilon)}}$.

Hence by above inequality, we have

$$\alpha\left(\ln \mu(\sigma,F)\right) \le \alpha\left(C\left(1/\sigma\right)^{\frac{(Q+\delta)(k+\varepsilon)}{(k-\varepsilon)}-1}\right),$$

On proceeding to limits, we have

$$\limsup_{\sigma\to 0^+} \frac{\alpha\left(\ln \mu(\sigma,F)\right)}{\alpha\left(1/\sigma\right)} \le \limsup_{\sigma\to 0^+} \frac{\alpha\left(C\left(1/\sigma\right)^{\frac{(Q+\delta)(k+\varepsilon)}{(k-\varepsilon)}-1}\right)}{\alpha\left(1/\sigma\right)}.$$

Now using (a) and (b) of Lemma A ,since $\varepsilon > 0$ is arbitrary, we get

$$\limsup_{\sigma\to 0^+} \frac{\alpha\left(\ln \mu(\sigma,F)\right)}{\alpha\left(1/\sigma\right)} \le Q-1.$$

To prove (i) above completely, suppose that

$$\limsup_{\sigma\to 0^+} \frac{\alpha\left(\ln \mu(\sigma,F)\right)}{\alpha\left(1/\sigma\right)} < Q-1.$$

Therefore for $\varepsilon_o \in \left(0,1/2\right)$ and σ sufficiently close to 0, we have,

$$\ln \mu(\sigma,F) < \alpha^{-1}\left[\alpha\left(1/\sigma\right)\left(Q-1-\varepsilon_o\right)\right],$$

$$\text{or } \ln A_n^* e^{-\lambda_n \sigma} \le \alpha^{-1}\left[\alpha(1/\sigma)\left(Q-1-\varepsilon_o\right)\right],\ n \ge 1$$

$$< \sigma^{-\left[\frac{(k+\varepsilon)(Q-1-\varepsilon_o)}{(k-\varepsilon)}\right]} \text{ ,using (2.2) above.}$$

Hence $\ln A_n^* \le \sigma^{-\left[\frac{(k+\varepsilon)(Q-1-\varepsilon_o)}{(k-\varepsilon)}\right]} + \sigma\lambda_n$.

Let us choose $\sigma = \lambda_n^{-\left[\frac{(k-\varepsilon)}{(k+\varepsilon)(Q-1-\varepsilon_o)-(k-\varepsilon)}\right]}$. Then we get $\ln A_n^* \le 2\lambda_n^{1-\left[\frac{(k-\varepsilon)}{(k+\varepsilon)(Q-1-\varepsilon_o)+(k-\varepsilon)}\right]}$ .On using (1.7), we have

$$\frac{\alpha\left(\lambda_n\right)}{\alpha\left(\lambda_n/\ln A_n^*\right)} \le \frac{\alpha\left(\lambda_n\right)}{\alpha\left(\lambda_n^{\left[\frac{(k-\varepsilon)}{(k+\varepsilon)(Q-1-\varepsilon_o)+(k-\varepsilon)}\right]}\right)}.$$

Proceeding to limits and using Lemma A, since $\varepsilon > 0$ is arbitrary, we have,

$$\limsup_{n\to\infty} \frac{\alpha\left(\lambda_n\right)}{\alpha\left(\lambda_n/\ln A_n^*\right)} \le \left(Q-\varepsilon_o\right) < Q,$$

which contradicts (2.1). Hence we have,

$$\limsup_{\sigma\to 0^+}\frac{\alpha\left(\ln\mu(\sigma,F)\right)}{\alpha(1/\sigma)}=Q-1.$$

Case II: Now we take $p=2,3,...,$. As in case I, we have for a given $\delta>0$ and $\forall n>N$,

$$\ln A_n^* e^{-\lambda_n\sigma}\le\lambda_n\left[\frac{1}{\alpha^{-1}\left[\alpha(\lambda_n)/(Q+\delta)\right]}-\sigma\right].$$

Since $\lambda_n\to\infty \text{ as } n\to\infty$, it is evident that for large values of *n*,

$\alpha^{-1}\left[\alpha(\lambda_n)/(Q+\delta)\right]\ge 1.$

Now let σ be a sufficiently small positive number satisfying,

$0<1/\alpha^{-1}\left[\alpha(\lambda_n)/(Q+\delta)\right]-\sigma\le 1.$

Let $I\equiv I(\sigma)=\alpha^{-1}\left[(Q+\delta)\alpha(2/\sigma)\right]$. If $\lambda_n\le I$, then

$$\ln A_n^* e^{-\lambda_n\sigma}\le I\left[\frac{1}{\alpha^{-1}\left[\alpha(\lambda_n)/(Q+\delta)\right]}-\sigma\right]<\alpha^{-1}\left[(Q+\delta)\alpha(2/\sigma)\right]. \qquad (2.3)$$

For $\lambda_n>I, i.e., \lambda_n>\alpha^{-1}\left[(Q+\delta)\alpha(2/\sigma)\right]$, we have

$\frac{2}{\sigma}<\alpha^{-1}\left[\alpha(\lambda_n)/(Q+\delta)\right].$

Hence $\ln A_n^* e^{-\lambda_n\sigma}\le\lambda_n(\sigma/2-\sigma)<0.$ (2.4)

By (2.3) and (2.4), we have,

$\ln\mu(\sigma,F)\le\alpha^{-1}\left[\alpha(2/\sigma)(Q+\delta)\right].$

Hence we have on using (1.7),

$$\limsup_{\sigma\to 0^+}\frac{\alpha\left(\ln\mu(\sigma,F)\right)}{\alpha(1/\sigma)}\le Q.$$

To prove the reverse inequality let us assume that

$$\limsup_{\sigma\to 0^+}\frac{\alpha\left(\ln\mu(\sigma,F)\right)}{\alpha(1/\sigma)}<Q\ .$$

Again for $\varepsilon_o\in(0,1/2)$, and for σ positive and sufficiently small, we have,

$$\ln A_n^* e^{-\lambda_n\sigma}\le\log\mu(\sigma)<\alpha^{-1}\left[\alpha(1/\sigma)(Q-2\varepsilon_o)\right]. \qquad (2.5)$$

From (2.1) it can be shown that there exists a sequence $\{n_k\}$ of positive integers such that

$$\alpha^{-1}\left[(Q-\varepsilon_o)\alpha\left(\frac{\lambda_{n_k}}{\ln A_{n_k}^*}\right)\right]<\lambda_{n_k}<\alpha^{-1}\left[(Q+\varepsilon_o)\alpha\left(\frac{\lambda_{n_k}}{\ln A_{n_k}^*}\right)\right]. \qquad (2.6)$$

Let us choose the sequence $\{\sigma_{n_k}\}$ satisfying

$$\alpha^{-1}\left[\left(Q-2\varepsilon_o\right)\alpha\left(1/\sigma_{n_k}\right)\right]=\left(\frac{\lambda_{n_k}}{2\alpha^{-1}\left[\alpha\left(\lambda_{n_k}\right)/\left(Q-\varepsilon_o\right)\right]}\right). \qquad (2.7)$$

We now have

$$\lim_{k\to\infty}\left(\frac{\lambda_{n_k}}{\alpha^{-1}\left[\alpha\left(\lambda_{n_k}\right)/\left(Q-\varepsilon_o\right)\right]}\right)=\infty.$$

If not, then for some positive constant P and all large values of *k*,

$$\alpha(\lambda_{n_k}/P) < \alpha(\lambda_{n_k})/(Q-\varepsilon_0)$$

which is a contradiction in view of (2.1).From (2.5) and (2.7), we have

$$\ln A^*_{n_k} \le \lambda_{n_k}\sigma_{n_k}+\frac{\lambda_{n_k}}{2\alpha^{-1}\left[\alpha\left(\lambda_{n_k}\right)/\left(Q-\varepsilon_o\right)\right]},$$

From above equation, using the definition of Q, we get,

$$\frac{1}{\sigma_{n_k}}\le\frac{\lambda_{n_k}}{\ln A^*_{n_k}}\frac{1}{\left[1-\lambda_{n_k}\Big/2\ln A^*_{n_k}\alpha^{-1}\left[\alpha\left(\lambda_{n_k}\right)/\left(Q-\varepsilon_o\right)\right]\right]}<\frac{2\lambda_{n_k}}{\ln A^*_{n_k}}.$$

From (1.9), (2.6) and (2.7) we get

$$\lambda_{n_k}=2\alpha^{-1}\left[\alpha\left(\lambda_{n_k}\right)/\left(Q-\varepsilon_o\right)\right]\alpha^{-1}\left[\left(Q-2\varepsilon_o\right)\alpha\left(1/\sigma_{n_k}\right)\right],$$

$$<2\alpha^{-1}\left[\frac{\left(Q+\varepsilon_o\right)\alpha\left(\lambda_{n_k}/\ln A^*_{n_k}\right)}{\left(Q-\varepsilon_o\right)}\right]\alpha^{-1}\left[\left(Q-2\varepsilon_o\right)\alpha\left(2\lambda_{n_k}/\ln A^*_{n_k}\right)\right],$$

$$<\alpha^{-1}\left[\left(Q-\varepsilon_o\right)\alpha\left(\lambda_{n_k}/\ln A^*_{n_k}\right)\right],$$

which contradicts the left hand inequality of (2.6), Hence we have,

$$\limsup_{\sigma\to 0^+}\frac{\alpha\left(\ln\mu\left(\sigma,F\right)\right)}{\alpha\left(1/\sigma\right)}\ge Q.$$

This completes the proof of Theorem 1.

Next we prove

Theorem 2: Suppose that Laplace Stieltjes Transform $F(s)$ given by (1.1) satisfies (1.2) and (1.3) and has the generalized order $\rho\in\left(1,\infty\right)$. Then

$$\limsup_{\sigma\to 0^+}\frac{\beta\left(\ln M_\mu\left(\sigma,F\right)\right)}{\left[\beta\left(1/\sigma\right)\right]^\rho}=\limsup_{n\to\infty}\frac{\beta\left(\lambda_n\right)}{\left[\beta\left(\lambda_n/\ln A^*_n\right)\right]^\rho}\ \text{for } p=1,2,3,.... \qquad (2.8)$$

where $\beta(\log x)=\alpha(x)$

Proof: Let us put

$$\limsup_{n\to\infty}\frac{\beta(\lambda_n)}{\left[\beta\left(\lambda_n/\ln A_n^*\right)\right]^\rho}=R \tag{2.9}$$

For a given $\varepsilon>0,\exists N_o(\varepsilon)>0$ such that $\forall n>N_o$,

$$\frac{\beta(\lambda_n)}{\left[\beta\left(\lambda_n/\ln A_n^*\right)\right]^\rho}<R+\varepsilon,$$

or $$\ln A_n^* e^{-\lambda_n\sigma}\le\lambda_n\left[\frac{1}{\beta^{-1}\left[\beta(\lambda_n)/(R+\varepsilon)\right]^{1/\rho}}-\sigma\right].$$

Now let σ be a sufficiently small positive number satisfying

$1/\beta^{-1}\left[\beta(\lambda_n)/(R+\varepsilon)\right]^{1/\rho}-\sigma\le 1$

and put $J\equiv J(\sigma)=\beta^{-1}\left[(R+\varepsilon)\{\beta(2/\sigma)\}^\rho\right]$. If $\lambda_n\le J$, then,

$$\ln A_n^* e^{-\lambda_n\sigma}\le J<\beta^{-1}\left[(R+\varepsilon)\{\beta(2/\sigma)\}^\rho\right]. \tag{2.10}$$

For $\lambda_n>J$, we have

$$\frac{2}{\sigma}\le\beta^{-1}\left[\beta(\lambda_n)/(R+\varepsilon)\right]^{1/\rho}.$$

Hence $\ln A_n^* e^{-\lambda_n\sigma}\le\lambda_n(\sigma/2-\sigma)<0.$ (2.11)

By (2.10) and (2.11), we have,

$$\ln A_n^* e^{-\lambda_n\sigma}\le\beta^{-1}\left[(R+\varepsilon)\beta^\rho(2/\sigma)\right],$$

i.e., $$\beta(\ln\mu(\sigma,F))\le(\mathrm{R}+\varepsilon)(\beta(2/\sigma))^\rho.$$

Proceeding to limits, since $\varepsilon>0$ is arbitrary,we get

$$\limsup_{\sigma\to0^+}\frac{\beta(\ln\mu(\sigma,F))}{\left[\beta(1/\sigma)\right]^\rho}\le R.$$

To prove the reverse inequality, we assume that

$$\limsup_{\sigma\to0^+}\frac{\beta(\ln\mu(\sigma,F))}{\left[\beta(1/\sigma)\right]^\rho}<R.$$

We can find $\varepsilon'\in(0,1/2)$ such that for σ sufficiently small. we have,

$$\ln A_n^* e^{-\lambda_n\sigma}\le\ln(\mu(\sigma))<\beta^{-1}\left[(R-2\varepsilon')\{\beta(1/\sigma)\}^\rho\right]. \tag{2.12}$$

From (2.9) we can find a sequence { λ_{n_k} }such that

$$\beta^{-1}\left[(R-\varepsilon_o)\beta^\rho\left(\frac{\lambda_{n_k}}{\ln A_{n_k}^*}\right)\right]<\lambda_{n_k}<\beta^{-1}\left[(R+\varepsilon_o)\beta^\rho\left(\frac{\lambda_{n_k}}{\ln A_{n_k}^*}\right)\right]. \tag{2.13}$$

Let σ_{n_k} be chosen satisfying

$$\beta^{-1}\left[\left(R-2\varepsilon'\right)\left(\beta\left(1/\sigma_{n_k}\right)\right)^{\rho}\right]=\left(\frac{\lambda_{n_k}}{2\beta^{-1}\left[\beta\left(\lambda_{n_k}\right)^{1/\rho}\Big/\left(R-\varepsilon'\right)\right]}\right). \tag{2.14}$$

From (2.12) and (2.13), we have

$$\ln A^{*}_{n_k}\le\lambda_{n_k}\sigma_{n_k}+\frac{\lambda_{n_k}}{2\beta^{-1}\left[\beta\left(\lambda_{n_k}\right)^{1/\rho}\Big/\left(R-\varepsilon'\right)\right]},$$

From above equation after a simple calculation we get,

$$\frac{1}{\sigma_{n_k}}\le\frac{\lambda_{n_k}}{\ln A^{*}_{n_k}}\frac{1}{\left[1-\lambda_{n_k}\Big/2\ln A^{*}_{n_k}\beta^{-1}\left[\beta\left(\lambda_{n_k}\right)^{1/\rho}\Big/\left(R-\varepsilon'\right)\right]\right]}<\frac{2\lambda_{n_k}}{\ln A^{*}_{n_k}}.$$

From (3.14) we get,

$$\lambda_{n_k}=2\beta^{-1}\left[\left(R-2\varepsilon'\right)\left(\beta\left\{2\lambda_{n_k}/(\ln A^{*}_{n_k})\right\}\right)^{\rho}\right]\beta^{-1}\left[\left(R+\varepsilon'\right)^{1/\rho}\beta\left(\lambda_{n_k}/\ln A^{*}_{n_k}\right)\Big/\left(R-\varepsilon'\right)\right]$$

$$<\beta^{-1}\left[\left(R-\varepsilon'\right)\beta^{\rho}\left(\frac{\lambda_{n_k}}{\ln A^{*}_{n_k}}\right)\right],$$

which is a contradiction of (2.13), Hence we have,

$$\limsup_{\sigma\to0^{+}}\frac{\beta\left(\ln\mu\left(\sigma,F\right)\right)}{\left[\beta\left(1/\sigma\right)\right]^{\rho}}\ge R.$$

This completes the proof of Theorem 2.

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