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Mixed Quadrature Over Sphere

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Abstract

A mixed quadrature rule of degree of precision seven is used for the evaluation of volume integral. Volume integral is converted to surface integral by using Divergence theorem of Gauss. We have applied mixed quadrature rule over a circular region to evaluate the typical volume integrals over a spherical region using monomial transformation with different radius. Finally the efficiency of this method is numerically verified and error bound is determined.

KEYWORDS- Finite element method, Degree of precision, Mixed quadrature, Spherical region, Monomial transformation.

1. INTRODUCTION

The finite element method is essentially a numerical method for the approximation solution of practical problem arising in engineering and scientific analysis. The advantages of finite element technique over other alternatives are more fully appreciated. An example of considerable importance is study of bodies of revolution, where a three dimensional problem is solved by two-dimensional analysis. In particular, they are used for problems involving calculation of mass of shell, center of mass, moment of inertia of a shell, fluid flow and mass flow of a cross surface, electric charge distributed over a surface, plane strain, plate bending, heat conduction over a plate and similar problem in other areas of engineering which are very difficult to analyze using analytical technique. These problems can be solved by finite element method. From literature review various methods for evaluating volume integral is given by Lee and Requicha[1] evaluation of volume integrals by transforming volume integral to surface integral and then into parametric line integral is given by Timmer and Stern [4], Cattani and Paoluzzi[2] gave a symbolic solution to both surface and volume integration of polynomials by using triangulation of solid boundary, Rathod, Nagaraja, Venkatesudu and Ramesh [5] have discussed the volume integral of a function is expressed as sum of four integrals over the unit triangle by using Divergence theorem of Gauss, Shivaram[7] evaluation of volume integral of arbitrary function over spherical region by using generalized Gaussian quadrature rule and Jena and Dash[3] mixed quadrature of real definite integral over a triangle. This Paper is organized as follows. In section II, the mixed quadrature rule (Lobatto-4 point and Clenshaw-curtis-5 point rule) of degree of precision seven is formulated along with error bound is evaluated. In section III volume of sphere is equal to 8 times the volume in first octant is represented. In section IV we will introduce the mixed quadrature rule by applying the monomial transformation to convert the interval $[0,1]$ to $[-1,1]$ over a circle region of various value of a. In section V we compare our numerical results with exact values and error bound is identified.

2. FORMULATION OF MIXED QUADRATURE RULE OF PRECISION SEVEN

We choose the Lobatto-4 point rule $R_{L4}(f)$,

$$I(f)=\frac{1}{6}\left[f(-1)+5f\left(-\frac{1}{\sqrt{5}}\right)+5f\left(\frac{1}{\sqrt{5}}\right)+f(1)\right] \tag{2.1}$$

And Clenshaw-curtis-5 point rule $R_{CC5}(f)$,

$$I(f)=\frac{1}{15}\left[f(-1)+8f\left(-\frac{1}{\sqrt{2}}\right)+12f(0)+8f\left(\frac{1}{\sqrt{2}}\right)+f(1)\right] \tag{2.2}$$

Each of the rules $R_{L4}(f)$ and $R_{CC5}(f)$ is of precision five. Let $E_{L4}(f)$ and $E_{CC5}(f)$ denotes the error in approximating the integral $I(f)$ by the rules $R_{L4}(f)$ and $R_{CC5}(f)$ respectively. Using Maclaurin's expansion of functions in equation (2.1) and (2.2), we get

$$I(f)=R_{L4}(f)+E_{L4}(f) \tag{2.3}$$

$$I(f)=R_{CC5}(f)+E_{CC5}(f) \tag{2.4}$$

$$E_{L4}(f)=-\frac{32}{6!\times 525}f^{vi}(0)-\frac{128}{8!\times 1125}f^{viii}(0)-\frac{3136}{10!\times 20625}f^{x}(0)-\frac{7296}{12!\times 40625}f^{xii}(0)-..... \tag{2.5}$$

$$E_{CC5}(f)=\frac{2}{6!\times 105}f^{vi}(0)+\frac{1}{8!\times 45}f^{viii}(0)+\frac{1}{10!\times 66}f^{x}(0)+\frac{1}{12!\times 260}f^{xii}(0)+...... \tag{2.6}$$

Multiplying equation (2.4) by $\left(\frac{16}{5}\right)$ and then adding it to equation (2.3), we obtain

$$I(f)=\frac{1}{21}\left[5R_{L4}(f)+16R_{CC5}(f)\right]+\frac{1}{21}\left[5E_{L4}(f)+16E_{CC5}(f)\right] \tag{2.7}$$

$$I(f)=R_{L4CC5}(f)+E_{L4CC5}(f) \tag{2.8}$$

$$R_{L4CC5}(f)=\frac{1}{21}\left[5R_{L4}(f)+16R_{CC5}(f)\right] \tag{2.9}$$

The mixed quadrature rule due to Lobatto-4 point rule and Clenshaw-Curtis-5 point rule $R_{L4CC5}(f)$ is

$$R_{L4CC5}(f)=\frac{1}{630}\begin{bmatrix}57f(-1)+256f\left(\frac{1}{\sqrt{2}}\right)+125f\left(\frac{1}{\sqrt{5}}\right)+384f(0)\\+125f\left(-\frac{1}{\sqrt{5}}\right)+256f\left(-\frac{1}{\sqrt{2}}\right)+57f(1)\end{bmatrix} \tag{2.10}$$

Equation (2.10) is the desired mixed quadrature rule of precision seven, as it is constructed from two different types of rules of the same precision five for the approximate evaluation of $I(f)$. The truncation error generated in the approximation is given by

$$\begin{aligned}E_{L4CC5}(f)&=\frac{1}{21}\left[5E_{L4}(f)+16E_{CC5}(f)\right]\\&=-\frac{16}{8!\times1575}f^{viii}(0)-\frac{712}{10!\times28875}f^{x}(0)-\frac{6796}{12!\times170625}f^{xii}(0)-\ldots\ldots\end{aligned} \tag{2.11}$$

$$\left|E_{L4CC5}(f)\right|\cong\frac{16}{8!\times1575}\left|f^{viii}(0)\right|+\ldots\ldots.. \tag{2.12}$$

3. MIXED QUADRATURE OVER A SPHERICAL REGION

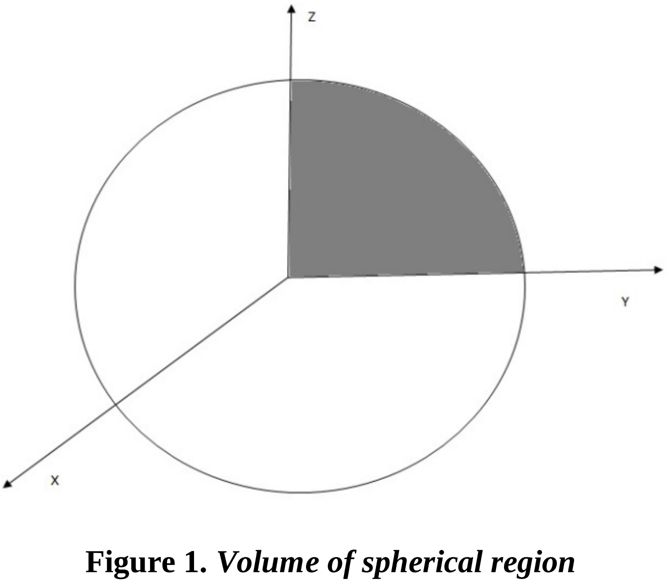

Figure 1. ***Volume of spherical region***

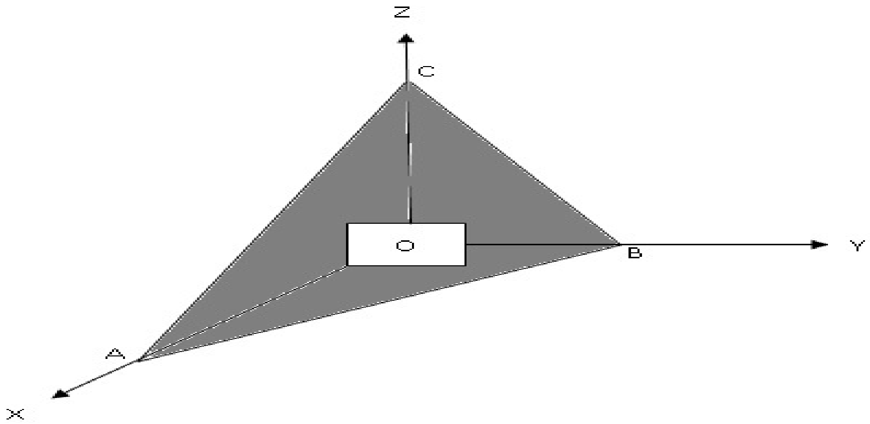

Figure 2. ***OABC is piecewise smooth and is composed of four surfaces***

The numerical integration of an arbitrary function $f(x,y,z)$ over a spherical region is given by

$$I=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} f(x,y,z)dz\,dy\,dx \tag{3.1}$$

$$I=8\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\int_{0}^{\sqrt{a^2-x^2-y^2}} f(x,y,z)dz\,dy\,dx \tag{3.2}$$

Mixed quadrature rule for integrating of volume integral bounded by spherical region

$$V=\left\{(x,y,z)/\ 0\le x\le a,\ 0\le y\le\sqrt{a^2-x^2},\ 0\le z\le\sqrt{a^2-x^2-y^2}\right\} \tag{3.3}$$

With radius $a=0.5,\ 1,\ 3$ and these volumes integral convert to surface integral using Divergence theorem of Gauss.

4. FORMULATION OF INTEGRALS OVER A QUARTER CIRCLE REGION

The numerical integration of an arbitrary function f over a quarter circle region is given by

$$I=\iint_C f(x,y)dx\,dy=\int_0^a\int_0^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx \tag{4.1}$$

The integral of the equation (4.1) can be transformed to the square $\{(u,v)/0\le u\le 1,\ 0\le v\le 1\}$ by the transformation

$$x=au,\ y=av\sqrt{1-u^2} \tag{4.2}$$

$$I=\int_0^1\int_0^1 f[x(u,v),y(u,v)]J\ du\,dv \tag{4.3}$$

Where $J(u,v)$ is the Jacobian of transformation

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u}\\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v}\end{vmatrix}=a^2\sqrt{1-u^2} \tag{4.4}$$

From equation (4.4) we can write as

$$I=\int_0^1\int_0^1 f\left(au\,,av\sqrt{1-u^2}\right)a^2\sqrt{1-u^2}\;du\,dv \tag{4.5}$$

The integral I of equation (4.5) can be transformed further into an integral over the standard 2-square

$\{(\xi,\eta)/-1\le\xi,\,\eta\le 1\}$ by monomial transformation

$$u=\frac{1+\xi}{2},\; v=\frac{1+\eta}{2} \tag{4.6}$$

Now the integral I of equation (4.5) with equation (4.6) transformed to

$$I=\int_{-1}^{1}\int_{-1}^{1} f\left\{a\left(\frac{1+\xi}{2}\right),a\left(\frac{1+\eta}{2}\right)\right\}\left\{\sqrt{1-\left(\frac{1+\xi}{2}\right)^2}\right\}a^2\sqrt{1-\left(\frac{1+\xi}{2}\right)^2}\;J\,d\xi\,d\eta \tag{4.7}$$

Where,

$$J=\frac{\partial(u,v)}{\partial(\xi,\eta)}=\begin{vmatrix}\dfrac{\partial u}{\partial \xi} & \dfrac{\partial v}{\partial \xi}\\ \dfrac{\partial u}{\partial \eta} & \dfrac{\partial v}{\partial \eta}\end{vmatrix}=\frac{1}{4}$$

Now equation (4.7) takes the form

$$I=\int_{-1}^{1}\int_{-1}^{1} f\left\{a\left(\frac{1+\xi}{2}\right),a\left(\frac{1+\eta}{2}\right)\right\}\left\{\sqrt{1-\left(\frac{1+\xi}{2}\right)^2}\right\}\frac{a^2}{4}\sqrt{1-\left(\frac{1+\xi}{2}\right)^2}\;d\xi\,d\eta \tag{4.8}$$

$$I=\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{a^2}{4}\sqrt{1-\left(\frac{1+\xi_i}{2}\right)^2}\right\}w_i\,w_j\,f\left\{x\left(\xi_i\,,\eta_j\right),y\left(\xi_i\,,\eta_j\right)\right\} \tag{4.9}$$

Where $\xi_i\,,\eta_j$ are the points obtained in the $\xi\,,\eta$ direction respectively and $w_i\,,w_j$ are corresponding weight from equation (2.10). We can write equation (4.9) as

$$I_{mix}=\sum_{k=1}^{N=n\times n}c_k\,f\left(x_k\,,y_k\right) \tag{4.10}$$

Where $c_k\,,x_k$ and y_k are obtained from the relation

$$c_k=c_{i\,j}=\frac{a^2}{4}\sqrt{1-\left(\frac{1+\xi_i}{2}\right)^2}\;w_i w_j \tag{4.11}$$

$$x_k = x_{ij} = a\left(\frac{1+\xi_i}{2}\right) \tag{4.12}$$

$$y_k = y_{ij} = a\left(\frac{1+\eta_j}{2}\right)\sqrt{1-\left(\frac{1+\xi_i}{2}\right)^2} \tag{4.13}$$

Where $i, j, k = 1, 2, 3.....n$. We have tabulated a sample of these weight coefficients c_k and sampling points x_k and y_k which are given in table-1, table-2 and table-3 using equation (4.11), equation (4.12), equation (4.13) for different $a = 0.5, 1, 3$ respectively. All the mathematical calculations are performed by using Mat lab format long.

Table-1: (x_k , y_k and c_k over the region with $a = 0.5$ and $N = 7$)

x_k	y_k	c_k
0.000000000000000	0.000000000000000	0.000511621315192
0.000000000000000	0.073223304703363	0.002297808012094
0.000000000000000	0.138196601125011	0.001121976568405
0.000000000000000	0.250000000000000	0.003446712018141
0.000000000000000	0.361803398874990	0.001121976568405
0.000000000000000	0.426776695296637	0.002297808012094
0.000000000000000	0.500000000000000	0.000511621315192
0.073223304703363	0.000000000000000	0.002273034369052
0.073223304703363	0.072433853189792	0.010208715762761
0.073223304703363	0.136706644937285	0.004984724493536
0.073223304703363	0.247304643935531	0.015313073644141
0.073223304703363	0.357902642933776	0.004984724493536
0.073223304703363	0.422175434681269	0.010208715762761
0.073223304703363	0.494609287871062	0.002273034369052
0.138196601125011	0.000000000000000	0.001078269565916
0.138196601125011	0.070370864419767	0.004842754541656
0.138196601125011	0.132813102610405	0.002364626241043
0.138196601125011	0.240261159697888	0.007264131812483
0.138196601125011	0.347709216785369	0.002364626241043
0.138196601125011	0.410151454976008	0.004842754541656
0.138196601125011	0.480522319395775	0.001078269565916
0.250000000000000	0.000000000000000	0.002984940167239
0.250000000000000	0.063413242022161	0.013406047066897
0.250000000000000	0.119681767290924	0.006545921419383
0.250000000000000	0.216506350946110	0.020109070600346
0.250000000000000	0.313330934601295	0.006545921419383
0.250000000000000	0.369599459870058	0.013406047066897
0.250000000000000	0.433012701892220	0.002984940167239

0.361803398874990	0.000000000000000	0.000744021767082
0.361803398874990	0.050539635269449	0.003478016793625
0.361803398874990	0.095385012253002	0.001698250387518
0.361803398874990	0.172553108174343	0.005217025190456
0.361803398874990	0.249721204095683	0.001698250387518
0.361803398874990	0.294566581079236	0.003478016793625
0.361803398874990	0.345106216348685	0.000744021767082
0.426776695296637	0.000000000000000	0.001197170344045
0.426776695296637	0.038149735932003	0.005376765053955
0.426776695296637	0.072001173137128	0.002625373561502
0.426776695296637	0.130251345819997	0.008065147580933
0.426776695296637	0.188501518502866	0.002625373561502
0.426776695296637	0.222352955707990	0.005376765053955
0.426776695296637	0.260502691639994	0.001197170344045
0.500000000000000	0.000000000000000	0.000000000000000
0.500000000000000	0.000000000000000	0.000000000000000
0.500000000000000	0.000000000000000	0.000000000000000
0.500000000000000	0.000000000000000	0.000000000000000
0.500000000000000	0.000000000000000	0.000000000000000
0.500000000000000	0.000000000000000	0.000000000000000
0.500000000000000	0.000000000000000	0.000000000000000

Table-2: (x_k , y_k and c_k over the region with $a=1$ and $N=7$)

x_k	y_k	c_k
0.000000000000000	0.000000000000000	0.002046485260771
0.000000000000000	0.146446609406726	0.009191232048375
0.000000000000000	0.276393202250021	0.004487906273621
0.000000000000000	0.500000000000000	0.013786848072562
0.000000000000000	0.723606797749979	0.004487906273621
0.000000000000000	0.853553390593274	0.009191232048375
0.000000000000000	1.000000000000000	0.002046485260771
0.146446609406727	0.000000000000000	0.009092137476209
0.146446609406727	0.144867706379584	0.040834863051043
0.146446609406727	0.273413289874570	0.019938897974142
0.146446609406727	0.494609287871061	0.061252294576564
0.146446609406727	0.715805285867553	0.019938897974142
0.146446609406727	0.844350869362538	0.040834863051043
0.146446609406727	0.989218575742123	0.009092137476209
0.276393202250021	0.000000000000000	0.004313078263662
0.276393202250021	0.140741728839534	0.019371018166622
0.276393202250021	0.265626205220811	0.009458504964171
0.276393202250021	0.480522319395775	0.029056527249933

0.276393202250021	0.695418433570738	0.009458504964171
0.276393202250021	0.820302909952015	0.019371018166622
0.276393202250021	0.961044638791549	0.004313078263662
0.500000000000000	0.000000000000000	0.011939760668956
0.500000000000000	0.126826484044322	0.053624188267590
0.500000000000000	0.239363534581848	0.026183685677534
0.500000000000000	0.433012701892219	0.080436282401385
0.500000000000000	0.626661869202590	0.026183685677534
0.500000000000000	0.739198919740117	0.053624188267590
0.500000000000000	0.866025403784439	0.011939760668956
0.723606797749979	0.000000000000000	0.003097608706833
0.723606797749979	0.101079270538898	0.013912067174500
0.723606797749979	0.190770024506003	0.006793001550073
0.723606797749979	0.345106216348685	0.020868100761825
0.723606797749979	0.499442408191366	0.006793001550073
0.723606797749979	0.589133162158472	0.013912067174500
0.723606797749979	0.690212432697370	0.003097608706833
0.853553390593274	0.000000000000000	0.004788681376179
0.853553390593274	0.076299471864006	0.021507060215821
0.853553390593274	0.144002346274255	0.010501494246006
0.853553390593274	0.260502691639993	0.032260590323731
0.853553390593274	0.377003037005732	0.010501494246006
0.853553390593274	0.444705911415981	0.021507060215821
0.853553390593274	0.521005383279987	0.004788681376179
1.000000000000000	0.000000000000000	0.000000000000000
1.000000000000000	0.000000000000000	0.000000000000000
1.000000000000000	0.000000000000000	0.000000000000000
1.000000000000000	0.000000000000000	0.000000000000000
1.000000000000000	0.000000000000000	0.000000000000000
1.000000000000000	0.000000000000000	0.000000000000000
1.000000000000000	0.000000000000000	0.000000000000000

Table-3: (x_k , y_k and c_k over the region with $a = 3$ and $N = 7$)

x_k	y_k	c_k
0.000000000000000	0.000000000000000	0.018418367346939
0.000000000000000	0.439339828220179	0.082721088435374
0.000000000000000	0.829179606750063	0.040391156462585
0.000000000000000	1.500000000000000	0.124081632653067
0.000000000000000	2.170820393249937	0.040391156462585
0.000000000000000	2.560660171779821	0.082721088435374
0.000000000000000	3.000000000000000	0.018418367346939
0.439339828220179	0.000000000000000	0.081829237285879

0.439339828220179	0.434603119138754	0.367513767459386
0.439339828220179	0.820239869623711	0.179450081767278
0.439339828220179	1.483827863613184	0.551270651189079
0.439339828220179	2.147415857602657	0.179450081767278
0.439339828220179	2.533052608087614	0.367513767459386
0.439339828220179	2.967655727226368	0.081829237285879
0.829179606750063	0.000000000000000	0.038817704372958
0.829179606750063	0.422225186518603	0.174339163499601
0.829179606750063	0.796878615662433	0.085126544677539
0.829179606750063	1.441566958187324	0.261508745249401
0.829179606750063	2.086255300712214	0.085126544677539
0.829179606750063	2.460908729856044	0.174339163499601
0.829179606750063	2.883133916374647	0.038817704372958
1.500000000000000	0.000000000000000	0.107457846020600
1.500000000000000	0.380479452132966	0.482617694408307
1.500000000000000	0.718090603745546	0.235653171097806
1.500000000000000	1.299038105676658	0.723926541612461
1.500000000000000	1.879985607607771	0.235653171097806
1.500000000000000	2.217596759220350	0.482617694408307
1.500000000000000	2.598076211353316	0.107457846020600
2.170820393249937	0.000000000000000	0.027878478361501
2.170820393249937	0.303237811616694	0.125208604570951
2.170820393249937	0.572310073518010	0.061137013950660
2.170820393249937	1.035318649046055	0.187812906856427
2.170820393249937	1.498327224574100	0.061137013950660
2.170820393249937	1.767399486475415	0.125208604570951
2.170820393249937	2.070637298092109	0.027878478361501
2.560660171779821	0.000000000000000	0.043098132385610
2.560660171779821	0.228898415592018	0.193563541942388
2.560660171779821	0.432007038842766	0.094513448214057
2.560660171779821	0.781508074919981	0.290345312913582
2.560660171779821	1.131009111017196	0.094513448214057
2.560660171779821	1.334117734247944	0.193563541942388
2.560660171779821	1.563016149839962	0.043098132385610
3.000000000000000	0.000000000000000	0.000000000000000
3.000000000000000	0.000000000000000	0.000000000000000
3.000000000000000	0.000000000000000	0.000000000000000
3.000000000000000	0.000000000000000	0.000000000000000
3.000000000000000	0.000000000000000	0.000000000000000
3.000000000000000	0.000000000000000	0.000000000000000
3.000000000000000	0.000000000000000	0.000000000000000

5. NUMERICAL VERIFICATION

The exact value of integrals I_1 , I_2 and I_3 along with numerical solution $I_{1RL4CC5}, I_{2RL4CC5}, I_{3RL4CC5}$ and its error bound $|E_{1RL4CC5}|, |E_{2RL4CC5}|, |E_{3RL4CC5}|$ are represented in table-4.

Where,

$$I_1 = \int_0^{0.5} \int_0^{\sqrt{0.5^2-x^2}} \int_0^{\sqrt{0.5^2-x^2-y^2}} xyz\ dz\,dy\,dx = \int_0^{0.5} \int_0^{\sqrt{0.5^2-x^2}} 0.5xy\left(0.25 - x^2 - y^2\right) dy\,dx$$

$$I_2 = \int_0^{1} \int_0^{\sqrt{1^2-x^2}} \int_0^{\sqrt{1^2-x^2-y^2}} \frac{xyz}{\sqrt{x^2+y^2+z^2}}\,dz\,dy\,dx = \int_0^{1} \int_0^{\sqrt{1^2-x^2}} xy\left(1 - \sqrt{x^2+y^2}\right) dy\,dx$$

$$I_3 = \int_0^{3} \int_0^{\sqrt{3^2-x^2}} \int_0^{\sqrt{3^2-x^2-y^2}} \frac{xyz}{\sqrt{x^2+y^2+z^2}}\,dz\,dy\,dx = \int_0^{3} \int_0^{\sqrt{3^2-x^2}} xy\left(3 - \sqrt{x^2+y^2}\right) dy\,dx$$

Table-4: Numerical Verification

a	Numerical Solution	Exact Value	Error Bound
0.5	$I_{1RL4CC5}$ =0.000325583054987	I_1 =0.000325520833333	$\lvert E_{1RL4CC5}\rvert$ =0.000000622216500
1	$I_{2RL4CC5}$ =0.025000622968335	I_2 =0.025000000000000	$\lvert E_{2RL4CC5}\rvert$ =0.000000622968335
3	$I_{3RL4CC5}$ =6.075046302397980	I_3 =6.075000000000000	$\lvert E_{3RL4CC5}\rvert$ =0.000046303979797

6. CONCLUSION

The numerical approximation of various integrals due to mixed quadrature rule R_{L4CC5} of degree of precision seven over sphere provides excellent convergence to their exact value. Our method is essential as the demand for higher order integration rules in the finite element method is increasing.

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