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Arithmetical properties of principal ideals in morphic rings

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Abstract

A lot of arithmetical properties of the principal ideals of the commutative Bezout and morphic rings are formulated in the terms of the morphic pairs and related constructions. It is proved that every morphic reduced ring is a von Neumann regular ring. In a case of an exchange ring we obtain the results on the structure of its principal ideals and the analogue of the Pythagorean theorem is proved for the morphic exchange rings.

AMS subject classification:
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1. Introduction

In [12] it is proved the lemma that allows us to determine when a commutative ring R is a morphic ring in terms of principal ideals of such rings. The mentioned statement allows us to make a partition on the pairs of the set of all principal ideals such that any element of this partition is uniquely determined by some pair of principal ideals. According to this fact it is naturally to ask: what can be said about the morphic pair if we consider the sum, product, intersection, quotient of two principal ideals as one of the components of the morphic pair? Moreover, since the annihilator operation is an involution on the lattice of all principal ideals of the morphic ring we only need to calculate the annihilator of the given sum, product, intersection, quotient of principal ideals in terms of morphic pairs of the given ones.

One of the proved below properties states that the GCD and LCM of the components of any morphic pair form a morphic pair too and the LCM is contained in the nilradical of R. Furthermore, the components of every morphic pair behave "locally" as the usual integers. As a consequence it will be shown that every reduced morphic ring is a von Neumann regular ring. Also, sometimes it is necessary to determine weather the quotient ring R/I of the commutative ring R is a morphic ring. We give an answer to this question and apply it to the cases when $I = J(R)$ and hence obtain the connection of the morphic and semiregular rings.

Moreover, not only the product, sum, intersection and quotient of the principal ideals are tightly connected between themself, but also the tensor product of two principal ideals of a morphic ring can be expressed as one of the components of some morphic pair. In fact, the tensor product of any two principal ideals is isomorphic to their intersection. As an application of the mentioned properties we will show some decompositions of the morphic exchange rings and prove the morphic analogue of the famous Pythagorean theorem. In addition we will find the generator of an intersection of two principal ideals and of their sum and quotient, that proves a well-known facts [8, 18] alternatively.

As it is proved in [15] that a commutative Bezout domain is an elementary divisor ring if and only if any quotient ring R/aR is so, where a is an arbitrary nonzero element of R. Since any finite homomorphic image R/aR of a commutative Bezout domain R is a morphic ring [20] then all the studies concerning morphic rings become related to the famous elementary divisor ring problem [8].

2. Preliminaries

All the rings considered in the article are supposed to be commutative with nonzero identity element. If S is a subset of a ring R then by $\langle S\rangle$ we mean an ideal generated by the set S. In a case when S consists of a single element $S = \{a\}$ we simply write $\langle a\rangle$. Let $U(R)$ be a set of all invertible elements of a ring R. By the *Jacobson radical* $J(R)$ of a ring R we mean the set $J(R) = \{x \in R | \forall a \in R : \ 1 - ax \in U(R)\}$, and the *nilradical* $Nil(R)$ is defined as a set of all nilpotent elements of a ring R. A ring R is called a *reduced* ring if $Nil(R) = \{0\}$. Also by $spec(R)$ (resp. $mspec(R)$, $mspec(a)$) we denote the space of all prime ideals (resp. maximal ideals, maximal ideals that contain an element a) in case of a commutative ring. By $V(I)$ and $U(I)$ we denote the closed and open sets of an ideal I of a ring R in Zarisky topology.

Suppose that A is a subset in a ring R. A set $\text{Ann}(A) = \{x | Ax = 0\}$ is called an *annihilator* of a set A. We say that an element $a \in R$ is a *regular element*, if it is not a zero divisor, that is $\text{Ann}(a) = \langle 0\rangle$. If an element $a \in R$ is a divisor of an element $b \in R$ then we will write $a|b$.

We start with recalling of some definitions and facts that we will need below in our proofs.

Definition 2.1. [2, 9] We say that a ring R is has *the stable range 1* if for any elements $a, b \in R$ the equality $\langle a, b\rangle = \langle 1\rangle$ implies that there is some $x \in R$ such that $\langle a + bx\rangle = \langle 1\rangle$. If such element $x \in R$ always can be taken to be an idempotent then it is said that R has *the idempotent stable range 1*.

Definition 2.2.

1) If any finitely generated ideal of a ring R is principal then a ring R is said to be a *Bezout ring*.

2) We say that a rectangular matrix A over a ring R *admits canonical diagonal reduction* if there are two invertible matrices P, Q of the appropriate sizes such that the matrix $PAQ = D = (d_i)$ is a diagonal matrix with an additional condition: for all indices we have the inclusion $\langle d_i \rangle \supseteq \langle d_{i+1} \rangle$. If every matrix over a ring R admits canonical diagonal reduction then R is said to be an *elementary divisor ring*.

3) A ring R is said to be a *P-injective ring* if any homomorphism from any principal ideal $\langle a \rangle$ into R can be extended to an endomorphism of the ring R, namely, using the multiplication by some fixed element of the ring R [13].

4) A ring R is called a *morphic ring* if for any $a \in R$ there is an isomorphism $R/\langle a \rangle \cong \mathrm{Ann}(a)$ of the left (right) modules [12].

Here we gather some results concerning our topic.

Theorem 2.3. For any elements a, b in a right Bezout ring R of stable range 1 one can find some elements $x, d \in R$ such that $a + bx = d$ and $\langle a, b \rangle = \langle d \rangle$ [19].

Here are some Nicholson's criterions for the P-injective and the morphic rings in a commutative case.

Theorem 2.4.

1) The following statements are equivalent for a ring R:

   a) R is a right P-injective ring;
   b) For any $a \in R$: $\mathrm{Ann}(\mathrm{Ann}(a)) = \langle a \rangle$ [13].

2) The following statements are equivalent for a ring R:

   a) R is a morphic ring;
   b) For any $a \in R$ one can find $b \in R$ such that $\mathrm{Ann}(a) = \langle b \rangle$, $\mathrm{Ann}(b) = \langle a \rangle$;
   c) For any $a \in R$ one can find $b \in R$ such that $\mathrm{Ann}(a) = \langle b \rangle$, $\mathrm{Ann}(b) \cong \langle a \rangle$ [12].

All von Neumann regular rings, $\mathbb{Z}_n$ and other finite homomorphic images of commutative Bezout domains are the examples of morphic rings [20]. For instance, a finite homomorphic image of Bezout domain $\hat{H} = \mathbb{Z} + x\mathbb{Q}[[x]]$ by the ideal $\langle x \rangle$ is isomorphic to the trivial extension $R = T(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$. The latter ring R is a morphic ring.

Also in [12] it is proved that every morphic ring is a P-injective one, and that the commutative von Neumann regular rings are morphic rings. In addition it is useful to mention that a pair (a, b) of elements of a ring R in the previous theorem is called a **morphic pair** and this fact will be denoted as $\langle a\rangle \sim \langle b\rangle$, since any morphic pair is determined by the pair of some principal ideals, but not elements.

Definition 2.5. An element a of a commutative ring R is said to be a *von Neumann regular element* if there is some $b \in R$ such that $a^2b = a$. If all elements of a ring R are von Neumann regular then R is called a *von Neumann regular ring*. A ring R is called a *semiregular ring* if $R/J(R)$ is a von Neumann regular ring and the idempotents can be lifted modulo $J(R)$.

Definition 2.6. [11, 16] An element a of a ring R is said to be an *exchange element* if there is some idempotent $e = e^2 \in R$ such that $\langle e\rangle \subseteq \langle a\rangle$, $\langle 1-e\rangle \subseteq \langle 1-a\rangle$. A ring R is called *an exchange ring* if all its elements the are exchange elements. An element a of a ring R is called a *clean element* if it can be expressed as a sum of an invertible element and an idempotent. If all elements of a ring R are clean then R is called a *clean ring*.

As it is proved in [9] for a commutative ring R the following conditions are equivalent:

- R is an exchange ring;
- R is a clean ring;
- R is an idempotent stable range 1 ring.

As it is stated in [20], a is an adequate element of Bezout domain R if and only if R/aR is a semiregular ring, hence it is a clean ring, and as it was mentioned it is morphic ring at the same time. At the same time not all clean rings are even Bezout rings: a trivial extension ring $T(k[[x]], k[[x]]/\langle x\rangle)$ is a such a ring, where k is any field. Moreover, a ring $R = T(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ mentioned above is morphic but not semiregular, since $R/J(R) \cong \mathbb{Z}$ that is not regular. Thus the class of all Bezout domains provides us a lot of tools to construct morphic and clean rings.

3. Arithmetics of morphic pairs and related questions

Lemma 3.1. [Ideal quotient properties] Let R be a commutative ring. Then the *ideals quotient* $(A : B) = \{x \in R |\ xB \subset A\}$ of the ideals A, B of a ring R satisfies the following properties:

1. $(A : B)B \subset A \subset (A : B)$;
2. $((A : B) : C) = (A : BC) = ((A : C) : B)$;

3. $(\bigcap_i A_i : B) = \bigcap_i (A_i : B)$;

4. $(A : \sum_i B_i) = \bigcap_i (A : B_i)$;

5. $\mathrm{Ann}(A) = (0 : A)$.

Lemma 3.2. [Arithmetical properties of morphic pairs] Let R be a morphic ring and $a, b, c, d, x \in R$. Then

1) If $\langle a\rangle \sim \langle b\rangle$ and $\langle c\rangle \sim \langle d\rangle$ then
   - $\langle a, c\rangle \sim \langle b\rangle \cap \langle d\rangle$;
   - $\langle a\rangle \cap \langle c\rangle \sim \langle b, d\rangle$;
   - $\langle ac\rangle \sim (b : c) = (d : a)$;
   - $(a : c) \sim \langle bc\rangle$.

2) If $\langle a\rangle \sim \langle b\rangle$ then $\langle a, b\rangle \sim \langle a\rangle \cap \langle b\rangle$, i.e. the GCD and LCM of the components of morphic pair also form a morphic pair and if $g = \mathrm{lcm}(a, b)$ then $g^2 = 0$.

3) If $\langle a\rangle \sim \langle b\rangle$ then
   - $\langle a^2\rangle \sim (b : a)$;
   - $\langle a^2, b^2\rangle \sim (a : b) \cap (b : a)$;
   - $\langle a^2, b\rangle = (a^3 : a) \sim a(b : a^2)$;
   - $\langle a, b^2\rangle = (b^3 : b) \sim b(a : b^2)$;
   - $(a^3 : a)(b^3 : b) = \langle a^3, b^3\rangle = \langle a, b\rangle^3$.

4) If $\langle a\rangle \sim \langle b\rangle$, $\langle a'\rangle \sim \langle c\rangle$ and $\langle a, a'\rangle = \langle 1\rangle$ then $a|c$, $a'|b$ and $\langle b\rangle \cap \langle c\rangle = 0$.

5) $\mathrm{Ann}(a) \subset \langle 1 - ax\rangle$, $\mathrm{Ann}(1 - ax) \subset \langle a\rangle$. Moreover, if $a^2x = a$ then $\langle a\rangle \sim \langle 1 - ax\rangle$.

6) If $\langle a, b\rangle = \langle 1\rangle$ then $\mathrm{Ann}(ab) \subseteq \langle a + tb\rangle$, for any element $t \in R$ such that $\langle a, t\rangle = \langle 1\rangle$.

Proof. For all groups of properties we will use the mentioned above properties of the ideals quotients.

1) It is sufficient to mention that:
   - $\mathrm{Ann}(a, c) = \mathrm{Ann}(a) \cap \mathrm{Ann}(c) = \langle b\rangle \cap \langle d\rangle$;

- $\mathrm{Ann}(ac) = (0 : ac) = ((0 : a) : c) = (b : c)$, $\mathrm{Ann}(ac) = (0 : ac) = ((0 : c) : a) = (d : a)$;
- $\mathrm{Ann}(bc) = (0 : bc) = (a : c)$.

Since $\mathrm{Ann}(\mathrm{Ann}(x)) = \langle a \rangle$ all the desired properties are proved.

2) According to the previous properties we know that $\langle a, b \rangle \sim \langle a \rangle \cap \langle b \rangle$. If we denote $\langle g \rangle = \langle a \rangle \cap \langle b \rangle$ and $\langle d \rangle = \langle a, b \rangle$ then $\langle g \rangle \subset \langle d \rangle$ and $\langle g \rangle \langle g \rangle \subset \langle gd \rangle = \langle 0 \rangle$, so $g^2 = 0$.

3) Similarly to 1) we obtain:

- $\mathrm{Ann}(a^2) = (0 : a^2) = ((0 : a) : a) = (b : a)$;
- $\mathrm{Ann}(a^2, b^2) = \mathrm{Ann}(a^2) \cap \mathrm{Ann}(b^2) = (a : b) \cap (b : a)$;
- $\mathrm{Ann}(a^2, b) = \mathrm{Ann}(a^2) \cap \mathrm{Ann}(b) = \langle a \rangle \cap \mathrm{Ann}(a^2) = \{ar | a^3 r = 0\} = a\mathrm{Ann}(a^3) = a(0 : a^3) = a(b : a^2)$;
- $\langle a^2, b \rangle = \mathrm{Ann}(a\mathrm{Ann}(a^3)) = (a^3 : a)$;
- $(a^3 : a)(b^3 : b) = \langle a^2, b \rangle \langle a, b^2 \rangle = \langle a^3, ab^2, a^2 b, b^3 \rangle = \langle a^3, 0, 0, b^2 \rangle = \langle a^3, b^3 \rangle = \langle a, b \rangle^3$.

All other inclusions and properties can be proved analogously.

4) Since a, a' are coprime then their annihilators have zero intersection: $\langle b \rangle \cap \langle c \rangle = \langle 0 \rangle$. Then $bc = 0$ and $b \in \mathrm{Ann}(c) = \langle a' \rangle$. Hence $a' | b$. Similarly we obtain $a | c$.

5) If $a^2 x = a$ then $\langle 1 - ax \rangle \subset \mathrm{Ann}(a) \subset \langle 1 - ax \rangle$ by the previous property. So $\mathrm{Ann}(a) = \langle 1 - ax \rangle$.

6) From $\langle 1 \rangle = \langle a, b \rangle \langle a, t \rangle = \langle a, tb \rangle$ we can conclude that there are some $u, v \in R$ such that $au + tbv = 1$. If we denote $\langle l \rangle = \mathrm{Ann}(a + tb)$ and $m = la = -ltb$ then multiplication of $au + tbv = 1$ by l gives us: $l = lau + ltbv = m(u - v)$. Hence $l \in \langle m \rangle \subseteq \langle l \rangle$, that is $\langle l \rangle = \langle m \rangle$. Therefore, $\langle l \rangle = \langle m \rangle = \langle la \rangle = \langle ltb \rangle \subseteq \langle a \rangle \cap \langle b \rangle = \langle ab \rangle$. Taking the annilators of the both sides of the inclusion we obtain the desired property.

The lemma is proved. ■

Lemma 3.3. A total ring of fractions $Q(R)$ of a morphic ring R coincides with R.

Proof. Suppose that $d \in R \setminus \{0\}$ is not a zero divisor. Then $\mathrm{Ann}(d) = \langle 0 \rangle$. But all morphic pairs are defined uniquely in the terms of the ideals of a morphic ring R, so $\langle 0 \rangle \sim \langle 1 \rangle$ and $\langle d \rangle = \langle 1 \rangle$. Hence d is an invertible element, and the localization of R by the set of its nonzero divisors is trivial. The lemma is proved. ■

Lemma 3.4. [Five elements lemma] Let R be a morphic ring and $\langle a\rangle \sim \langle b\rangle$. Suppose that $\langle d\rangle = \langle a, b\rangle$, $a = a_0 d$, $b = b_0 d$ and $\langle g\rangle = \langle a\rangle \cap \langle b\rangle$. Then the product of elements of any nontrivial subset of $\{a_0, b_0, d, d, 1\}$ is morphic to the product of the other elements. The latter means that:

1. $\langle a_0 b_0\rangle \sim \langle d^2\rangle$;
2. $\langle a_0 d^2\rangle \sim \langle b_0\rangle$;
3. $\langle b_0 d^2\rangle \sim \langle a_0\rangle$;
4. $\langle g\rangle = \langle a_0 b_0 d\rangle \sim \langle d\rangle$.

Proof. By Lemma 2 we know that $\langle a\rangle \sim \langle b\rangle$ and $\langle d\rangle \sim \langle g\rangle$. Then by the same lemma $\langle ad\rangle \sim (b : d)$. Since $b_0 d = b$ then $\langle b_0\rangle \subset (b : d)$. If we take any $x \in (b : d)$ then $xd = bt$, for some $t \in R$. Then $xd = b_0 dt$ and hence $x - b_0 t \in \mathrm{Ann}(d) = \langle g\rangle \subset \langle b\rangle \subset \langle b_0\rangle$. So $x \in \langle b_0\rangle$ and $(b : d) \subset \langle b_0\rangle$. As a result $\langle ad\rangle \sim \langle b_0\rangle$. Similarly it can be proved that $\langle bd\rangle \sim \langle a_0\rangle$.

Again by Lemma 2 we have: $\langle a_0 b_0\rangle \sim (d^2 b_0 : b_0)$. Obviously $\langle d^2\rangle \subset (d^2 b_0 : b_0)$. As it was before we again take an element $x \in (d^2 b_0 : b_0)$ and obtain $x b_0 = d^2 b_0 t$, for some $t \in R$. Therefore, $x - d^2 t \in \mathrm{Ann}(b_0) = \langle a_0 d^2\rangle \subset \langle d^2\rangle$. Thus $x \in \langle d^2\rangle$ and $(d^2 b_0 : b_0) = \langle d^2\rangle$. So $\langle a_0 b_0\rangle \sim \langle d^2\rangle$ as was desired.

In order to obtain the latter morphic pair we again use Lemma 2. Since $\langle a_0 b_0\rangle \sim \langle d^2\rangle$ and $\langle d\rangle \sim \langle g\rangle$ then $\langle a_0 b_0 d\rangle \sim (d^2 : d)$. If $x \in (d^2 : d)$ then $xd = d^2 t$, for some $t \in R$. Hence $x - dt \in \mathrm{Ann}(d) = \langle g\rangle \subset \langle d\rangle$ and $x \in \langle d\rangle$. So, $\langle d\rangle \subset (d^2 : d) \subset \langle d\rangle$. As a result $\langle g\rangle = \langle a_0 b_0 d\rangle \sim \langle d\rangle$.

The lemma is proved. ■

Remark 3.5. Five elements lemma tells us the next fact: the elements of any morphic pair locally have the same behavior as the usual integers; the transferring of the cofactors from one part of the morphic pair to another one preserves the morphic pair.

Theorem 3.6. Let R be a commutative ring and I is some its ideal. Then a ring R/I is morphic if and only if for any element $\overline{a} \in R/I$ there is some element $\overline{b} \in R/I$ such that

$$(I : a) = I + \langle b\rangle, \ (I : b) = I + \langle a\rangle.$$

Proof. Suppose that $\overline{R} = R/I$ is a morphic ring. Then for any element $\overline{a} \in R/I$ there is some element $\overline{b} \in R/I$ such that $\langle \overline{a}\rangle = \mathrm{Ann}(\overline{b})$. Then

$$\langle \overline{b}\rangle = \mathrm{Ann}(\overline{a}) = \{\overline{x} \mid \overline{xa} = \overline{0}\} = \{\overline{x} \mid xa \in I\}.$$

Thus $\overline{x} \in \langle \overline{b}\rangle$ if and only if $x \in (I : a)$. Hence $(I : a) = I + \langle b\rangle$. Analogously we obtain the another equality.

Conversely, suppose that for any element $\overline{a} \in R/I$ there is some element $\overline{b} \in R/I$ such that $(I : a) = I + \langle b\rangle$, $(I : b) = I + \langle a\rangle$. If we denote by $f : R \to \overline{R} = R/I$

a canonical homomorphism from R onto the ring $\overline{R}$, then $\mathrm{Ann}(\overline{a}) = \{\overline{x} \mid xa \in I\} = f((I : a))$. However, $f((I : a)) = f(I + \langle b\rangle) = \langle\overline{b}\rangle$. Similarly we obtain that $\mathrm{Ann}(\overline{b}) = \overline{a}$. The theorem is proved. ■

Theorem 3.7. For any commutative ring R the following properties hold:

(i) If $\overline{a}, \overline{b} \in \overline{R} = R/J(R)$ then $\langle\overline{a}\rangle \sim \langle\overline{b}\rangle$ if and only if $\bigcap\{M \mid M \notin \mathrm{mspec}(a)\} \subset J(R) + \langle b\rangle$, $\bigcap\{M \mid M \notin \mathrm{mspec}(b)\} \subset J(R) + \langle a\rangle$.

(ii) If $\overline{a}, \overline{b} \in \overline{R} = R/Nil(R)$ then $\langle\overline{a}\rangle \sim \langle\overline{b}\rangle$ if and only if $\bigcap\{(P : a) \mid P \in \mathrm{spec}(R)\} \subset Nil(R) + \langle b\rangle$, $\bigcap\{(P : b) \mid P \in \mathrm{spec}(R)\} \subset Nil(R) + \langle a\rangle$.

(iii) If R is a morphic ring and $\langle a\rangle \sim \langle b\rangle$ then $\mathrm{mspec}(R) = U(a) \sqcup U(b) \sqcup V(\langle a, b\rangle)$.

Proof. Since $J(R) = \bigcap\{M \mid M \in \mathrm{mspec}(R)\}$ then $(J(R) : x) = \bigcap\{(M : x) \mid M \in \mathrm{mspec}(R)\}$. If $x \in M \in \mathrm{mspec}(R)$ then $(M : x) = R$, if $x \notin M \in \mathrm{mspec}(R)$ then $(M : x) \supset M$. By the maximality of M we conclude that $M = (M : x)$. Thus by Theorem 1 we obtain the statement of (i). Similarly we can prove (ii).

In order to prove (iii) we notice that $0 = ab \in M \in \mathrm{mspec}(R)$ implies that if $M \in U(a)$ then $M \in V(b)$. Thus $M \notin (U(a) \sqcup U(b))$ if and only if $M \in V(a) \cap V(b) = V(\langle a, b\rangle)$. Finally, we obtain that $\mathrm{mspec}(R) = U(a) \sqcup U(b) \sqcup V(\langle a, b\rangle)$. The theorem is proved. ■

As a corollary we obtain one well-known result. However we present here its equivalent proof.

Corollary 3.8. Every morphic reduced ring R is a von Neumann regular ring.

Proof. Let $a \in R$ be an arbitrary element of R. Since R is a morphic ring then there is some $b \in R$ such that $\langle a\rangle \sim \langle b\rangle$. Then by Lemma 2 we obtain that $\langle a, b\rangle \sim \langle a\rangle \cap \langle b\rangle$, $\langle g\rangle = \langle a\rangle \cap \langle b\rangle$ and $g^2 = 0$. Since R is a reduced ring and $\langle 1\rangle \sim \langle 0\rangle$ then $\langle a, b\rangle = \langle 1\rangle$. Multiplying the last equality by $\langle a\rangle$ we obtain that $\langle a^2\rangle = \langle a\rangle$, that means that R is a von Neumann regular ring. ■

Corollary 3.9. Using the result of Theorem 3 we have:

(i) If idempotents in R lifts modulo $J(R)$ then $R/J(R)$ is a morphic ring if and only if R is a semiregular ring.

(ii) The stable range of a morphic reduced ring equals to 1.

In a case of the rings of idempotent stable range 1 we can obtain the following result that can be interpreted as a one famous theorem from the elementary geometry.

Theorem 3.10. [Morphic Pythagorean theorem] Let R be a commutative Bezout ring of idempotent stable range 1. If $ab = 0$ then there exists $d \in R$ such that $a^2+b^2 = d^2,\ \langle d\rangle = \langle a, b\rangle$.

Proof. By the first theorem of Preliminaries section there are two idempotents $e, f \in R$, an element $d \in R$ and $w \in U(R)$ such that $a + be = d,\ af + b = dw,\ \langle a, b\rangle = \langle d\rangle$. Then

$$a + be = d = afw^{-1} + bw^{-1}.$$

The latter double equality implies that $a(1 - fw^{-1}) = b(w^{-1} - e)$. After the multiplication by b we obtain that $b^2e = b^2w^{-1}$. Thus

$$d^2 = (a + be)^2 = a^2 + b^2e^2 = a^2 + (b^2e)e = a^2 + b^2ew^{-1} = a^2 + b^2(w^{-1})^2.$$

However,

$$d^2 = a^2 + b^2e^2 = a^2 + b^2e = a^2 + b^2w^{-1}.$$

After the subtraction we will have:

$$0 = d^2 - d^2 = a^2 + b^2(w^{-1})^2 - a^2 - b^2w^{-1}.$$

Hence $b^2(w^{-1})^2 = b^2w^{-1}$ and then $b^2w^{-1} = b^2$. Finally,

$$d^2 = a^2 + b^2w^{-1} = a^2 + b^2.$$

The theorem is proved. ■

Corollary 3.11. Similarly we can prove the analogue of the latter theorem for the cubes: $a^3 + b^3 = d^3$. Since 2 and 3 are coprime integers then for any positive integer $n \geq 2$ we have also $a^n + b^n = d^n$, where $\langle a\rangle \sim \langle b\rangle$ and $\langle a, b\rangle = \langle d\rangle$ in a commutative idempotent stable range 1 morphic ring R.

In the following 2 lemmas we will prove a few additional arithmetical properties of morphic and P-injective rings, that can be used in other researches concerning this classes of rings.

4. Arithmetical properties of principal ideals in Bezout and morphic rings

We begin with one well-known property of the tensor product of modules over some commutative ring, that will be used several times below.

Proposition 4.1. Let M be a R-module and I, J are some ideals of a commutative ring R. Then

(i) $M \otimes_R R/I \cong M/IM$;

(ii) $R/I \otimes_R R/J \cong R/(I+J)$.

As a corollary we obtain the following statement:

Proposition 4.2. The following properties hold in any commutative morphic ring R:

(i) $\langle a \rangle \otimes_R \langle b \rangle \cong \langle a \rangle \cap \langle b \rangle$, for every $a, b \in R$;

(ii) $\langle a \rangle \otimes_R \langle a \rangle \cong \langle a \rangle$, for every $a \in R$;

(iii) If $\langle a \rangle \sim \langle b \rangle$ and $\langle c \rangle \sim \langle d \rangle$ then $\langle a \rangle \otimes_R \langle c \rangle \cong \langle a \rangle / \langle ad \rangle \cong \langle c \rangle / \langle cb \rangle$;

(iv) $\langle a \rangle \otimes_R \mathrm{Ann}(a) \cong \langle a \rangle / \langle a^2 \rangle$ for every $a \in R$;

(v) If $\langle a \rangle \subset \langle b \rangle$ then $\langle a \rangle \otimes_R \langle b \rangle \cong \langle a \rangle$.

(vi) If $\langle a, b \rangle = \langle 1 \rangle$ then $\langle a \rangle \otimes_R \langle b \rangle \cong \langle ab \rangle$.

Proof. By the previous statement $\langle a \rangle \otimes_R \langle b \rangle \cong R/\mathrm{Ann}(a) \otimes_R R/\mathrm{Ann}(b) \cong R/(\mathrm{Ann}(a) + \mathrm{Ann}(b)) \cong \mathrm{Ann}(\mathrm{Ann}(a) + \mathrm{Ann}(b)) \cong \langle a \rangle \cap \langle b \rangle$.

Also, if $\langle a \rangle \sim \langle b \rangle$ and $\langle c \rangle \sim \langle d \rangle$ then $\langle a \rangle \otimes_R \langle c \rangle \cong \langle a \rangle \otimes_R R/\langle d \rangle \cong \langle a \rangle / \langle ad \rangle$. Similarly $\langle a \rangle \otimes_R \langle c \rangle \cong \langle c \rangle / \langle cb \rangle$. All other desired properties follow immediately from these ones. The statement is proved. ∎

Remark 4.3. We conclude that an element a in a morphic ring R is a von Neumann regular element if and only if $\langle a \rangle \otimes_R \mathrm{Ann}(a) = 0$.

Proposition 4.4. Let R be a commutative morphic exchange ring and a is its arbitrary element. Then there is an idempotent $e \in R$ such that

(i) $\langle a \rangle \otimes_R \langle e \rangle \cong \langle e \rangle$;

(ii) $\langle 1-a \rangle \otimes_R \langle 1-e \rangle \cong \langle 1-e \rangle$;

(iii) $R \cong \langle 1-a \rangle / \langle (1-a)e \rangle \oplus \langle a \rangle / \langle a(1-e) \rangle$;

(iv) $\langle a \rangle \otimes_R \langle 1-a \rangle \cong \langle a \rangle / \langle a^2 x \rangle \oplus \langle 1-a \rangle / \langle (1-a)^2 y \rangle$, for some $x, y \in R$.

Proof. Since R is an exchange ring then there is an idempotent $e^2 = e \in R$ such that

$$e \in \langle a \rangle,\ 1-e \in \langle 1-a \rangle, e = ax, 1-e = (1-a)y,$$

for some $x, y \in R$. Then by the previous statement $\langle a \rangle \otimes_R \langle e \rangle \cong \langle e \rangle$ and $\langle 1-a \rangle \otimes_R \langle 1-e \rangle \cong \langle 1-e \rangle$. Therefore

$$R = \langle e \rangle \oplus \langle 1-e \rangle \cong (\langle a \rangle \otimes_R \langle e \rangle) \oplus (\langle 1-a \rangle \otimes_R \langle 1-e \rangle) \cong \langle a \rangle / \langle a(1-e) \rangle \oplus \langle 1-a \rangle / \langle (1-a)e \rangle.$$

At last,

$$\begin{aligned}\langle a\rangle \otimes_R \langle 1-a\rangle &\cong \langle a\rangle \otimes_R \langle 1-a\rangle \otimes_R R = \langle a\rangle \otimes_R \langle 1-a\rangle \otimes_R (\langle e\rangle \oplus \langle 1-e\rangle)\\ &\cong (\langle a\rangle \otimes_R \langle 1-a\rangle \otimes_R \langle e\rangle) \oplus (\langle a\rangle \otimes_R \langle 1-a\rangle \otimes_R \langle 1-e\rangle)\\ &\cong (\langle 1-a\rangle \otimes_R \langle e\rangle) \oplus (\langle a\rangle \otimes_R \langle 1-e\rangle)\\ &\cong \langle 1-a\rangle/\langle (1-a)(1-e)\rangle \oplus \langle a\rangle/\langle ae\rangle\\ &\cong \langle a\rangle/\langle a^2x\rangle \oplus \langle 1-a\rangle/\langle (1-a)^2y\rangle.\end{aligned}$$

The statement is proved. ■

Below we use the following notation: if $a, b, d \in R$ are such that $\langle a, b\rangle = \langle d\rangle$ then there are some $a_0, b_0 \in R$ that satisfy the equalities $a = a_0 d,\ b = b_0 d$. The element a_0 we will sometimes write as $a_0 = \dfrac{a}{(a,b)}$, meaning that (a, b) is a GCD of elements a, b.

Lemma 4.5. Let R be a Bezout ring and $a, b \in R$. Then $(a : b) = \langle \dfrac{a}{(a,b)}\rangle + \mathrm{Ann}(\langle a\rangle + \langle b\rangle)$ and $\langle a\rangle \cap \langle b\rangle = \langle \dfrac{ab}{(a,b)}\rangle$.

Proof. Suppose that $\langle a, b\rangle = \langle d\rangle,\ a = a_0 d,\ b = b_0 d,\ au + bv = d$ for some elements $a_0, b_0, d, u, v \in R$. Then since $a_0 b = b_0 a \in (a : b)$, $\mathrm{Ann}(d) \subseteq \mathrm{Ann}(b)$ then $\langle a_0\rangle + \mathrm{Ann}(d) \subseteq (a : b)$. Moreover, if $xb \in \langle a\rangle$ then there is some $s \in R$ such that $xb = as$. Hence $xbv = asv$ and $xd(1 - a_0 u) = da_0 sv$ implies that $x(1 - a_0 u) - a_0 sv \in \mathrm{Ann}(d)$. As a result we obtain $x \in \langle a_0\rangle + \mathrm{Ann}(d)$ and $(a : b) \subseteq \langle a_0\rangle + \mathrm{Ann}(d)$ as was desired.

For the next part of lemma we suppose that some $x \in \langle a\rangle \cap \langle b\rangle$. Then there are such $r, s \in R$ such that $r \in (b : a),\ s \in (a : b)$. Using this we have that $x \in a(b : a) \cap b(a : b) \subseteq \langle a\rangle \cap \langle b\rangle$. So, $\langle a\rangle \cap \langle b\rangle = a(b : a) \cap b(a : b) = a(\langle b_0\rangle + \mathrm{Ann}(d)) \cap b(\langle a_0\rangle + \mathrm{Ann}(d)) = \langle ab_0\rangle \cap \langle ba_0\rangle = \langle a_0 b_0 d\rangle$. The lemma is proved. ■

As a simple corollary of the latter lemma we obtain:

Corollary 4.6. Let R be a Bezout ring and $a, b \in R$. Then $R/(a : b) \cong \langle a, b\rangle/\langle a\rangle$ as R-modules.

Proof. Combining proposition 1 and the previous lemma we obtain:

$$\begin{aligned}\frac{R}{(a:b)} &= \frac{R}{\langle \frac{a}{(a,b)}\rangle + \mathrm{Ann}(\langle a,b\rangle)} \cong \frac{R}{\langle \frac{a}{(a,b)}\rangle} \otimes_R \frac{R}{\langle \mathrm{Ann}(\langle a,b\rangle)\rangle}\\ &\cong \frac{R}{\langle \frac{a}{(a,b)}\rangle} \otimes_R \langle a,b\rangle \cong \frac{\langle a,b\rangle}{\langle a,b\rangle\langle \frac{a}{(a,b)}\rangle} = \frac{\langle a,b\rangle}{\langle a\rangle}.\end{aligned}$$

The corollary is proved. ■

So, if we denote $\overline{R} = R/aR$ then $\overline{bR} \cong R/(a : b)$ as R-modules, and this means that principal ideals of a finite homomorphic images of a Bezout ring are isomorphic as modules to homomorphic images of this ring.

In the latter lemma we have obtained not only that intersection and quotient of two principal ideals in a commutative Bezout ring R are again principal ideals (that is well-known fact, see [8, 18]), but we have found the generators of these ideals.

The following theorem states that every finite homomorphic image of a Bezout ring R by any regular ideal is a morphic ring.

Theorem 4.7. Let R be a Bezout ring and $a \in R \setminus \{0\}$ is not a zero divisor. Then R/aR is a morphic ring.

Proof. Let we have any $\overline{b} \in R/aR$. Suppose that $\langle a, b\rangle = \langle d\rangle,\ a = a_0 d,\ b = b_0 d$ for some elements $a_0, b_0, d \in R$. Then by the previous lemma $(a : b) = \langle a_0\rangle + \mathrm{Ann}(d) = \langle a_0\rangle$, since $\mathrm{Ann}(d) \subseteq \mathrm{Ann}(a) = 0$. Taking $c = a_0$ we have that $(a : b) = \langle c\rangle = \langle c, a\rangle$, and

$$(a : c) = (a : a_0) = \langle \frac{a}{(a, a_0)} \rangle + \mathrm{Ann}(\langle a_0\rangle + \langle a\rangle) = \langle d\rangle + \mathrm{Ann}(a_0) = \langle d\rangle = \langle b, a\rangle.$$

Hence the conditions of Theorem 1 are satisfied and R/aR is a morphic ring.

The theorem is proved. ■

The following property connects the GCD and LCM of any two principal ideals is any *arithmetical ring*, i.e. the commutative ring with distributive lattice of ideals. We will prove it in a commutative case, but it is still valid in a noncommutative one with a few additional assumptions.

Proposition 4.8. Let R be an arithmetical ring and $a, b \in R$. Then $\langle a, b\rangle = \langle a + tb\rangle + \langle a\rangle \cap \langle b\rangle$, for any element $t \in R$ such that $\langle a, t\rangle = \langle 1\rangle$.

Proof. The inclusion of right hand side in a left one is evident. Now suppose that $x = au + bv \in \langle a, b\rangle$ is an arbitrary element. Then $x = (a + tb)u + b(v - ut) \in \langle a + tb\rangle + \langle b\rangle$. Since $\langle a, t\rangle = \langle 1\rangle$ then $ap + tq = 1$, for some elements $p, q \in R$. Moreover, $x = au + (ap + tq)bv = a(u + pbv - qv) + (a + tb)qv \in \langle a + tb\rangle + \langle a\rangle$. By the distributivity of the lattice of ideals of R we obtain

$$\langle a, b\rangle \subseteq (\langle a + tb\rangle + \langle b\rangle) \cap (\langle a + tb\rangle + \langle a\rangle) = \langle a + tb\rangle + \langle a\rangle \cap \langle b\rangle.$$

Combining both inclusions we obtain the desired property. The statement is proved. ■

If we take $t = 1$ then the latter statement can be formulated as:

$$GCD(a, b) = GCD(a + b, LCM(a, b)),$$

or even $GCD(a + b, ab) = 1$, if a, b are coprime.

The last property will connects two subsets that are associated with any element of almost Baer Bezout ring in the set of all its principal ideals.

Definition 4.9. A ring R is said to be an *almost Baer ring* if an annihilator of each element in R is a principal ideal [18].

Theorem 4.10. Let R be an almost Baer Bezout ring and $a \in R$. Then

$$(a : (a : \mathrm{Ann}(a))) = \langle a \rangle + \mathrm{Ann}(a),\ (a : (a : (a : \mathrm{Ann}(a)))) = (a : \mathrm{Ann}(a))$$

. This means that there is a bijective correspondence between the sets

$$D(R) = \{(a : \mathrm{Ann}(a)) = a_D,\ a \in R\},\ S(R) = \{\langle a \rangle + \mathrm{Ann}(a) = a_S,\ a \in R\}$$

given by $a_D \leftrightarrow a_S$.

Proof. Let $\langle b \rangle = \mathrm{Ann}(a),\ \langle a, b \rangle = \langle d \rangle,\ a = a_0 d,\ b = b_0 d,\ \langle a \rangle + \mathrm{Ann}(d) = \langle g \rangle$, for some elements $b, a_0, a_0, d, g \in R$. Then

$$(a : (a : b)) = (a : \langle a_0 \rangle + \mathrm{Ann}(d)) = (a : a_0) \cap (a : \mathrm{Ann}(d)) = (\langle d \rangle + \mathrm{Ann}(a_0)) \cap (a : g) =$$

$$(\langle a \rangle + \mathrm{Ann}(a) + \mathrm{Ann}(a_0)) \cap (\langle \frac{a}{g} \rangle + \mathrm{Ann}(g)) = (\langle a \rangle + \mathrm{Ann}(a)) \cap (\langle \frac{a}{g} \rangle + \mathrm{Ann}(g)) \subseteq \langle d \rangle = \langle a, b \rangle.$$

For the reverse inclusion note that $a \in (a : (a : b))$, so we only need to prove that $b \in (a : (a : b))$. Since $\mathrm{Ann}(g) \subseteq \mathrm{Ann}(a) = \langle b \rangle \subseteq \langle d \rangle$, then $\mathrm{Ann}(g) \cap \langle d \rangle = \mathrm{Ann}(g)$. As far as $g = ar + d_0$, for some $r \in R, d_0 \in \mathrm{Ann}(d)$, then $bg = bar + bd_0 = 0$ and $\langle b \rangle \subseteq \mathrm{Ann}(g) \subseteq \langle d \rangle \cap \langle \frac{a}{g} \rangle + \mathrm{Ann}(g) = (a : (a : b))$, that was desired.

We finish the proof with the observation: $(a : (a : (a : \mathrm{Ann}(a)))) = (a : \langle a \rangle + \mathrm{Ann}(a)) = (a : a) \cap (a : \mathrm{Ann}(a)) = (a : \mathrm{Ann}(a))$. The theorem is proved. ■

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